KYUNGPOOK Math. J. 58(2018), 689-695
https://doi.org/10.5666/KMJ.2018.58.4.689
pISSN 1225-6951 eISSN 0454-8124

On Zeros and Fixed Points of Differences of Meromorphic Functions

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Abstract. In this paper, we give some results on the zeros and fixed points of the difference and the divided difference of transcendental meromorphic functions. This improves on results of Langley.

1. Introduction

We assume that the reader is familiar with the basic notions of Nevanlinna's value distribution theory (see [7, 9, 10]). Let f be a function that is transcendental and meromorphic in the plane. The forward differences $\Delta^n f$ are defined in the standard way [8] by

$$(1.1) \qquad \Delta f(z) = f(z+1) - f(z), \quad \Delta^{n+1} f(z) = \Delta^n f(z+1) - \Delta^n f(z)$$

* Corresponding Author.
Received December 13, 2017; revised October 8, 2018; accepted October 16, 2018.
2010 Mathematics Subject Classification: 30D35, 30D45.
Key words and phrases: zeros, fixed points, difference, entire function, meromorphic function.
This work was supported by the key scientific research project for higher education institutions of Henan Province, China (no. 18A110002) and training program for young backbone teachers of colleges and universities in Henan Province, China (no. 2017GGJS126).

and the divided difference is defined by

$$G_n(z) = \frac{\Delta^n f(z)}{f(z)}. \tag{1.2}$$

Recently, a number of papers focus on complex difference equations and differences analogues of Nevanlinna's theory. Bergweiler and Langley [1] firstly investigated the existed of zeros of $\Delta f(z)$ and $\frac{\Delta f(z)}{f(z)}$, and obtained many profound results. The results may be viewed as discrete analogues of the following existing theorem on the zeros of f'.

Theorem A.([BL]) *Let f be transcendental and meromorphic in the plane with $\liminf_{r\to\infty} \frac{T(r,f)}{r} = 0$. Then f' has infinitely many zeros.*

In [1], Bergweiler and Langley proved the following theorems.

Theorem B.([1]) *Let $n \in \mathbb{N}$ and f be transcendental entire function of order $\sigma(f) = \sigma < \frac{1}{2}$ and $G_n(z)$ is defined by (1.2). If G_n is transcendental, then $G_n(z)$ has infinitely many zeros. In particular, if f has order less than $\min[\frac{1}{n}, \frac{1}{2}]$, then $G_n(z)$ is transcendental and has infinitely many zeros.*

For the first difference $\Delta f(z)$ and divided difference $\frac{\Delta f(z)}{f(z)}$, they also have the following theorem.

Theorem C.([1]) *Let f be a function transcendental and meromorphic function in the plane which satisfies $\underline{\lim}_{r\to\infty} \frac{T(r,f)}{r} = 0$, then $\Delta f(z), \frac{\Delta f(z)}{f(z)}$ are both transcendental.*

But for some $n \geq 2$, $G_n(z)$ fails to be transcendental if f is an entire function of order less than $\frac{1}{2}$; see [5].

In [4], Langley extended Theorem A. In fact, he got the following theorem.

Theorem D.([4]) *Let $n \in \mathbb{N}$ and f be a transcendental meromorphic function of order of growth $\sigma(f) = \sigma < 1$ in the plane and assume that $G_n(z)$ as defined by (1.2) is transcendental.*

(1) *If $G_n(z)$ has lower order $\mu < \alpha < 1/2$, which holds in particular if $\sigma < 1/2$, then $\delta(0, G_n(z)) \leq 1 - \cos\pi\alpha$ or $\delta(\infty, f) \leq \frac{\mu}{\alpha}$.*

(2) *If $\sigma = 1/2$, then either $G_n(z)$ has infinitely many zeros or $\delta(\infty, f) < 1$.*

(3) *If f is entire and $\sigma < \frac{1}{2} + \delta_0$, then $G_n(z)$ has infinitely many zeros: here δ_0 is a small positive absolute constant.*

In [2], Chen and Shon studied the fixed points of differences and divided differences of meromorphic functions and got some results. One of them is the following.

