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# Finslerian Hypersurface and Generalized $\beta-$Conformal Change of Finsler Metric

Shiv Kumar Tiwari* and Anamika Rai
*Department of Mathematics, K. S. Saket Post Graduate College, Ayodhya, Faizabad-224 123, India*
*e-mail* : sktiwarisaket@yahoo.com *and* anamikarai2538@gmail.com

Abstract. In the present paper, we have studied the Finslerian hypersurfaces and generalized $\beta-$conformal change of Finsler metric. The relations between the Finslerian hypersurface and the other which is Finslerian hypersurface given by generalized $\beta-$conformal change have been obtained. We have also proved that generalized $\beta-$conformal change makes three types of hypersurfaces invariant under certain conditions.

## 1. Introduction

Let $(M^n, L)$ be an $n-$dimensional Finsler space on a differentiable manifold $M^n$ equipped with the fundamental function $L(x, y)$. In 1984, Shibata [12] introduced the transformation of Finsler metric:

$$\overline{L}(x, y) = f(L, \beta), \tag{1.1}$$

where $\beta = b_i(x)\, y^i$, $b_i(x)$ are components of a covariant vector in $(M^n, L)$ and $f$ is positively homogeneous function of degree one in $L$ and $\beta$. This change of metric is called a $\beta-$change. In 2013, Prasad, B. N. and Kumari, Bindu [10] have considered the $\beta-$change of Finsler metric. In the year 2014 [13], we studied generalized $\beta-$change defining as

$$L(x, y) \to \overline{L}(x, y) = f(L, \beta^{1)}, \beta^{2)}, \ldots, \beta^{m)}), \tag{1.2}$$

where $f$ is any positively homogeneous function of degree one in $L$, $\beta^{1)}$, $\beta^{2)}$, …, $\beta^{m)}$, where $\beta^{1)}$, $\beta^{2)}$, …, $\beta^{m)}$ are linearly independent one-form.

* Corresponding Author.
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The conformal theory of Finsler spaces has been initiated by M. S. Knebelman [7] in 1929 and has been investigated in detail by many authors [1, 2, 3, 6] etc. The conformal change is defined as

$$L(x,y) \to e^{\sigma(x)} L(x,y), \tag{1.3}$$

where $\sigma(x)$ is a function of position only and known as conformal factor.

We also studied the generalized $\beta-$conformal change of Finsler metric by taking

$$\overline{L} = f(e^{\sigma(x)}\, L(x,y), \beta^{1)}, \beta^{2)}, \dots, \beta^{m)}), \tag{1.4}$$

where $f$ is any positively homogeneous function of degree one in $e^{\sigma}L$, $\beta^{1)}$, $\beta^{2)}$, ..., $\beta^{m)}$.

On the other hand, in 1985, M. Matsumoto investigated the theory of Finslerian hypersurface [8]. He has defined three types of hypersurfaces that were called a hyperplane of the first, second and third kinds.

In the year 2009, B. N. Prasad and Gauri Shanker [11] studied the Finslerian hypersurfaces and $\beta-$change of Finsler metric and obtained different results in his paper. In the present paper, using the field of linear frame [5, 4, 9], we shall consider Finslerian hypersurfaces given by a generalized $\beta-$conformal change of a Finsler metric. Our purpose is to give some relations between the original Finslerian hypersurface and the other which is Finslerian hypersurface given by generalized $\beta-$conformal change. We have also obtained that a generalized $\beta-$conformal change makes three types of hypersurfaces invariant under certain conditions.

## 2. Finslerian Hypersurfaces

Let $M^n$ be an $n-$dimensional manifold and $F^n = (M^n, L)$ be an $n-$dimensional Finsler space equipped with the fundamental function $L(x,y)$ on $M^n$. The metric tensor $g_{ij}(x,y)$ and Cartan's $C-$tensor $C_{ijk}(x,y)$ are given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i\, \partial y^j}, \qquad C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k},$$

respectively and we introduce the Cartan's connection $C\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$ in $F^n$.

