KYUNGPOOK Math. J. 58(2018), 733-746
https://doi.org/10.5666/KMJ.2018.58.4.733
pISSN 1225-6951 eISSN 0454-8124

Density by Moduli and Korovkin Type Approximation Theorem of Boyanov and Veselinov

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Abstract. The concept of f-statistical convergence which is, in fact, a generalization of statistical convergence, has been introduced recently by Aizpuru et al. (Quaest. Math. 37: 525-530, 2014). The main object of this paper is to prove an f-statistical analog of the classical Korovkin type approximation theorem of Boyanov and Veselinov. It is shown that the f-statistical analog is intermediate between the classical theorem and its statistical analog. As an application, we estimate the rate of f-statistical convergence of the sequence of positive linear operators defined from $C^*[0,\infty)$ into itself.

1. Introduction

The idea of statistical convergence was given by Zygmund [36] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was formally introduced by Steinhaus [35] and Fast [16] and later reintroduced by Schoenberg [34]. Various generalizations and applications of statistical convergence have been studied by *Šalát* [33], Fridy [17], Aizpuru *et al*. [1], Gadjiev and Orhan [18], Mursaleen and Alotaibi [28] and many others.

Statistical convergence depends on the natural density of subsets of the set $\mathbb{N} = \{1, 2, 3, \ldots\}$. The natural density $d(K)$ of set $K \subseteq \mathbb{N}$ (see [29, Chapter 11]) is defined by

$$d(K) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \le n \colon k \in K\}\right|, \tag{1.1}$$

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Received April 4, 2018; revised August 9, 2018; accepted August 13, 2018.
2010 Mathematics Subject Classification: 40A35, 41A36, 41A25, 47B38, 40A30.
Key words and phrases: modulus function, statistical convergence, positive linear operator, rate of convergence, Korovkin type approximation theorem.

where $|\{ k \leq n : k \in K \}|$ denotes the number of elements of K not exceeding n. Obviously we have $d(K) = 0$ provided that K is finite.

The concept of statistical convergence was studied by Kolk [22] in the more general setting of normed spaces.

Definition 1.1. Let X be a normed space. A sequence $x = (x_k)$ in X is said to be *statistically convergent* to some $x \in X$ if, for each $\epsilon > 0$ the set $\{ k \in \mathbb{N} : \|x_k - x\| \geq \epsilon \}$ has natural density zero, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} |\{ k \leq n : \|x_k - x\| \geq \epsilon \}| = 0,$$

and we write it as $st - \lim x_k = x$.

Recall [25, 32] that a modulus f is a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(y)$ for $x \geq 0, \quad y \geq 0$,

(iii) f is increasing,

(iv) f is continuous from the right at 0.

If f, g are moduli and a, b are positive real numbers, then $f \circ g$, $af + bg$, and $f \vee g$ are moduli. A modulus may be unbounded or bounded. For example, the modulus $f(x) = x^p$ where $0 < p \leq 1$, is unbounded, but $g(x) = \frac{x}{(1+x)}$ is bounded. For the work related to sequence spaces defined by a modulus one may refer to [1, 5, 6, 7, 9, 10, 25] and many others.

Aizpuru *et al*. [1] have recently introduced a new concept of density by moduli and consequently obtained a new concept of non-matrix convergence, namely, f-statistical convergence which is, in fact, a generalization of the concept of statistical convergence and intermediate between the ordinary convergence and the statistical convergence. This idea of replacing natural density with density by moduli has also been recently used to study the concepts of f-statistical convergence of order α [5], f-lacunary statistical convergence [6] and f-statistical boundedness [10].

Definition 1.2.([1]) For any unbounded modulus f, the *f-density* of a set $K \subset \mathbb{N}$ is denoted by $d^f(K)$ and is defined by

$$d^f(K) = \lim_{n\to\infty} \frac{f(|\{k \leq n : k \in K\}|)}{f(n)}$$

in case this limit exists.

