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Coefficient Bounds for Bi-spirallike Analytic Functions

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Abstract. In the present paper, we introduce and investigate two new subclasses, namely; the class of *strongly* $\alpha-$*bi-spirallike functions of order* β and $\alpha-$*bi-spirallike functions of order* ρ, of the function class Σ; of normalized analytic and *bi-univalent* functions in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We find estimates on the coefficients $|a_2|$, $|a_3|$ and $|a_4|$ for functions in these two subclasses.

1. Introduction and Definitions

Let $\mathcal{A}$ be the class of analytic functions $f(z)$ in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and represented by the *normalized* series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1.1}$$

We denote by $\mathcal{S}$ the family of univalent functions in $\mathcal{A}$. (see, for details,[4, 27]).
It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

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and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}) \quad [4].$$

The inverse function $f^{-1}(w)$ is given by

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

The function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if $(i) f \in \mathcal{S}$ and $(ii) f^{-1}(w)$ has an univalent *analytic continuation* to $|w| < 1$. Let Σ be the class of bi-univalent functions in $\mathbb{U}$. Initial pioneering work on the class Σ were done in [9, 16]. Srivastava *et al.* [26] mentioned some interesting examples of functions in the class Σ. Recently, Mishra and Soren [14] were add two more examples which are well demonstrated there in.

Špaček [19] and Libera [11] introduced the families of α-*spirallike* functions $(-\frac{\pi}{2} < \alpha < \frac{\pi}{2})$ and α-*spirallike functions of order* ρ $(-\frac{\pi}{2} < \alpha < \frac{\pi}{2},\ 0 \leq \rho \leq 1)$ respectively. Libera [11] completely settled the coefficient estimate problem for α-*spirallike functions of order* ρ. In this paper we introduce the families of α-*bi-spirallike functions of order* ρ and *strongly* α-*bi-spirallike functions of order* β. We find estimates for $|a_2|$, $|a_3|$ and $|a_4|$ for functions, of the form (1.1), in both these classes. Through out in this section also, we continue to denote by g the analytic continuation of the inverse of the function f to $\mathbb{U}$. We now have the following definitions:

Definition 1.1. The function $f(z)$, given by (1.1), is said to be a member of $\alpha - \mathcal{SP}_{\Sigma}^{\beta}$, the class of strongly α-*bi-spirallike functions of order* β $(|\alpha| \leq \frac{\pi}{2}, 0 \leq \beta < 1)$, if each of the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \left|\arg\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right)\right| < \beta\frac{\pi}{2} \qquad (z \in \mathbb{U}) \tag{1.3}$$

and

$$\left|\arg\left(e^{i\alpha}\frac{wg'(w)}{g(w)}\right)\right| < \beta\frac{\pi}{2} \qquad (w \in \mathbb{U}). \tag{1.4}$$

Definition 1.2. The function $f(z)$, given by (1.1), is said to be a member of $\alpha - \mathcal{SP}_{\Sigma}(\rho)$, the class of α-*bi-spirallike functions of order* ρ $(|\alpha| \leq \frac{\pi}{2}, 0 \leq \rho < 1)$, if each of the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \Re\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right) > \rho\cos\alpha \qquad (z \in \mathbb{U}) \tag{1.5}$$

and

$$\Re\left(e^{i\alpha}\frac{wg'(w)}{g(w)}\right) > \rho\cos\alpha \qquad (w \in \mathbb{U}). \tag{1.6}$$

Furthermore, let $\mathcal{P}$ be the class of analytic functions $p(z)$ of the form:

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \qquad (z \in \mathbb{U})$$

and satisfy $\Re(p(z)) > 0 \quad (z \in \mathbb{U})$. We shall need this class to describe the classes $\alpha - \mathcal{SP}_{\Sigma}^{\beta}$ and $\alpha - \mathcal{SP}_{\Sigma}(\rho)$.

As follow up of the work of Mishra and Soren [14], at present there is renewed interest in the study of the class Σ and its many new subclasses. For example see [1, 2, 3, 5, 7, 8, 10, 17, 18, 20, 21, 29, 30, 31]. Many researchers are still working upon finding an upper bound for a_n for the functions in subclasses of Σ. However, not much was known about the bound of the general coefficients $a_n \quad (n \geq 4)$ of subclasses of bi-univalent functions up until the publication of the article by Mishra and Soren [14]. See [6, 13, 12, 15, 22, 23, 24, 25, 28]. For a brief history on the developments regarding the class Σ see [26].