A hypersurface $M^{n-1}$ of the underlying smooth manifold $M^n$ may be parametrically represented by the equation $x^i = x^i(u^\alpha)$, where $u^\alpha$ are Gaussian coordinates on $M^{n-1}$ and Greek indices vary from 1 to $n-1$. Here, we shall assume that the matrix consisting of the projection factors $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ is of rank $n-1$. The following notations are also employed:

$$B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha\, \partial u^\beta}, \qquad B^i_{0\beta} = v^\alpha B^i_{\alpha\beta}.$$

If the supporting element $y^i$ at a point $(u^\alpha)$ of $M^{n-1}$ is assumed to be tangential to $M^{n-1}$, we may then write $y^i = B^i_\alpha(u)v^\alpha$, i.e. $v^\alpha$ is thought of as the supporting element of $M^{n-1}$ at the point $(u^{\alpha})$. Since the function $\overline{L}(u,v) = L\{x(u), y(u,v)\}$ gives rise to a Finsler metric of $M^{n-1}$, we get a $(n-1)-$dimensional Finsler space $F^{n-1} = \{M^{n-1}, \overline{L}(u,v)\}$.

At each point $(u^{\alpha})$ of $F^{n-1}$, the unit normal vector $N^i(u,v)$ is defined by

$$g_{ij} B^i_{\alpha} N^j = 0, \qquad g_{ij} N^i N^j = 1. \tag{2.1}$$

If $B^{\alpha}_i$, $N_i$ is the inverse matrix of $(B^i_{\alpha}, N^i)$, we have

$$B^i_{\alpha} B^{\beta}_i = \delta^{\beta}_{\alpha}, \quad B^i_{\alpha} N_i = 0, \quad N^i N_i = 1 \quad \text{and} \quad B^i_{\alpha} B^{\alpha}_j + N^i N_j = \delta^i_j.$$

Making use of the inverse matrix $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get

$$B^{\alpha}_i = g^{\alpha\beta} g_{ij} B^j_{\beta}, \quad N_i = g_{ij} N^j. \tag{2.2}$$

For the induced Cartan's connection $IC\Gamma = (F^{\alpha}_{\beta\gamma}, N^{\beta}_{\alpha}, C^{\alpha}_{\beta\gamma})$ on $F^{n-1}$, the second fundamental $h-$tensor $H_{\alpha\beta}$ and the normal curvature $H_{\alpha}$ are respectively given by [9]

$$\begin{aligned} H_{\alpha\beta} &= N_i (B^i_{\alpha\beta} + F^i_{jk} B^j_{\alpha} B^k_{\beta}) + M_{\alpha} H_{\beta}, \\ H_{\alpha} &= N_i (B^i_{0\beta} + N^i_j B^j_{\beta}), \end{aligned} \tag{2.3}$$

where

$$M_{\alpha} = C_{ijk} B^i_{\alpha} N^j N^k.$$

Contracting $H_{\alpha\beta}$ by $v^{\alpha}$, we immediately get $H_{0\beta} = H_{\alpha\beta} v^{\alpha} = H_{\beta}$. Furthermore the second fundamental $v-$tensor $M_{\alpha\beta}$ is given by [8]

$$M_{\alpha\beta} = C_{ijk} B^i_{\alpha} B^j_{\beta} N^k. \tag{2.4}$$

## 3. Finsler Space with Generalized $\beta-$Conformal Change

Let $(M^n, L)$ be a Finsler space $F^n$, where $M^n$ is an $n-$dimensional differentiable manifold equipped with a fundamental function $L$. A change in fundamental metric $L$, defined by equation (1.4), is called generalized $\beta-$conformal change, where $\sigma(x)$ is conformal factor and function of position only and $\beta^{1)}$, $\beta^{2)}$, ..., $\beta^{m)}$ all are linearly independent one-form and defined as $\beta^{r)} = b^{r)}_i y^i$.

Homogeneity of $f$ gives

$$e^{\sigma} L f_0 + f_r \beta^{r)} = f, \tag{3.1}$$

where the subscripts '0' and '$r$' denote the partial derivative with respect to $L$ and $\beta^{r)}$ respectively. The letters $r$, $s$, $t$, $r'$ and $s'$ vary from 1 to $m$ throughout the paper. Summation convention is applied for the indices $r$, $s$, $t$, $r'$ and $s'$. If we write $\overline{F}^n = (M^n, \overline{L})$, then the Finsler space $\overline{F}^n$ is said to be obtained from $F^n$ by generalized $\beta-$conformal change. The quantities corresponding to $\overline{F}^n$ are denoted by putting bar on those quantities.