Definition 1.3.([1]) Let f be an unbounded modulus and X be a normed space. A sequence $x = (x_k)$ in X is said to be *f-statistically convergent* to x, if, for each $\varepsilon > 0$,

$$d^f(\{k \in \mathbb{N} : \|x_k - x\| \geq \varepsilon\}) = 0,$$

$$\textit{i.e.,} \lim_{n\to\infty} \frac{1}{f(n)} f(|\{ k \leq n : \|x_k - x\| \geq \varepsilon\}|) = 0,$$

and we write it as $f - st \lim x_k = x$ or $S^f - \lim x_k = x$.

Remark 1.4.([1]) For any unbounded modulus f, every convergent sequence is f-statistically convergent which in turn is statistically convergent, but not conversely.

The theory of approximation is an area of mathematical analysis, which, at its core, is concerned with the approximation of functions by simpler and more easily calculated functions. In the fifties, the theory of approximation of functions by positive linear operators developed a lot, when Popoviciu [31], Bohman [11] and Korvokin [23, 24], discovered, independently, a simple and easily applicable criterion to check if a sequence of positive linear operators converges uniformly to the function to be approximated. This criterion says that the necessary and sufficient condition for the uniform convergence of the sequence (L_n) of positive linear operators to the continuous function g on the compact interval $[a, b]$, is the uniform convergence of the sequence $(L_n g)$ to g for the only three functions $e_n(x) = x^n, n = 0, 1, 2$. This classical result of approximation theory is mostly known under the name of Bohman-Korovkin theorem, because Popoviciu's contribution in [31] remained unknown for a long time.

Due to this classical result, the monomials e_n, $n = 0, 1, 2$, play an important role in the approximation theory of linear and positive operators on spaces of continuous functions. These monomials are often called Korovkin test-functions. This elegant and simple result has inspired many mathematicians to extend this result in different directions, generalizing the notion of sequence and considering different spaces. In this way a special branch of approximation theory arose, called K*orovkin-type approximation theory.* A complete and comprehensive exposure on this topic can be found in [2].

Statistical convergence had not been examined in approximation theory until 2002. The Korovkin first and second approximation theorems were first proved via statistical convergence by Gadjiev and Orhan [18] and Duman [14], in years 2002 and 2003, respectively. After this, Korovkin-type approximation theorems have been studied via various summability methods by many mathematicians [4, 13, 21, 26, 27, 28]. Quite recently Bhardwaj and Dhawan [8] have obtained f-statistical analogs of the Korovkin first and second approximation theorem.

In 1970, Boyanov and Veselinov [12] have proved the Korovkin theorem on $C^*[0, \infty)$, by using the test functions $1, e^{-x}, e^{-2x}$, where $C^*[0, \infty)$ denotes the Banach space of all real-valued continuous functions defined on $[0, \infty)$, with the property that $\lim_{x \to \infty} f(x)$ exists, with the uniform norm $\|.\|_\infty$. The statistical analog of this well-known result was given by Duman et al. [15]. In this paper, we generalize the classical result of Boyanov and Veselinov by using the notion of f-statistical convergence. A relationship between the classical theorem of Boyanov and Veselinov and their statistical and f-statistical analogs has also been studied. We also estimate the rate of f-statistical convergence of the sequence of positive linear operators defined from $C^*[0, \infty)$ into itself.

Before proceeding to establish the proposed results, we recall [20, 30] that for any linear spaces X, Y of real functions

(1) the mapping $L : X \to Y$ is called *linear operator* if

$$L(\alpha g_1 + \beta g_2) = \alpha L(g_1) + \beta L(g_2) \quad \text{for} \quad g_1, g_2 \in X \quad \text{and} \quad \alpha, \beta \in \mathbb{R},$$

(2) if $g \geq 0, g \in X \Rightarrow Lg \geq 0$, then L is a positive linear operator,

(3) in order to highlight the argument of the function $Lg \in Y$ we use the notation $L(g, x)$,

(4) if L is a positive linear operator, then for every $g \in X$ we have $|Lg| \leq L(|g|)$.