Motivated by the aforementioned work [14], in the present paper we have introduced two new subclasses of the function class Σ and we find estimates for $|a_2|$, $|a_3|$ and $|a_4|$ for functions, of the form (1.1), when $f \in \alpha - \mathcal{SP}_{\Sigma}^{\beta}$ and $\alpha - \mathcal{SP}_{\Sigma}(\rho)$.

2. Coefficient Bounds for the Class of Bi-spirallike Functions

We state and prove the following:

Theorem 2.1. *Let the function $f(z)$, represented by the series (1.1), be in the class $\alpha - \mathcal{SP}_{\Sigma}^{\beta}$ $(|\alpha| \leq \frac{\pi}{2}, 0 \leq \beta < 1)$. Then*

$$|a_2| \leq \frac{2\beta}{\sqrt{(1+\beta)}} \sqrt{\cos(\alpha/\beta)}, \tag{2.1}$$

$$|a_3| \leq \begin{cases} \beta \cos\left(\frac{\alpha}{\beta}\right), & 0 \leq \beta \leq \frac{1}{3}, \\ \frac{4\beta^2}{1+\beta} \cos\left(\frac{\alpha}{\beta}\right), & \frac{1}{3} \leq \beta < 1 \end{cases} \tag{2.2}$$

and

$$|a_4| \leq \begin{cases} \frac{2\beta}{3}\left(1 - \frac{2}{3}\frac{16\beta^2-3\beta-1}{\sqrt[3]{1+\beta}}\sqrt{\cos(\alpha/\beta)}\right)\cos(\alpha/\beta), & 0 \leq \beta < \frac{3+\sqrt{73}}{32} \\ \frac{2\beta}{3}\left(1 + \frac{2}{3}\frac{16\beta^2-3\beta-1}{\sqrt[3]{1+\beta}}\sqrt{\cos(\alpha/\beta)}\right)\cos(\alpha/\beta), & \frac{3+\sqrt{73}}{32} \leq \beta < \frac{2}{5} \\ \frac{2\beta}{3}\left(\frac{15\beta}{5\beta+4} + \frac{2}{3}\frac{16\beta^2-3\beta-1}{\sqrt[3]{1+\beta}}\sqrt{\cos(\alpha/\beta)}\right)\cos(\alpha/\beta), & \frac{2}{5} \leq \beta < 1. \end{cases} \tag{2.3}$$

Proof. We write

$$f'(z) = \frac{f(z)}{z} e^{-i\alpha} h(z) \quad \left(z \in \mathbb{U}; -\beta\frac{\pi}{2} < \alpha < \beta\frac{\pi}{2}\right) \tag{2.4}$$

where $h(z)$ is analytic in $\mathbb{U}$ and satisfies

$$h(0) = e^{i\alpha} \quad \text{and} \quad |\arg h(z)| < \beta\frac{\pi}{2} \quad (z \in \mathbb{U}).$$

It can be checked that the function $q(z)$ defined by:

$$h(z)^{\frac{1}{\beta}} = \cos\left(\frac{\alpha}{\beta}\right) q(z) + i \sin\left(\frac{\alpha}{\beta}\right) \qquad (z \in \mathbb{U})$$

is a member of the class $\mathcal{P}$. Suppose that

$$q(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{U}).$$

By comparing coefficients in (2.4), we have

$$a_2 = \beta c_1 e^{-i(\frac{\alpha}{\beta})} \cos\left(\frac{\alpha}{\beta}\right), \tag{2.5}$$

$$2a_3 - a_2^2 = \beta c_2 e^{-i(\frac{\alpha}{\beta})} \cos\left(\frac{\alpha}{\beta}\right) + \frac{\beta(\beta-1)}{2} c_1^2 e^{-2i(\frac{\alpha}{\beta})} \cos^2\left(\frac{\alpha}{\beta}\right) \tag{2.6}$$

and

$$\begin{aligned} 3a_4 - 3a_2a_3 + a_2^3 = \beta c_3 e^{-i(\frac{\alpha}{\beta})} \cos\left(\frac{\alpha}{\beta}\right) &+ \beta(\beta-1) c_1 c_2 e^{-2i(\frac{\alpha}{\beta})} \cos^2\left(\frac{\alpha}{\beta}\right) \\ &+ \frac{\beta(\beta-1)(\beta-2)}{6} c_1^3 e^{-3i(\frac{\alpha}{\beta})} \cos^3\left(\frac{\alpha}{\beta}\right). \end{aligned} \tag{2.7}$$