Theorem E.([2]) *Let $n \in \mathbb{N}, c \in \mathbb{C}$ and f be a function transcendental meromorphic function of order of growth $\sigma(f) = \sigma < 1$. If $G_n(z) = \frac{\Delta^n f(z)}{f(z)}$ is transcendental, then $G_n(z)$ has infinitely many fixed points.*

Comparing Theorem B with Theorem D (1), we know that the latter give a precise estimation of the zeros of $G_n(z)$. For the fixed points of $G_n(z)$, can we estimate it precisely like Theorem D (1)? In fact, Theorem E is extended as following.

Theorem 1. *Let $n \in \mathbb{N}$, let f be a transcendental and meromorphic function of order $\sigma(f) < 1$ in the plane and assume that $G_n(z)$ as defined by (1.2) is transcendental. Then $2\delta(0, G_n(z)-z)+\delta(0, G_n(z)) \leq 2$. Furthermore, if $G_n(z)$ has lower order $\mu < \alpha < \frac{1}{2}$, which holds in particular if $\sigma(f) < \frac{1}{2}$, and $\delta(0, G_n(z)) > 1 - \cos\pi\alpha$, then $\delta(0, G_n(z) - z) < \frac{1}{2}(1 + \cos\pi\alpha)$ and $\delta(\infty, f) \leq \frac{\mu}{\alpha}$.*

In Theorem D, there is a condition that $G_n(z)$ is transcendental. Can we remove it for the case (2) and (3)? The question is studied and the following results are obtained.

Theorem 2. *Let n be odd, let f be a transcendental meromorphic function of order $\sigma(f) = \frac{1}{2}$ and f has finite many poles in the plane, $G_n(z)$ is defined by* (1.2). *Then $G_n(z)$ has infinitely many zeros.*

Theorem 3. *Let $n \in \mathbb{N}$, let f be a transcendental meromorphic function of order $\sigma(f) < \frac{1}{2} + \delta_0$ and $\sigma(f) \neq \frac{n-k}{n}, k \in \mathbb{N}, n(\frac{1}{2} - \delta_0) < k < n$, where δ_0 is a small positive absolute constant, and $G_n(z)$ is defined by* (1.2). *Then $G_n(z)$ has infinitely many zeros.*

The final theorem from [1] showed that for transcendental meromorphic function satisfying the very strong growth restriction $T(r, f) = O(\log r)^2$ as $r \to \infty$, either the first differential or the first divided difference has infinitely many zeros. Langley improved it in [4] as following.

Theorem F.([4]) *Let f be a transcendental meromorphic function in the plane of order less than* 1/6, *and define $G = \Delta f/f$. Then at least one of G and Δf has infinitely many zeros.*

Obviously, we have the following corollary by Theorem A, C and E.

Corollary 1. *Under the hypothesis of Theorem F, G has infinity many fixed points.*

Thus, there exists a natural question that how about that fixed points of Δf under the hypothesis of Theorem F. It deserves for further study.

2. Lemmas for the Proofs of Theorems

Remark 1. Following Hayman ([11], p75-p76), we define an $\varepsilon-$*set* to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If E is an $\varepsilon-$*set*, then the set of $r \geq 1$ for which the circle $S(0, r)$ meets E has finite logarithmic measure, and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.

Lemma 1.([1]) *Let $n \in \mathbb{N}$ and f be transcendental and meromorphic of order less than 1 in the plane. Then there exists an $\varepsilon - set$ E_n such that*

$$\Delta^n f(z) \sim f^{(n)}(z)$$

as $z \to \infty$ *in* $\mathbb{C} \setminus E_n$.

Lemma 2.([6]) *Let* f *be a function transcendental and meromorphic function with order* $\sigma(f) = \sigma < \infty$, $H = \{(k_1, j_1), (k_2, j_2), \cdots, (k_q, j_q)\}$ *be a finite set of distinct pairs of integers that satisfy* $k_i > j_i \geq 0$, *for* $i = 1, \cdots, q$ *and let* $\varepsilon > 0$ *be a given constant. Then there exists a set* $E \subset (1, \infty)$ *with finite logarithmic measure such that for all* z *satisfying* $|z| \notin E \cup [0, 1]$ *and for all* $(k, j) \in H$, *we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma - 1 + \varepsilon)}.$$