2. Korovkin Type Approximation Theorem of Boyanov and Veselinov via f-statistical Convergence

We begin this section by recalling the classical Korovkin type approximation theorem due to of Boyanov and Veselinov [12].

Theorem 2.1. *If the sequence* (L_n) *of positive linear operators* $L_n : C^*[0, \infty) \to C^*[0, \infty)$ *satisfies the conditions*

$$\begin{aligned} \lim \|L_n(1, x) - 1\|_\infty &= 0, \\ \lim \|L_n(e^{-t}, x) - e^{-x}\|_\infty &= 0, \\ \lim \|L_n(e^{-2t}, x) - e^{-2x}\|_\infty &= 0, \end{aligned}$$

then for any function $g \in C^*[0, \infty)$*, we have*

$$\lim \|L_n(g, x) - g(x)\|_\infty = 0.$$

The statistical analog of this theorem, given by Duman et al. [15], is as follows.

Theorem 2.2. *Let* (L_n) *be a sequence of positive linear operators* $L_n : C^*[0, \infty) \to C^*[0, \infty)$*. Then, for all* $g \in C^*[0, \infty)$*,*

$$st - \lim \|L_n(g, x) - g(x)\|_\infty = 0$$

if and only if

$$\begin{aligned} st - \lim \|L_n(1, x) - 1\|_\infty &= 0, \\ st - \lim \|L_n(e^{-t}, x) - e^{-x}\|_\infty &= 0, \\ st - \lim \|L_n(e^{-2t}, x) - e^{-2x}\|_\infty &= 0. \end{aligned}$$

We now state and prove an f-statistical analog of the classical theorem of Boyanov and Veselinov.

Theorem 2.3. *Let f be an unbounded modulus and (L_n) be a sequence of positive linear operators $L_n : C^*[0,\infty) \to C^*[0,\infty)$. Then, for all $g \in C^*[0,\infty)$,*

$$f - st\lim \|L_n(g,x) - g(x)\|_\infty = 0 \tag{2.1}$$

if and only if

$$f - st\lim \|L_n(1,x) - 1\|_\infty = 0, \tag{2.2}$$

$$f - st\lim \|L_n(e^{-t},x) - e^{-x}\|_\infty = 0, \tag{2.3}$$

$$f - st\lim \|L_n(e^{-2t},x) - e^{-2x}\|_\infty = 0. \tag{2.4}$$

Proof. Since each of $1, e^{-t}, e^{-2t}$ belongs to $C^*[0,\infty)$, conditions (2.2) - (2.4) follow immediately from (2.1). Now, let the conditions (2.2) - (2.4) hold. Let $g \in C^*[0,\infty)$. We follow the proof of Theorem 2 of Boyanov and Veselinov [12] up to a certain stage. Since the function g is bounded on $[0,\infty)$, we can write

$$|g(t) - g(x)| < 2M, \quad 0 \le t, x < \infty. \tag{2.5}$$

It is easy to prove that for a given $\epsilon > 0$ there is a $\delta > 0$ such that

$$|g(t) - g(x)| < \epsilon \quad \text{whenever} \quad |e^{-t} - e^{-x}| < \delta \quad \text{for all} \quad t, x \in [0,\infty). \tag{2.6}$$

Putting $\psi(t) = (e^{-t} - e^{-x})^2$ (x an arbitrary number but fixed in the interval $[0,\infty)$) and using inequalities (2.5) and (2.6), we have

$$|g(t) - g(x)| < \epsilon + \frac{2M}{\delta^2}\psi(t) \quad \text{for all} \quad t \in [0,\infty).$$

This means,

$$-\epsilon - \frac{2M}{\delta^2}\psi(t) < g(t) - g(x) < \epsilon + \frac{2M}{\delta^2}\psi(t) \quad \text{for all} \quad t \in [0,\infty). \tag{2.7}$$