Similarly, we take

$$g'(w) = \frac{g(w)}{w} e^{-i\alpha} H(w) \quad \left(w \in \mathbb{U}; -\beta\frac{\pi}{2} < \alpha < \beta\frac{\pi}{2}\right) \tag{2.8}$$

where $H(w)$ is analytic in $\mathbb{U}$ and satisfies

$$H(0) = e^{i\alpha} \quad \text{and} \quad |\arg H(w)| < \beta\frac{\pi}{2} \quad (w \in \mathbb{U}).$$

The function $p(w)$ defined by

$$H(w)^{\frac{1}{\beta}} = \cos\left(\frac{\alpha}{\beta}\right) p(w) + i \sin\left(\frac{\alpha}{\beta}\right) \qquad (w \in \mathbb{U})$$

is a member of the class $\mathcal{P}$. If

$$p(w) = 1 + l_1 w + l_2 w^2 + \cdots \quad (w \in \mathbb{U}),$$

then again by comparing the coefficients in (2.8), we have the following:

$$-a_2 = \beta l_1 e^{-i(\frac{\alpha}{\beta})} \cos\left(\frac{\alpha}{\beta}\right), \tag{2.9}$$

$$3a_2^2 - 2a_3 = \beta l_2 e^{-i(\frac{\alpha}{\beta})} \cos\left(\frac{\alpha}{\beta}\right) + \frac{\beta(\beta-1)}{2} l_1^2 e^{-2i(\frac{\alpha}{\beta})} \cos^2\left(\frac{\alpha}{\beta}\right) \tag{2.10}$$

and

$$\begin{aligned} -(10a_2^3 - 12a_2a_3 + 3a_4) = \beta l_3 e^{-i(\frac{\alpha}{\beta})} \cos\left(\frac{\alpha}{\beta}\right) &+ \beta(\beta-1) l_1 l_2 e^{-2i(\frac{\alpha}{\beta})} \cos^2\left(\frac{\alpha}{\beta}\right) \\ &+ \frac{\beta(\beta-1)(\beta-2)}{6} l_1^3 e^{-3i(\frac{\alpha}{\beta})} \cos^3\left(\frac{\alpha}{\beta}\right). \end{aligned} \tag{2.11}$$

From (2.5) and (2.9), gives

$$l_1 = -c_1. \tag{2.12}$$

We shall obtain a refined estimate on $|c_1|$ for use in the estimates of $|a_3|$ and $|a_4|$. For this purpose we first add (2.6) with (2.10); then use the relations (2.12) and get the following:

$$2a_2^2 = \beta(c_2 + l_2) e^{-i(\frac{\alpha}{\beta})} \cos\left(\frac{\alpha}{\beta}\right) + \frac{\beta(\beta-1)}{2}(c_1^2 + l_1^2) e^{-2i(\frac{\alpha}{\beta})} \cos^2\left(\frac{\alpha}{\beta}\right).$$

Putting $a_2 = \beta c_1 e^{-i(\frac{\alpha}{\beta})} \cos\left(\frac{\alpha}{\beta}\right)$ from (2.5), we have after simplification:

$$c_1^2 = \frac{c_2 + l_2}{(1+\beta) e^{-i(\frac{\alpha}{\beta})} \cos\left(\frac{\alpha}{\beta}\right)}. \tag{2.13}$$

By applying the familiar inequalities $|c_2| \le 2$ and $|l_2| \le 2$ we get:

$$|c_1| \le \sqrt{\frac{4}{(1+\beta)\cos\left(\frac{\alpha}{\beta}\right)}} = \frac{2}{\sqrt{(1+\beta)\cos\left(\frac{\alpha}{\beta}\right)}} \tag{2.14}$$

and

$$|a_2| \le \beta |c_1| \cos(\alpha/\beta) = \frac{2\beta}{\sqrt{(1+\beta)}} \sqrt{\cos(\alpha/\beta)}.$$

We have thus obtained (2.1).