Lemma 3.([9]) *Suppose that* f *is a meromorphic function in the complex plane, and* $a_1(z), a_2(z)$ *and* $a_3(z)$ *are three distinct small functions of* $f(z)$. *Then*

$$T(r, f) < \sum_{j=1}^{3} \overline{N}\left(r, \frac{1}{f - a_j(z)}\right) + S(r, f).$$

Lemma 4.([3]) *Suppose that* $f(z) = \frac{g(z)}{d(z)}$ *is a meromorphic function with* $\sigma(f) = \sigma$, *where* $g(z)$ *is an entire function and* $d(z)$ *is a polynomial. Then there exists a sequence* $\{r_j\}, r_j \to \infty$, *such that for all* z *satisfying* $|z| = r_j, |g(z)| = M(r_j, g)$, *when* j *sufficiently large, we have*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\upsilon_g(z)}{z}\right)^n (1 + o(1)), \quad n \geq 1,$$

and

$$\sigma(f) = \lim_{j \to \infty} \frac{\log \upsilon_g(r_j)}{\log r_j}.$$

3. Proof of Theorems

Proof of Theorem 1. Assume $n, c, \sigma, f, G_n(z)$ are as in the hypotheses. Set $G_n^*(z) = G_n(z) - z$, then $\sigma(G_n^*(z)) = \sigma(G_n(z)) \leq \sigma(f) < 1$, G_n^* is transcendental. By Lemma 1, there exists an $\varepsilon - set$ E_n, such that, as $z \to \infty$ in $C \backslash E_n$,

$$G_n(z) = \frac{\Delta^n f(z)}{f(z)} \sim \frac{f^{(n)}(z)}{f(z)}, \tag{3.1}$$

where E_n contains all zeros and poles of $G_n(z)$. So, there exists a subset $F_1 \subset (1, \infty)$ of finite logarithmic measure such that for large $|z| = r$ not in F_1, $z \notin E_n$ and

$$G_n^*(z) \sim \frac{f^{(n)}(z)}{f(z)} - z. \tag{3.2}$$

By Lemma 2, for any given $\varepsilon(0 < 2\varepsilon < 1-\sigma)$, there exists a subset $F_2 \subset (1,\infty)$ of finite logarithmic measure such that for large $|z| = r$ not in F_2,

$$\left|\frac{f^{(n)}(z)}{f(z)}\right| \leq |z|^{n(\sigma-1+\varepsilon)}. \tag{3.3}$$

Set an $\varepsilon - set$ E_n^* consists of all zeros and poles of $G_n^*(z)$, then there exists a subset $F_3 \subset (1,\infty)$ of finite logarithmic measure such that if $z \in E_n^*$, then $|z| = r \in F_3$. Thus, by (3.2) and (3.3), we see that for large $|z| = r \notin [0,1] \cup F_1 \cup F_2 \cup F_3$, $G_n^*(z)$ has no zero and pole on $|z| = r$, and

$$|G_n^*(z) + z| = \left|\frac{f^{(n)}(z)}{f(z)}(1+o(1))\right| \leq |z|^{\varepsilon} < |G_n^*(z)| + |z| \tag{3.4}$$

holds on $|z| = r$. Applying the Rouché's theorem to function z and $G_n^*(z)$, we obtain that

$$n\left(r, \frac{1}{G_n^*}\right) - n\left(r, G_n^*\right) = n\left(r, \frac{1}{z}\right) - n(r,z) = 1. \tag{3.5}$$

Applying Lemma 3 (Generation of second fundamental theorem) to function $G_n(z)$, we have

$$\begin{aligned} T(r, G_n(z)) < \quad & \overline{N}\left(r, \frac{1}{G_n(z)}\right) + \overline{N}\left(r, \frac{1}{G_n(z)-z}\right) \\ & + \overline{N}(r, G_n(z)) + S(r, G_n(z)). \end{aligned} \tag{3.6}$$

Since $\overline{N}(r, G_n(z)) = \overline{N}(r, G_n^*(z))$, $T(r, G_n^*(z)) = T(r, G_n(z)) + S(r, G_n)$, by (3.5) and (3.6), we have

$$T(r, G_n^*(z)) < \overline{N}\left(r, \frac{1}{G_n(z)}\right) + 2\overline{N}\left(r, \frac{1}{G_n^*(z)}\right) + S(r, G_n(z)). \tag{3.7}$$