To find the relation between fundamental quantities of $(M^n, L)$ and $(M^n, \overline{L})$, we use the following results:

$$\dot{\partial}_i \beta^{r)} = b_i^{r)}, \quad \dot{\partial}_i L = l_i, \quad \dot{\partial}_j l_i = L^{-1} h_{ij}, \tag{3.2}$$

where $\dot{\partial}_i$ stands for $\frac{\partial}{\partial y^i}$ and $h_{ij}$ are components of angular metric tensor of $(M^n, L)$ given by

$$h_{ij} = g_{ij} - l_i l_j = L \dot{\partial}_i \dot{\partial}_j L.$$

Differentiating (3.1) with respect to $L$ and $\beta^{s)}$ respectively, we get

$$e^\sigma L f_{00} + f_{0r} \beta^{r)} = 0 \tag{3.3}$$

and

$$e^\sigma L f_{0s} + f_{rs} \beta^{r)} = 0. \tag{3.4}$$

The successive differentiation of (1.4) with respect to $y^i$ and $y^j$ give

$$\bar{l}_i = e^\sigma f_0 l_i + f_r b_i^{r)}, \tag{3.5}$$

$$\overline{h}_{ij} = e^\sigma \frac{f f_0}{L} h_{ij} + e^{2\sigma} f f_{00} l_i l_j + e^\sigma f f_{0r} (b_j^{r)} l_i + b_i^{r)} l_j) + f f_{rs} b_i^{r)} b_j^{s)}. \tag{3.6}$$

Using equations (3.3) and (3.4) in equation (3.6), we have

$$\overline{h}_{ij} = e^\sigma \frac{f f_0}{L} h_{ij} + f f_{rs} \left( b_i^{r)} - \frac{\beta^{r)}}{L} l_i \right) \left( b_j^{s)} - \frac{\beta^{s)}}{L} l_j \right). \tag{3.7}$$

If we put $m_i^{r)} = b_i^{r)} - \frac{\beta^{r)}}{L} l_i$, equation (3.7) may be written as

$$\overline{h}_{ij} = e^\sigma \frac{f f_0}{L} h_{ij} + f f_{rs} m_i^{r)} m_j^{s)}. \tag{3.8}$$

From equations (3.5) and (3.8), we get the following relation between metric tensors of $(M^n, L)$ and $(M^n, \overline{L})$

$$\begin{aligned} \overline{g}_{ij} =& e^\sigma \frac{f f_0}{L} g_{ij} + e^\sigma \left( e^\sigma f_0^2 - \frac{f f_0}{L} \right) l_i l_j + f f_{rs} m_i^{r)} m_j^{s)} \\ &+ e^\sigma f_0 f_r (b_i^{r)} l_j + b_j^{r)} l_i) + f_r f_s b_i^{r)} b_j^{s)}. \end{aligned} \tag{3.9}$$

Now,

$$\begin{aligned} &\text{(a)} \quad \dot{\partial}_i m_j^{r)} = -\frac{1}{L} \left( m_i^{r)} l_j + \frac{\beta^{r)}}{L} h_{ij} \right), \\ &\text{(b)} \quad \dot{\partial}_i f = e^\sigma f_0 l_i + f_r b_i^{r)}, \\ &\text{(c)} \quad \dot{\partial}_i f_{rs} = e^\sigma f_{rs0} l_i + f_{rst} b_i^{t)}. \end{aligned} \tag{3.10}$$

Differentiating equation (3.8) with respect to $y^k$ and using equations (3.2), (3.3), (3.4), (3.5), (3.9) and (3.10), we get

$$\overline{C}_{ijk} = p_0 C_{ijk} + p_1(h_{ij}m_k^{r)} + h_{jk}m_i^{r)} + h_{ki}m_j^{r)}) + p_2 m_i^{r)} m_j^{s)} m_k^{t)}, \tag{3.11}$$

where

$$\begin{aligned} p_0 &= e^{\sigma}\frac{ff_0}{L}C_{ijk}, \qquad p_1 = \frac{e^{\sigma}}{2L}(f_0 f_r + f f_{0r}), \\ p_2 &= \frac{1}{2}(f_{rs}f_t + f_{st}f_r + f_{tr}f_s + f f_{rst}). \end{aligned} \tag{3.12}$$