We next find a bound on $|a_3|$. For this we substract (2.10) from (2.6) and get

$$4a_3 = 4a_2^2 + \beta(c_2 - l_2)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right) + \frac{\beta(\beta-1)}{2}(c_1^2 - l_1^2)e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right).$$

The relation $c_1^2 = l_1^2$ from (2.12), reduces the above expression to

$$4a_3 = 4a_2^2 + \beta(c_2 - l_2)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right). \tag{2.15}$$

Next putting that $a_2 = \beta c_1 e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)$ and using (2.13), we obtain

$$\begin{aligned}
4a_3 &= 4\beta^2 c_1^2 e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right) + \beta(c_2 - l_2)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)\\
&= 4\beta^2\left(\frac{c_2 + l_2}{(1+\beta)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)}\right)e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right)\\
&\quad + \beta(c_2 - l_2)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)\\
&= \frac{\beta}{1+\beta}\left[(5\beta+1)c_2 + (3\beta-1)l_2\right]e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right).
\end{aligned}$$

Therefore, the inequalities $|c_2| \le 2$ and $|l_2| \le 2$ give the following:

$$4|a_3| \le \begin{cases} \frac{2\beta}{1+\beta}(5\beta+1+1-3\beta)\cos\left(\frac{\alpha}{\beta}\right) = 4\beta\cos\left(\frac{\alpha}{\beta}\right), & 0 \le \beta \le \frac{1}{3} \\ \\ \frac{2\beta}{1+\beta}(5\beta+1+3\beta-1)\cos\left(\frac{\alpha}{\beta}\right) = \frac{16\beta^2}{1+\beta}\cos\left(\frac{\alpha}{\beta}\right), & \frac{1}{3} \le \beta < 1 \end{cases}$$

which simplifies to:

$$|a_3| \le \begin{cases} \beta\cos\left(\frac{\alpha}{\beta}\right), & 0 \le \beta \le \frac{1}{3} \\ \frac{4\beta^2}{1+\beta}\cos\left(\frac{\alpha}{\beta}\right), & \frac{1}{3} \le \beta < 1. \end{cases}$$

This is precisely the assertion of (2.2).

We shall next find an estimate on $|a_4|$. At first we shall derive a relation connecting c_1, c_2, c_3, l_2 and l_3. To this end, we first add the equations (2.7) and (2.11) and get

$$\begin{aligned}
-9a_2^3 + 9a_2a_3 = \beta(c_3 + l_3)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right) + \beta(\beta-1)(c_1c_2 + l_1l_2)e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right)\\
+\frac{\beta(\beta-1)(\beta-2)}{6}(c_1^3 + l_1^3)e^{-3i(\frac{\alpha}{\beta})}\cos^3\left(\frac{\alpha}{\beta}\right).
\end{aligned}$$

By putting $l_1 = -c_1$ the above expression reduces to the following:
(2.16)
$$-9a_2^3 + 9a_2a_3 = \beta(c_3 + l_3)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right) + \beta(\beta-1)c_1(c_2 - l_2)e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right).$$

Substituting $a_3 = a_2^2 + \frac{\beta}{4}(c_2 - l_2)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)$ from (2.15) into (2.16) we get after simplification:

$$\begin{aligned}\frac{9\beta a_2}{4}(c_2 - l_2)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right) =& \beta(c_3 + l_3)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)\\ &+ \beta(\beta-1)c_1(c_2 - l_2)e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right).\end{aligned}$$

Since $a_2 = \beta c_1 e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)$, (see 2.5) we have

$$\begin{aligned}\frac{9\beta^2}{4}c_1(c_2 - l_2)e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right) =& \beta(c_3 + l_3)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)\\ &+ \beta(\beta-1)c_1(c_2 - l_2)e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right).\end{aligned}$$

Or equivalently:

$$c_1(c_2 - l_2) = \frac{4(c_3 + l_3)}{5\beta + 4}e^{i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right). \tag{2.17}$$

We wish to express a_4 in terms of the first three coefficients of $q(z)$ and $p(w)$. Now substracting (2.11) from (2.7), we get

$$\begin{aligned}&6a_4 = -11a_2^3 + 15a_2a_3 + \beta(c_3 - l_3)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)\\ &+\beta(\beta-1)(c_1c_2 - l_1l_2)e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right) + \frac{\beta(\beta-1)(\beta-2)}{6}(c_1^3 - l_1^3)e^{-3i(\frac{\alpha}{\beta})}\cos^3\left(\frac{\alpha}{\beta}\right).\end{aligned}$$