Thus, by the definition of deficiency, we have

$$2\delta(0, G_n^*(z)) + \delta(0, G_n(z)) \leq 2. \tag{3.8}$$

Assume further that $\delta(0, G_n) > 1 - \cos\pi\alpha$ and by the proof of Theorem D(1)(see[4]), we have

$$\delta(0, G_n^*) < \frac{1}{2}(1+\cos\pi\alpha), \quad \delta(\infty, f) > \frac{\mu}{\alpha}. \qquad \square$$

Proof of Theorem 2. Since f is a transcendental meromorphic function of order of growth $\sigma(f) = \frac{1}{2}$ and f has finite many poles, we set $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ is an entire function and $d(z)$ is a polynomial. By Lemma 1, we know that there exists an $\varepsilon - set$ E_n, such that

$$\Delta^n f(z) \sim f^{(n)}(z) \tag{3.9}$$

as $z \to \infty$ in $C \setminus E_n$. By Lemma 4, there exists a sequence $\{r_j\}, r_j \to \infty$, such that for all z satisfying $|z| = r_j, |g(z)| = M(r_j, g)$, when j sufficiently large, we have

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\upsilon_g(r_j)}{z}\right)^n (1 + o(1)), \quad n \geq 1, \tag{3.10}$$

$$\sigma(f) = \lim_{j \to \infty} \frac{\log \upsilon_g(r_j)}{\log r_j}, \tag{3.11}$$

where $\upsilon_g(r)$ is the central index of $g(z)$. By (3.10) and (3.11), we have

$$G_n(z) = \left(\frac{\upsilon_g(r_j)}{z}\right)^n (1 + o(1)) \tag{3.12}$$

for the sequence $\{r_j\}, r_j \to \infty$. Assume that $G_n(z)$ is a rational function. Set $H = \{|z| = r : r \in E_n\}$. Then by Remark 1, H is of finite logarithmic measure. Set the logarithmic measure of H by $lm(H) = \log \kappa < \infty$, then for the above sequence $\{r_j\}$, there is a point $r'_j \in [r_j, (1+\kappa)r_j] \backslash H$. Since

$$\frac{\log \upsilon_g(r'_j)}{\log r'_j} \geq \frac{\log \upsilon_g(r_j)}{\log[(1+\kappa)r_j]} = \frac{\log \upsilon_g(r_j)}{\log r_j \left[1 + \frac{\log(1+\kappa)}{\log r_j}\right]}. \tag{3.13}$$

We have

$$\sigma(f) = \lim_{r'_j \to \infty} \frac{\log \upsilon_g(r'_j)}{\log r'_j}. \tag{3.14}$$

By (3.14), for any given $\varepsilon (0 < \varepsilon < 1 - \sigma)$, we get that for sufficiently large j,

$$(r'_j)^{(\sigma - 1 - \varepsilon)n} \leq \left(\frac{\upsilon_g(r'_j)}{r'_j}\right)^n \leq (r'_j)^{(\sigma - 1 + \varepsilon)n}. \tag{3.15}$$

Since $(\sigma - 1 + \varepsilon)n < 0$ and $G_n(z)$ is rational function, by (3.12) and (3.15), we can deduce that, as $z \to \infty$,

$$G_n(z) \sim \beta z^{-k} \tag{3.16}$$

where $\beta \neq 0$ is a constant and k is a positive integer. Since ε is arbitrary, by (3.13), (3.15) and (3.16), we have

$$\sigma = 1 - \frac{k}{n} = \frac{1}{2}. \tag{3.17}$$

Thus, $n = 2k$. Since k is a positive integer, n is even, it is contradicts the hypothesis. Hence, $G_n(z)$ is transcendental. Since the poles of f is finite, we have $\delta(\infty, f) = 1$. By Theorem D(2), $G_n(z)$ has infinitely many zeros. □

Proof of Theorem 3. Using the Wiman-Valiron theory and by the same argument of the proof of Theorem 2, we can prove it. □

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