## 4. Hypersurfaces Given by a Generalized $\beta-$Conformal Change

Consider a Finslerian hypersurface $F^{n-1} = \{M^{n-1}, \overline{L}(u,v)\}$ of the $F^n$ and another Finslerian hypersurface $\overline{F}^{n-1} = \{M^{n-1}, \overline{L}(u,v)\}$ of the $\overline{F}^n$ given by generalized $\beta-$conformal change. Let $N^i$ be the unit vector at each point of $F^{n-1}$ and $(B_i^{\alpha}, N_i)$ be the inverse matrix of $(B_{\alpha}^i, N^i)$. The function $B_i^{\alpha}$ may be considered as components of $(n-1)$ linearly independent tangent vectors of $F^{n-1}$ and they are invariant under generalized $\beta-$conformal change. Thus, we shall show that a unit normal vector $\overline{N}^i(u,v)$ of $\overline{F}^{n-1}$ is uniquely determined by

$$\overline{g}_{ij}B_{\alpha}^i\overline{N}^j = 0, \qquad \overline{g}_{ij}\overline{N}^i\,\overline{N}^j = 1. \tag{4.1}$$

Contracting (3.9) by $N^iN^j$ and paying attention to (2.1) and the fact that $l_iN^i = 0$, we have

$$\overline{g}_{ij}N^iN^j = p_0 + p(b_i^{r)}b_j^{s)}N^iN^j), \tag{4.2}$$

where $p = ff_{rs} + f_rf_s$. Therefore, we obtain

$$\overline{g}_{ij}\left(\pm\frac{N^i}{\sqrt{p_0 + p(b_i^{r)}b_j^{s)}N^iN^j)}}\right)\left(\pm\frac{N^j}{\sqrt{p_0 + p(b_i^{r)}b_j^{s)}N^iN^j)}}\right) = 1.$$

Hence, we can put

$$\overline{N}^i = \frac{N^i}{\sqrt{p_0 + p(b_i^{r)}b_j^{s)}N^iN^j)}},$$

where we have chosen the positive sign in order to fix an orientation.

Using equations (3.9), (4.3) and from first condition of (4.1), we have

$$B_{\alpha}^i(2p_1Ll_i + pb_i^{r)}).\frac{b_j^{s)}\,N^j}{\sqrt{p_0 + p(b_i^{r)}b_j^{s)}N^iN^j)}} = 0. \tag{4.4}$$

If $B^i_\alpha(2p_1 L l_i + p b^{r)}_i) = 0$, then contracting it by $v^\alpha$ and using $y^i = B^i_\alpha v^\alpha$, we get $L = 0$ or $\beta^{r)} = 0$ which is a contradiction with the assumption that $L > 0$. Hence $b^{s)}_j N^j = 0$. Therefore equation (4.3) is written as

$$\overline{N}^i = \frac{N^i}{\sqrt{p_0}}. \tag{4.5}$$

Summarizing the above, we obtain

**Proposition 4.1.** *For a field of linear frame* $(B^i_1, B^i_2, \ldots, B^i_{n-1}, N^i)$ *of* $F^n$ *there exists a linear frame* $(B^i_1, B^i_2, \ldots, B^i_{n-1}, \overline{N}^i = \frac{N^i}{\sqrt{p_0}})$ *of* $\overline{F}^n$ *such that (4.1) is satisfied along* $\overline{F}^{n-1}$ *and then* $b^{r)}_i$ *is tangential to both of the hypersurfaces* $F^{n-1}$ *and* $\overline{F}^{n-1}$.

The quantities $\overline{B}^\alpha_i$ are uniquely defined along $\overline{F}^{n-1}$ by

$$\overline{B}^\alpha_i = \overline{g}^{\alpha\beta} \overline{g}_{ij} B^j_\beta$$

where $\overline{g}^{\alpha\beta}$ is the inverse matrix of $\overline{g}_{\alpha\beta}$. Let $(\overline{B}^\alpha_i, \overline{N}^i)$ be the inverse matrix of $(B^i_\alpha, \overline{N}^i)$, then we have

$$B^i_\alpha \overline{B}^\beta_i = \delta^\beta_\alpha, \qquad B^i_\alpha \overline{N}_i = 0, \qquad \overline{N}^i \overline{N}_i = 1.$$