Observing that $l_1 = -c_1$ we have $c_1^3 - l_1^3 = 2c_1^3$ and therefore

$$\begin{aligned}6a_4 =& -9a_2^3 + 9a_2a_3 - 2a_2^3 + 6a_2a_3 + \beta(c_3 - l_3)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)\\ &+ \beta(\beta-1)c_1(c_2 + l_2)e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right)\\ &+ \frac{\beta(\beta-1)(\beta-2)}{3}c_1^3e^{-3i(\frac{\alpha}{\beta})}\cos^3\left(\frac{\alpha}{\beta}\right).\end{aligned}$$

We replace $-9a_2^3 + 9a_2a_3$ by the right hand side of (2.16), put $a_3 = \beta^2c_1^2e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right) + \frac{\beta}{4}(c_2 - l_2)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)$ (see (2.15)) and $a_2 = \beta c_1e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)$.

This gives

$$\begin{aligned}6a_4 &= \beta(c_3+l_3)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)\\&\quad+\beta(\beta-1)c_1(c_2-l_2)e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right)-2\beta^3c_1^3e^{-3i(\frac{\alpha}{\beta})}\cos^3\left(\frac{\alpha}{\beta}\right)\\&\quad+6\beta c_1e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)\left(\beta^2c_1^2e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right)+\frac{\beta}{4}(c_2-l_2)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)\right)\\&\quad+\beta(c_3-l_3)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)+\beta(\beta-1)c_1(c_2+l_2)e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right)\\&\quad+\frac{\beta(\beta-1)(\beta-2)}{3}c_1^3e^{-3i(\frac{\alpha}{\beta})}\cos^3\left(\frac{\alpha}{\beta}\right)\\&=2\beta c_3e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)+\frac{\beta(5\beta-2)}{2}c_1(c_2-l_2)e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right)\\&\quad+\beta(\beta-1)c_1(c_2+l_2)e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right)+\frac{13\beta^3-3\beta^2+2\beta}{3}c_1^3e^{-3i(\frac{\alpha}{\beta})}\cos^3\left(\frac{\alpha}{\beta}\right).\end{aligned}$$

Next, replacing $c_1(c_2-l_2)$ by the expression in the right hand side of (2.17) and c_1^2 by (2.13) we finally get

$$\begin{aligned}6a_4 &= 2\beta c_3e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)+\frac{\beta(5\beta-2)}{2}\frac{4(c_3+l_3)}{5\beta+4}e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)\\&\quad+\beta(\beta-1)c_1(c_2+l_2)e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right)\\&\quad+\frac{13\beta^3-3\beta^2+2\beta}{3}c_1\frac{(c_2+l_2)}{1+\beta}e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right)\\&=2\beta c_3e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)+\frac{2\beta(5\beta-2)}{5\beta+4}(c_3+l_3)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)\\&\quad+\frac{16\beta^3-3\beta^2-\beta}{3(1+\beta)}c_1(c_2+l_2)e^{-2i(\frac{\alpha}{\beta})}\cos^2\left(\frac{\alpha}{\beta}\right)\\&=\beta\left[\frac{4(5\beta+1)}{5\beta+4}c_3+\frac{2(5\beta-2)}{5\beta+4}l_3\right.\\&\qquad\left.+\frac{16\beta^2-3\beta-1}{3(1+\beta)}c_1(c_2+l_2)e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right)\right]e^{-i(\frac{\alpha}{\beta})}\cos\left(\frac{\alpha}{\beta}\right).\end{aligned}$$

This gives

$$\begin{aligned}|a_4|\leq&\frac{\beta}{6}\left\{\left|\frac{4(5\beta+1)}{5\beta+4}\right||c_3|+\left|\frac{2(5\beta-2)}{5\beta+4}\right||l_3|\right.\\&\qquad\left.+\left|\frac{16\beta^2-3\beta-1}{3(1+\beta)}\right||c_1||(c_2+l_2)|\cos\left(\frac{\alpha}{\beta}\right)\right\}\cos\left(\frac{\alpha}{\beta}\right).\end{aligned}$$