In fact, if $|e^{-t} - e^{-x}| < \delta$, then (2.6) implies (2.7) since $\psi(t) = (e^{-t} - e^{-x})^2 \ge 0$ and if $|e^{-t} - e^{-x}| \ge \delta$, then

$$\frac{2M}{\delta^2}\psi(t) \ge \frac{2M}{\delta^2}\delta^2 = 2M,$$

and (2.7) follows from (2.5) since $\epsilon > 0$.
In view of monotonicity and linearity of the operators $L_n(g,x)$, the inequality (2.7) implies

$$-\epsilon L_n(1,x) - \frac{2M}{\delta^2}L_n(\psi,x) \le L_n(g,x) - L_n(g(x),x) \le \epsilon L_n(1,x) + \frac{2M}{\delta^2}L_n(\psi,x).$$

Note that x is fixed and so $g(x)$ is a constant number. Therefore,

$$-\epsilon L_n(1,x) - \frac{2M}{\delta^2}L_n(\psi,x) \le L_n(g,x) - g(x)L_n(1,x) \le \epsilon L_n(1,x) + \frac{2M}{\delta^2}L_n(\psi,x). \tag{2.8}$$

But

$$L_n(g,x) - g(x) = [L_n(g,x) - g(x)L_n(1,x)] + g(x)[L_n(1,x) - 1]. \tag{2.9}$$

Using (2.8) and (2.9), we have

$$L_n(g,x) - g(x) \le \epsilon L_n(1,x) + \frac{2M}{\delta^2} L_n(\psi,x) + g(x)[L_n(1,x) - 1]. \tag{2.10}$$

Now, let us estimate $L_n(\psi,x)$.

$$\begin{aligned} L_n(\psi,x) &= L_n(e^{-2t} - 2e^{-t}e^{-x} + e^{-2x},x) \\ &= L_n(e^{-2t},x) - 2e^{-x}L_n(e^{-t},x) + e^{-2x}L_n(1,x) \\ &= [L_n(e^{-2t},x) - e^{-2x}] - 2e^{-x}[L_n(e^{-t},x) - e^{-x}] + e^{-2x}[L_n(1,x) - 1]. \end{aligned} \tag{2.11}$$

Using (2.11) in (2.10), we have

$$\begin{aligned} & L_n(g,x) - g(x) \le \epsilon L_n(1,x) \\ & + \frac{2M}{\delta^2}\left\{[L_n(e^{-2t},x) - e^{-2x}] - 2e^{-x}[L_n(e^{-t},x) - e^{-x}] + e^{-2x}[L_n(1,x) - 1]\right\} \\ & + g(x)[L_n(1,x) - 1] \\ & = \left(\epsilon + \frac{2M}{\delta^2}e^{-2x} + g(x)\right)[L_n(1,x) - 1] + \frac{2M}{\delta^2}[L_n(e^{-2t},x) - e^{-2x}] \\ & - \frac{4M}{\delta^2}e^{-x}[L_n(e^{-t},x) - e^{-x}] + \epsilon. \end{aligned}$$

Therefore, using the fact that $|e^{-t}| \le 1$ and $|e^{-2t}| \le 1$, for all $t \in [0,\infty)$, we have

$$\begin{aligned} \|L_n(g,x) - g(x)\|_\infty &\le \left(\epsilon + \frac{2M}{\delta^2} + M\right)\|L_n(1,x) - 1\|_\infty \\ &\quad + \frac{2M}{\delta^2}\|L_n(e^{-2t},x) - e^{-2x}\|_\infty + \frac{4M}{\delta^2}\|L_n(e^{-t},x) - e^{-x}\|_\infty + \epsilon \\ &\le K(\|L_n(1,x) - 1\|_\infty + \|L_n(e^{-2t},x) - e^{-2x}\|_\infty \\ &\quad + \|L_n(e^{-t},x) - e^{-x}\|_\infty) + \epsilon, \end{aligned} \tag{2.12}$$

where $K = \max\left\{\epsilon + \frac{2M}{\delta^2} + M, \frac{4M}{\delta^2}, \frac{2M}{\delta^2}\right\}$. For any $\epsilon' > 0$, choose $\epsilon > 0$ such that $\epsilon < \epsilon'$. Now, from inequality (2.12), we have

$$\begin{aligned} |\{n \le N : \|L_n(g,x) - g(x)\|_\infty \ge \epsilon'\}| \le \Bigg| \Bigg\{ & n \le N : \|L_n(1,x) - 1\|_\infty + \|L_n(e^{-t},x) - e^{-x}\|_\infty \\ & + \|L_n(e^{-2t},x) - e^{-2x}\|_\infty \ge \frac{\epsilon' - \epsilon}{K} \Bigg\} \Bigg|. \end{aligned} \tag{2.13}$$