Furthermore $B^i_\alpha \overline{B}^\alpha_j + \overline{N}^i \overline{N}_j = \delta^i_j$. We also get $\overline{N}_i = \overline{g}_{ij} \overline{N}^j$ which in view of (3.5), (3.9) and (4.5) gives

$$\overline{N}_i = \sqrt{p_0}\, N_i. \tag{4.6}$$

We denote the Cartan's connection of $F^n$ and $\overline{F}^n$ by $(F^i_{jk}, N^i_j, C^i_{jk})$ and $(\overline{F}^i_{jk}, \overline{N}^i_j, \overline{C}^i_{jk})$ respectively and put $D^i_{jk} = \overline{F}^i_{jk} - F^i_{jk}$ which will be called difference tensor. We choose the vector field $b^{r)i}$ in $F^n$ such that

$$D^i_{jk} = A_{jk} b^{r)i} + B_{jk} l^i + \delta^i_j D_k + \delta^i_k D_j, \tag{4.7}$$

where $A_{jk}$ and $B_{jk}$ are components of a symmetric covariant tensor of second order and $D_i$ are components of a covariant vector. Since $N_i b^{r)i} = 0$, $N_i l^i = 0$ and $\delta^i_j N_i B^j_\alpha = 0$, from (4.7), we get

$$N_i D^i_{jk} B^j_\alpha B^k_\beta = 0 \quad \text{and} \quad N_i D^i_{0k} B^k_\beta = 0. \tag{4.8}$$

Therefore, from (2.3) and (4.6), we get

$$\overline{H}_\alpha = \sqrt{p_0}\, H_\alpha. \tag{4.9}$$

If each path of a hypersurface $F^{n-1}$ with respect to the induced connection also a path of the enveloping space $F^n$, then $F^{n-1}$ is called a hyperplane of the first kind. A hyperplane of the first kind is characterized by $H_\alpha = 0$ [8]. Hence from (4.9), we have

**Theorem 4.1.** *If $b_i^{r)}(x)$ be a vector field in $F^n$ satisfying (4.7), then a hypersurface $F^{n-1}$ is a hyperplane of the first kind if and only if the hypersurface $\overline{F}^{n-1}$ is a hyperplane of the first kind.*

Next contracting (3.11) by $B^i_\alpha \overline{N}^j \overline{N}^k$ and paying attention to (4.5), $m_i^{r)} N^i = 0$, $h_{jk} N^j N^k = 1$ and $h_{ij} B^i_\alpha N^j = 0$, we get

$$\overline{M}_\alpha = M_\alpha + \frac{p_1}{p_0} m_i^{r)} B^i_\alpha.$$

From (2.3), (4.6), (4.8), we have

$$\overline{H}_{\alpha\beta} = \sqrt{p}_0\, H_{\alpha\beta}. \tag{4.10}$$

If each $h-$path of a hypersurface $F^{n-1}$ with respect to the induced connection is also $h-$path of the enveloping space $F^n$, then $F^{n-1}$ is called a hyperplane of the second kind. A hyperplane of the second kind is characterized by $H_{\alpha\beta} = 0$ [8]. Since $H_{\alpha\beta} = 0$ implies that $H_\alpha = 0$ from (4.9) and (4.10), we have the following:

**Theorem 4.2.** *If $b_i^{r)}(x)$ be a vector field in $F^n$ satisfying (4.7), then a hypersurface $F^{n-1}$ is a hyperplane of the second kind if and only if the hypersurface $\overline{F}^{n-1}$ is a hyperplane of the second kind.*

Finally contracting (3.11) by $B^i_\alpha B^j_\beta \overline{N}^k$ and paying attention to (4.5), we have

$$\overline{M}_{\alpha\beta} = \sqrt{p}_0\, M_{\alpha\beta}. \tag{4.11}$$

If the unit normal vector of $F^{n-1}$ is parallel along each curve of $F^{n-1}$, then $F^{n-1}$ is called a hyperplane of third kind. A hyperplane of the third kind is characterized by $H_{\alpha\beta} = 0$, $M_{\alpha\beta} = 0$ [8]. From (4.10) and (4.11), we have:

**Theorem 4.3.** *If $b_i^{r)}(x)$ be a vector field in $F^n$ satisfying (4.7), then a hypersurface $F^{n-1}$ is a hyperplane of the third kind if and only if the hypersurface $\overline{F}^{n-1}$ is a hyperplane of the third kind.*

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