Therefore,

$$|a_4| \leq \begin{cases} \frac{2\beta}{3}\left(1 - \frac{2}{3}\frac{16\beta^2-3\beta-1}{\sqrt[3]{1+\beta}}\sqrt{\cos(\alpha/\beta)}\right)\cos(\alpha/\beta), & 0 \leq \beta < \frac{3+\sqrt{73}}{32} \\ \frac{2\beta}{3}\left(1 + \frac{2}{3}\frac{16\beta^2-3\beta-1}{\sqrt[3]{1+\beta}}\sqrt{\cos(\alpha/\beta)}\right)\cos(\alpha/\beta), & \frac{3+\sqrt{73}}{32} \leq \beta < \frac{2}{5} \\ \frac{2\beta}{3}\left(\frac{15\beta}{5\beta+4} + \frac{2}{3}\frac{16\beta^2-3\beta-1}{\sqrt[3]{1+\beta}}\sqrt{\cos(\alpha/\beta)}\right)\cos(\alpha/\beta), & \frac{2}{5} \leq \beta < 1. \end{cases}$$

We get the assertion (2.3). The proof of Theorem 2.1 is, thus, completed. □

Remark 2.2. Taking $\alpha = 0$ in the above Theorem 2.1, we readily arrive at Mishra and Soren [14] of Theorem 2.1.

Theorem 2.3. *Let $f(z)$, given by (1.1), be in the class $\mathcal{SP}_{\Sigma}^{\alpha}(\rho)$ $(|\alpha| \leq \frac{\pi}{2}, 0 \leq \rho < 1)$. Then*

$$|a_2| \leq \sqrt{2(1-\rho)\cos\alpha}, \tag{2.18}$$

$$|a_3| \leq 2(1-\rho)\cos\alpha \tag{2.19}$$

and

$$|a_4| \leq \frac{2(1-\rho)\cos\alpha}{3}\left[1 + 13\sqrt{2(1-\rho)\cos\alpha}\right]. \tag{2.20}$$

Proof. Let $f \in \mathcal{SP}_{\Sigma}^{\alpha}(\rho)$. Then by Definition 1.2, we have

$$e^{i\alpha}\frac{zf'(z)}{f(z)} = Q_1(z)\cos\alpha + i\sin\alpha \tag{2.21}$$

and

$$e^{i\alpha}\frac{wg'(w)}{g(w)} = P_1(w)\cos\alpha + i\sin\alpha \tag{2.22}$$

respectively, where $\Re(Q_1(z)) > \rho$,

$$Q_1(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{U})$$

and $\Re(P_1(w)) > \rho$,

$$P_1(w) = 1 + l_1 w + l_2 w^2 + \cdots \quad (w \in \mathbb{U}).$$

As in the proof of Theorem 2.1, by suitably comparing coefficients in (2.21) and (2.22) we have

$$a_2 e^{i\alpha} = c_1\cos\alpha, \tag{2.23}$$

$$(2.24)\qquad (2a_3 - a_2^2)e^{i\alpha} = c_2\cos\alpha,$$

$$(2.25)\qquad (3a_4 - 3a_2a_3 + a_2^3)e^{i\alpha} = c_3\cos\alpha$$

and

$$(2.26)\qquad -a_2e^{i\alpha} = l_1\cos\alpha,$$

$$(2.27)\qquad (3a_2^2 - 2a_3)e^{i\alpha} = l_2\cos\alpha,$$

$$(2.28)\qquad -(10a_2^3 - 12a_2a_3 + 3a_4)e^{i\alpha} = l_3\cos\alpha.$$

In order to express c_1 interms of c_2 and l_2 we first add (2.24) and (2.27) and get

$$(2.29)\qquad 2a_2^2 = (c_2 + l_2)\frac{\cos\alpha}{e^{i\alpha}}.$$

Again putting $a_2e^{i\alpha} = c_1\cos\alpha$ from (2.23) we have

$$2c_1^2\frac{\cos^2\alpha}{e^{2i\alpha}} = (c_2 + l_2)\frac{\cos\alpha}{e^{i\alpha}}.$$

Or equivalently:

$$(2.30)\qquad c_1^2 = (c_2 + l_2)\frac{e^{i\alpha}}{2\cos\alpha}.$$

The familiar inequalities $|c_2| \leq 2(1-\rho)$, $|l_2| \leq 2(1-\rho)$ yield

$$|c_1^2| \leq \frac{4(1-\rho)}{2\cos\alpha} = \frac{2(1-\rho)}{\cos\alpha}$$

which implies that

$$(2.31)\qquad |c_1| \leq \sqrt{\frac{2(1-\rho)}{\cos\alpha}}$$

and

$$\begin{aligned}|a_2| &\leq |c_1|\cos\alpha\\ &\leq \sqrt{\frac{2(1-\rho)}{\cos\alpha}}\cos\alpha = \sqrt{2(1-\rho)\cos\alpha}.\end{aligned}$$

This proves (2.18).