Now, write

$$D := \left\{ n : \|L_n(1,x) - 1\|_\infty + \|L_n(e^{-t},x) - e^{-x}\|_\infty + \|L_n(e^{-2t},x) - e^{-2x}\|_\infty \geq \frac{\epsilon' - \epsilon}{K} \right\},$$

$$D_1 := \left\{ n : \|L_n(1,x) - 1\|_\infty \geq \frac{\epsilon' - \epsilon}{3K} \right\},$$

$$D_2 := \left\{ n : \|L_n(e^{-t},x) - e^{-x}\|_\infty \geq \frac{\epsilon' - \epsilon}{3K} \right\},$$

$$D_3 := \left\{ n : \|L_n(e^{-2t},x) - e^{-2x}\|_\infty \geq \frac{\epsilon' - \epsilon}{3K} \right\}.$$

It is easy to see that $D \subset D_1 \cup D_2 \cup D_3$. Now, from (2.13), we have

$$\begin{aligned} |\{n \leq N : \|L_n(g,x) - g(x)\|_\infty \geq \epsilon'\}| \leq & \left| \left\{ n \leq N : \|L_n(1,x) - 1\|_\infty \geq \frac{\epsilon' - \epsilon}{3K} \right\} \right| \\ & + \left| \left\{ n \leq N : \|L_n(e^{-t},x) - e^{-x}\|_\infty \geq \frac{\epsilon' - \epsilon}{3K} \right\} \right| \\ & + \left| \left\{ n \leq N : \|L_n(e^{-2t},x) - e^{-2x}\|_\infty \geq \frac{\epsilon' - \epsilon}{3K} \right\} \right|, \end{aligned}$$

which yields that

$$\begin{aligned} & \frac{1}{f(N)} f(|\{n \leq N : \|L_n(g,x) - g(x)\|_\infty \geq \epsilon'\}|) \\ & \leq \frac{1}{f(N)} f\left(\left| \left\{ n \leq N : \|L_n(1,x) - 1\|_\infty \geq \frac{\epsilon' - \epsilon}{3K} \right\} \right| \right) \\ & + \frac{1}{f(N)} f\left(\left| \left\{ n \leq N : \|L_n(e^{-t},x) - e^{-x}\|_\infty \geq \frac{\epsilon' - \epsilon}{3K} \right\} \right| \right) \\ & + \frac{1}{f(N)} f\left(\left| \left\{ n \leq N : \|L_n(e^{-2t},x) - e^{-2x}\|_\infty \geq \frac{\epsilon' - \epsilon}{3K} \right\} \right| \right) \end{aligned}$$

and using (2.2) – (2.4), we get

(2.14)
$$f - st \lim \|L_n(g,x) - g(x)\|_\infty = 0, \quad \text{for all} \quad g \in C^*[0,\infty). \qquad \square$$

Remark 2.4. Since every convergent sequence is f-statistically convergent, it immediately follows that any sequence satisfying the conditions of the classical theorem of Boyanov and Veselinov (Theorem 2.1) automatically satisfies the conditions of its f-statistical analog (Theorem 2.3).

Our next example shows that there may exist a sequence of positive linear operators which satisfies the conditions of Theorem 2.3 but does not satisfy the conditions of Theorem 2.1, thereby showing that our result is stronger than the classical one.