Following the lines of proof of Theorem 2.1, with appropriate changes, we get that

$$4a_3 = (3c_2 + l_2)\frac{\cos\alpha}{e^{i\alpha}}.$$

The inequalities $|c_2| \le 2(1-\rho)$, $|l_2| \le 2(1-\rho)$, yield

$$|a_3| \le 2(1-\rho)\cos\alpha. \tag{2.32}$$

This is precisely the estimate (2.19).

We shall next find an estimate on $|a_4|$. By substracting (2.28) from (2.25) we get

$$6a_4 = -11a_2^3 + 15a_2a_3 + (c_3 - l_3)\frac{\cos\alpha}{e^{i\alpha}}.$$

A substitution of the value of a_2 from the relation (2.23) gives

$$6a_4 = -11c_1^3\frac{\cos^3\alpha}{e^{3i\alpha}} + 15c_1\frac{\cos\alpha}{e^{i\alpha}}a_3 + (c_3 - l_3)\frac{\cos\alpha}{e^{i\alpha}}.$$

Therefore, using the inequalities $|c_3| \le 2(1-\rho)$, $|l_3| \le 2(1-\rho)$, the estimate for $|c_1|$ from (2.31) and the estimate for $|a_3|$ from (2.32), we get

$$\begin{aligned}
6|a_4| &\le 11|c_1^3|\cos^3\alpha + 15|c_1|\cos\alpha|a_3| + |c_3 - l_3|\cos\alpha \\
&\le 11\cos^3\alpha\frac{2(1-\rho)}{\cos\alpha}\sqrt{\frac{2(1-\rho)}{\cos\alpha}} \\
&\quad + 15\cos\alpha\sqrt{\frac{2(1-\rho)}{\cos\alpha}}2(1-\rho)\cos\alpha + 4(1-\rho)\cos\alpha \\
&\le 4(1-\rho)\cos\alpha[1 + 13\sqrt{2(1-\rho)\cos\alpha}].
\end{aligned}$$

Or equivalently,

$$|a_4| \le \frac{2(1-\rho)\cos\alpha}{3}[1 + 13\sqrt{2(1-\rho)\cos\alpha}].$$

We get the assertion (2.20). This completes the proof of the Theorem 2.3. □

Remark 2.4. Taking $\alpha = 0$ in the above Theorem 2.3, we readily arrive at Mishra and Soren [14] of Theorem 2.3.

References

[1] R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramaniam, *Coefficient estimates for bi-univalent function Ma-Minda starlike and convex functions,* Appl. Math. Lett., **25**(2012), 344–351.

[2] Ş. Altinkaya and S. Yalçin, *Coefficient estimates for two new subclasses of bi-univalent functions with respect to symmetric points,* J. Funct. Spaces, (2015), Art. ID 145242, 5 pp.

[3] M. Çağlar, H. Orhan and N. Yağmur, *Coefficient bounds for new subclasses of bi-univalent functions,* Filomat, **27**(2013), 1165–1171.

[4] P. L. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften **259**, Springer, New York, 1983.

[5] B. A. Frasin and M. K. Aouf, *New subclasses of bi-univalent functions,* Appl. Math. Lett., **24**(2011), 1569–1573.

[6] J. M. Jahangiri and S. G. Hamidi, *Coefficient estimates for certain classes of bi-univalent functions,* Int. J. Math. Math. Sci., (2013), Art. ID 190560, 4 pp.

[7] S. S. Kumar, V. Kumar and V. Ravichandran, *Estimates for the initial coefficients of bi-univalent functions,* Tamsui Oxf. J. Inf. Math. Sci., **29**(2013), 487–504.

[8] S. K. Lee, V. Ravichandran and S. Supramaniam, *Initial coefficients of bi-univalent functions,* Abs. Appl. Anal., (2014), Article ID 640856, 6 pp.

[9] M. Lewin, *On a coefficient problem for bi-univalent functions,* Proc. Amer. Math. Soc., **18**(1967), 63–68.