Example 2.5. Consider the sequence $Q_n : C^*[0,\infty) \to C^*[0,\infty)$ of positive linear operators defined by

$$Q_n(g,x) = (1+\alpha_n)V_n(g,x),$$

where (V_n) is the sequence of classical Baskakov operators [3] defined by

$$V_n(g,x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k (1+x)^{-n-k} g\left(\frac{k}{n}\right), \quad x \geq 0, n \geq 1.$$

and (α_n) is the sequence of scalars which is f-statistically convergent to zero for some unbounded modulus f but not convergent to zero. From Corollary 3.3 of [19] it follows that

$$\lim \|V_n(1,x) - 1\|_\infty = 0, \tag{0.15}$$

$$\lim \|V_n(e^{-t},x) - e^{-x}\|_\infty = 0, \tag{0.16}$$

$$\lim \|V_n(e^{-2t},x) - e^{-2x}\|_\infty = 0. \tag{0.17}$$

Now making use of the fact (Remark 1.4) that every convergent sequence is f-statistically convergent, we see that the sequence (Q_n) satisfies conditions (2.2) — (2.4) of Theorem 2.3 and hence we have

$$f - st \lim \|Q_n(g,x) - g(x)\|_\infty = 0 \quad \text{for all} \quad g \in C^*[0,\infty).$$

On the other hand,

$$Q_n(1,x) = (1+\alpha_n)V_n(1,x) = (1+\alpha_n)$$

and so,

$$\lim \|Q_n(1,x) - 1\|_\infty = \lim \|1+\alpha_n - 1\| = \lim \|\alpha_n\| = \lim |\alpha_n| \neq 0,$$

from where it follows that (Q_n) does not satisfy the conditions of Theorem 2.1.

Remark 2.6 Since every f-statistically convergent sequence is statistically convergent, it immediately follows that any sequence satisfying the conditions of the f-statistical analog of the classical theorem of Boyanov and Veselinov (Theorem 2.3) automatically satisfies the conditions of the statistical analog of the classical theorem (Theorem 2.2).

Our next example shows that there may exist a sequence of positive linear operators which satisfies the conditions of Theorem 2.2 but not of Theorem 2.3, thereby showing that the statistical analog is stronger than the f-statistical analog.

Example 2.7. Consider the sequence $Q_n : C^*[0,\infty) \to C^*[0,\infty)$ of positive linear operators defined by

$$Q_n(g,x) = (1+\alpha_n)V_n(g,x),$$

where (V_n) is the sequence of Baskakov operators and (α_n) is the sequence of scalars which is statistically convergent to zero but not f-statistically convergent to zero for some unbounded modulus f. Proceeding as in Example 2.5, it is easy to see that the sequence (Q_n) satisfies the conditions of the statistical analog of the classical theorem of Boyanov and Veselinov but does not satisfy the conditions of the f-statistical analog of the classical theorem of Boyanov and Veselinov.

Remark 2.8. From Remarks 2.4, 2.6 and Examples 2.5, 2.7 it follows that the classical Korovkin type approximation theorem of Boyanov and Veselinov implies its f-statistical analog which in turn implies its statistical analog but not conversely. Hence our result is intermediate between the classical theorem and its statistical analog.

3. Rate of f-statistical Convergence

In this section, using f-statistical convergence we study the rate of convergence of a sequence of positive linear operators defined from $C^*[0,\infty)$ into itself with the help of modulus of continuity. We begin by presenting the following definition.

Definition 3.1. Let X be a normed space and (a_n) be a positive sequence of real numbers such that $a_n \to 0$ as $n \to \infty$. A sequence (x_k) in X is said to be *f-statistically convergent* to some $x \in X$ with the rate $o(a_n)$ if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{a_n\, f(n)} f(|\{\, k \le n : \|x_k - x\| \ge \epsilon\}|) = 0.$$

In this case, we write $x_k - x = S^f - o(a_n)$.

Lemmma 3.2. *Let f be an unbounded modulus, (a_n) and (b_n) be two sequences of positive real numbers. Let (x_k) and (y_k) be two sequences of real numbers such that $x_k - x = S^f - o(a_n)$ and $y_k - y = S^f - o(b_n)$. Let $c_n = \max\{a_n, b_n\}$. Then we have*

(i) $(x_k - x) \pm (y_k - y) = S^f - o(c_n)$,

(ii) $(x_k - x)(y_k - y) = S^f - o(c_n)$,

(iii) $\alpha(x_k - x) = S^f - o(a_n)$, *for any scalar* α.