[10] X.-F. Li and A.-P. Wang, *Two new subclasses of bi-univalent functions,* Int. Math. Forum, **7**(2012), 1495–1504.

[11] R. J. Libera, *Univalent α-spiral functions,* Canad. J. Math., **19**(1967), 449–456.

[12] A. K. Mishra and S. Barik, *Estimation for initial coefficients of bi-univalent λ-convex analytic functions in the unit disc,* J. Class. Anal., **7**(2015), 73–81.

[13] A. K. Mishra and S. Barik, *Estimation of initial coefficients of certain λ-bi-starlike analytic functions,* Asian-Eur. J. Math., **9**(2016), 1650066, 14 pp.

[14] A. K. Mishra and M. M. Soren, *Coefficient bounds for bi-starlike analytic functions,* Bull. Belg. Math. Soc. Simon Stevin, **21**(2014), 157–167.

[15] G. Murugusundaramoorthy and T. Bulboacă, *Estimate for initial MacLaurin coefficients of certain subclasses of bi-univalent functions of complex order associated with the Hohlov operator,* Ann. Univ. Paedagog. Crac. Stud. Math., **17** (2018), 27–36.

[16] E. Netanyahu, *The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$,* Arch. Ration Mech. Anal., **32**(1969), 100–112.

[17] Z. G. Peng and Q. Q. Han, *On the coefficients of several classes of bi-univalent functions,* Acta Math. Sci. Ser. B (Engl. Ed.), **34**(2014), 228–240.

[18] S. Siregar and S. Raman, *Certain subclasses of analytic and bi-univalent functions involving double zeta functions,* Int. J. Adv. Sci. Eng. Inform. Tech., **2**(2012), 16–18.

[19] L. Špaček, *Contributions à la thèorie des fonctions univalentes,* Časopis Pěst. Mat.-Fys., **62**(1933), 12–19.

[20] H. M. Srivastava, *Some inequalities and other results associated with certain subclasses of univalent and bi-univalent analytic functions,* Nonlinear Analysis, 607–630, Springer Optim. Appl. **68**, Springer, Berlin, New York and Heidelberg, 2012.

[21] H. M. Srivastava, S. S. Eker and R. M. Ali, *Coefficient bounds for a certain class of analytic and bi-univalent functions,* Filomat, **29**(2015), 1839–1845.

[22] H. M. Srivastava, S. S. Eker, S. G. Hamidi and J. M. Jahangiri, *Faber Polynomial Coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator,* Bull. Iran. Math. Soc., **44**(2018), 149–157.

[23] H. M. Srivastava, S. Gaboury and F. Ghanim, *Coefficient estimates for some general subclasses of analytic and bi-univalent functions,* Afr. Mat., **28**(2017), 693–706.

[24] H. M. Srivastava, S. Gaboury and F. Ghanim, *Coefficient estimates for a general subclass of analytic and bi-univalent functions of the Ma-Minda type,* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **112**(2018), 1157–1168.

[25] H. M. Srivastava, S. B. Joshi, S. S. Joshi, H. Pawar, *Coefficient estimates for certain subclasses of meromorphically bi-univalent functions,* Palest. J. Math., **5**(2016), 250–258.

[26] H. M. Srivastava, A. K. Mishra and P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions,* Appl. Math. Lett., **23**(2010), 1188–1192.

[27] H. M. Srivastava and S. Owa,(Eds.) *Current topics in analytic function theory,* World Scientific Publishing Co., Singapore, New Jersey, London and Hong Kong, 1992.

[28] H. M. Srivastava, F. M. Sarkar and H. Ö. Güney, *Some general coefficient estimates for a new class of analytic and bi-univalent functions defined by a linear combination,* Filomat, **32**(2018), 1313–1322.

[29] H.-G. Xiao and Q.-H. Xu, *Coefficient estimates for the generalized subclass of analytic and bi-univalent functions,* Eur. J. Pure Appl. Math., **10**(2017), 638–644.

[30] Q.-H. Xu, Y.-C. Gui and H. M. Srivastava, *Coefficient estimates for a certain subclass of analytic and bi-univalent functions,* Appl. Math. Lett., **25**(2012), 990–994.

[31] Q.-H. Xu, H.-G. Xiao and H. M. Srivastava, *A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems,* Appl. Math. Comput., **218**(2012), 11461–11465.