Proof. (i) Assume that $x_k - x = S^f - o(a_n)$ and $y_k - y = S^f - o(b_n)$. Then, for every $\epsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{c_n\, f(n)} f(|\{\, k \le n : |(x_k - x) \pm (y_k - y)| \ge \epsilon\}|) \\
&\le \frac{1}{c_n\, f(n)} f(|\{\, k \le n : |x_k - x| \ge \frac{\epsilon}{2}\}|) + \frac{1}{c_n\, f(n)} f(|\{\, k \le n : |y_k - y| \ge \frac{\epsilon}{2}\}|) \\
&\le \frac{1}{a_n\, f(n)} f(|\{\, k \le n : |x_k - x| \ge \frac{\epsilon}{2}\}|) + \frac{1}{b_n\, f(n)} f(|\{\, k \le n : |y_k - y| \ge \frac{\epsilon}{2}\}|) \qquad (3.1)
\end{aligned}$$

Now by taking the limit as $n \to \infty$ in (3.1) and using the hypothesis, we conclude that

$$\lim_{n\to\infty} \frac{1}{c_n\, f(n)} f(|\{\, k \le n : |(x_k - x) \pm (y_k - y)| \ge \epsilon\}|) = 0,$$

where $c_n = \max\{a_n, b_n\}$. Since the other assertions can be proved similarly, we omit the details. □

Recall [20] that for a continuous function $g \in C(I)$ defined on an interval $I \subset \mathbb{R}$, the function $\omega : C(I) \times [0,\infty) \to \mathbb{R} \cup \{\infty\}$, defined by:

$$\omega(g,\delta) = \sup\{\, |g(t) - g(x)| : t, x \in I, |t - x| \le \delta \,\}$$

is called its usual modulus of continuity.

To estimate the rate of convergence of a sequence of positive linear operators defined from $C^*[0,\infty)$ into itself, we make use of the following modulus of continuity:

$$\omega^*(g,\delta) = \sup_{\substack{t,x\ge 0 \\ |e^{-t} - e^{-x}| \le \delta}} |g(t) - g(x)|,$$

defined for every $\delta \ge 0$ and every function $g \in C^*[0,\infty)$. In fact, this modulus (known as weighted modulus of continuity) can be expressed in terms of the usual modulus of continuity by the relation [19]:

$$\omega^*(g,\delta) = \omega(g^*,\delta),$$

where g^* is the continuous function defined on $[0,1]$ by

$$g^*(x) = \begin{cases} g(-\ln x), & x \in (0,1], \\ \lim_{t\to\infty} f(t), & x = 0 \end{cases}$$

It is well known (e.g. see [20]) that for any $\delta > 0$

$$|g(t) - g(x)| \le \left(1 + \frac{(e^{-t} - e^{-x})^2}{\delta^2}\right) \omega^*(g,\delta) \quad \text{for all} \quad g \in C^*[0,\infty).$$

We are now in a position to give the promised estimation.

Theorem 3.3. *Let (L_n) be a sequence of positive linear operators from $C^*[0,\infty)$ into $C^*[0,\infty)$. Suppose that*

(1) $L_n(1,x) - 1 = S^f - o(a_n)$,

(2) $\omega^*(g,\lambda_n) = S^f - o(b_n)$, *where* $\lambda_n = \sqrt{\|L_n(\Psi_x, x)\|_\infty}$ *and* $\Psi_x(t) = (e^{-t} - e^{-x})^2$.

Then for all $g \in C^*[0,\infty)$, we have

$$L_n(g,x) - g(x) = S^f - o(c_n),$$

where $c_n = \max\{a_n, b_n\}$.

Proof. Let $g \in C^*[0,\infty)$ and $x \in [0,\infty)$. Then using the monotonicity of (L_n), we see (for any $\delta > 0$ and $n \in \mathbb{N}$) that

$$\begin{aligned}
|L_n(g,x) - g(x)| &= |L_n(g(t),x) - g(x)| \\
&= |L_n(g(t) - g(x),x) + L_n(g(x),x) - g(x)| \\
&\le |L_n(g(t) - g(x),x)| + |L_n(g(x),x) - g(x)| \\
&\le L_n(|g(t) - g(x)|,x) + |g(x)L_n(1,x) - g(x)| \\
&\le L_n\left(1 + \frac{(e^{-t} - e^{-x})^2}{\delta^2}, x\right)\omega^*(g,\delta) + |g(x)|\,|L_n(1,x) - 1| \\
&= \left[L_n(1,x) + \frac{1}{\delta^2}L_n((e^{-t} - e^{-x})^2,x)\right]\omega^*(g,\delta) \\
&\quad + |g(x)|\,|L_n(1,x) - 1| \\
&= [L_n(1,x) - 1]\,\omega^*(g,\delta) + \omega^*(g,\delta) \\
&\quad + \frac{1}{\delta^2}L_n((e^{-t} - e^{-x})^2,x)\,\omega^*(g,\delta) + |g(x)|\,|L_n(1,x) - 1|
\end{aligned}$$

Putting $\delta = \lambda_n = \sqrt{\|L_n((e^{-t} - e^{-x})^2,x)\|_\infty}$, we get

$$\begin{aligned}
\|L_n(g,x) - g(x)\|_\infty &\le \|(L_n(1,x) - 1)\|_\infty\,\omega^*(g,\lambda_n) + 2\omega^*(g,\lambda_n) \\
&\quad + \|g\|_\infty\,\|L_n(1,x) - 1\|_\infty \\
&\le K\{\|(L_n(1,x) - 1)\|_\infty\,\omega^*(g,\lambda_n) + \omega^*(g,\lambda_n) \\
&\quad + \|L_n(1,x) - 1\|_\infty\},
\end{aligned} \tag{3.2}$$

where $K = \max\{\|g\|_\infty, 2\}$.

Now, for a given $\epsilon > 0$, we consider the following sets

$$\begin{aligned}
D &:= \{n \le N : \|L_n(g,x) - g(x)\|_\infty \ge \epsilon\}, \\
D_1 &:= \left\{n \le N : \|L_n(1,x) - 1\|_\infty\,\omega^*(g,\lambda_n) \ge \frac{\epsilon}{3K}\right\}, \\
D_2 &:= \left\{n \le N : |\omega^*(g,\lambda_n)| \ge \frac{\epsilon}{3K}\right\}, \\
D_3 &:= \left\{n \le N : \|L_n(1,x) - 1\|_\infty \ge \frac{\epsilon}{3K}\right\}.
\end{aligned}$$

Then it follows from (3.2) that $D \subset D_1 \cup D_2 \cup D_3$.

Now, since $c_n = \max\{a_n, b_n\}$, for every $N \in \mathbb{N}$, we have

$$\frac{1}{c_N f(N)} f(|\{n \le N : \|L_n(g,x) - g(x)\|_\infty \ge \epsilon\}|)$$
$$\le \frac{1}{c_N f(N)} f\left(\left|\left\{n \le N : \|L_n(1,x) - 1\|_\infty \, \omega^*(g,\lambda_n) \ge \frac{\epsilon}{3K}\right\}\right|\right)$$
$$+ \frac{1}{b_N f(N)} f\left(\left|\left\{n \le N : |\omega^*(g,\lambda_n)| \ge \frac{\epsilon}{3K}\right\}\right|\right)$$
$$+ \frac{1}{a_N f(N)} f\left(\left|\left\{n \le N : \|L_n(1,x) - 1\|_\infty \ge \frac{\epsilon}{3K}\right\}\right|\right)$$

from where it follows that $L_n(g,x) - g(x) = S^f - o(c_n)$.

Acknowledgements. The authors wish to thank the referee for his valuable suggestions, which have improved the presentation of the paper.

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