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Existence of Solutions of Integral and Fractional Differential Equations Using α-type Rational F-contractions in Metric-like Spaces

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Abstract. We present α-type rational F-contractions in metric-like spaces, and respective fixed and common fixed point results for weakly α-admissible mappings. Useful examples illustrate the effectiveness of the presented results. As applications, we obtain sufficient conditions for the existence of solutions of a certain type of integral equations followed by examples of nonlinear fractional differential equations that are verified numerically.

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1. Introduction and Preliminaries

The Banach Contraction Principle (BCP) is the most famous elementary result in the metric fixed point theory. A huge amount of literature contains applications, generalizations and extensions of this principle carried out by several authors in different directions, e.g., by weakening the hypotheses, using different setups, considering various types of mappings and generalized form of metric spaces, see, e.g., [2, 6, 7, 10, 12, 14, 15, 18].

In this context, Matthews [12] introduced the notion of a partial metric space as a part of the study of denotational semantics of data-flow networks. He showed that BCP can be generalized to the partial metric context for applications in program verifications. Note that in partial metric spaces, self-distance of an arbitrary point need not be equal to zero.

Hitzler and Seda [6], resp. Amini-Harandi [2] made a further generalization under the name of dislocated, resp. metric-like space, also having the property of "non-zero self-distance". Amini-Harandi defined σ-completeness of these spaces. Further, Shukla et al. introduced in [15] the notion of 0-σ-complete metric-like space and proved some fixed point theorems in such spaces, as improvements of Amini-Harandi's results.

We recall some definitions and facts which will be used throughout the paper.

Definition 1.1.([2]) A mapping $\sigma : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$, for a nonempty set $\mathcal{X}$, is called *metric-like* if, for all $x, y, z \in \mathcal{X}$,

(σ_1) $\sigma(x, y) = 0 \Rightarrow x = y$;

(σ_2) $\sigma(x, y) = \sigma(y, x)$;

(σ_3) $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$.

In this case, the pair $(\mathcal{X}, \sigma)$ is called a *metric-like space*.

Example 1.2.([2]) Let $\mathcal{X} = \{0, 1\}$ and $\sigma : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be defined by

$$\sigma(x, y) = \begin{cases} 2, & \text{if } \ x = y = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Then $(\mathcal{X}, \sigma)$ is a metric-like space, but it is neither a metric space nor a partial metric space, since $\sigma(0, 0) > \sigma(0, 1)$.

Example 1.3.([14]) Let $\mathcal{X} = \mathbb{R}$. Then the mappings $\sigma_i : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ ($i \in \{1, 2, 3\}$), defined by

$$\sigma_1(x, y) = |x| + |y| + a, \quad \sigma_2(x, y) = |x - b| + |y - b|, \quad \sigma_3(x, y) = x^2 + y^2,$$

where $a \geq 0$ and $b \in \mathbb{R}$, are metric-like on $\mathcal{X}$.

Definition 1.4.([2, 15]) Let $(\mathcal{X}, \sigma)$ be a metric-like space. Then:

(1) A sequence $\{x_n\}$ in $\mathcal{X}$ converges to a point $x \in \mathcal{X}$ (denoted as $x_n \to x$ as $n \to \infty$) if $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x)$. A sequence $\{x_n\}$ is said to be *σ-Cauchy* if $\lim_{n,m\to\infty} \sigma(x_n, x_m)$ exists and is finite. The space $(\mathcal{X}, \sigma)$ is called *σ-complete* if for each σ-Cauchy sequence $\{x_n\}$, there exists $x \in \mathcal{X}$ such that

$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x) = \lim_{n,m\to\infty} \sigma(x_n, x_m).$$

(2) A sequence $\{x_n\}$ in $(\mathcal{X}, \sigma)$ is called a *0-σ-Cauchy sequence* if $\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0$. The space $(\mathcal{X}, \sigma)$ is said to be *0-σ-complete* if every 0-σ-Cauchy sequence in $\mathcal{X}$ converges to a point $x \in \mathcal{X}$ such that $\sigma(x, x) = 0$.

(3) A mapping $f : \mathcal{X} \to \mathcal{X}$ is *σ-continuous*, if $x_n \to x$ as $n \to \infty$ in $(\mathcal{X}, \sigma)$ implies that $fx_n \to fx$ as $n \to \infty$.

It is easy to show (e.g., [2]) that the limit of a sequence in a metric-like space might not be unique.

Lemma 1.5.([9]) *Let $(\mathcal{X}, \sigma)$ be a metric-like space.*

(a) *If $x, y \in \mathcal{X}$ then $\sigma(x, y) = 0$ implies that $\sigma(x, x) = \sigma(y, y) = 0$.*

(b) *If a sequence $\{x_n\}$ in $\mathcal{X}$ converges to some $x \in \mathcal{X}$ with $\sigma(x, x) = 0$ then $\lim_{n\to\infty} \sigma(x_n, y) = \sigma(x, y)$ for all $y \in \mathcal{X}$.*

It is clear that if a metric-like space is σ-complete, then it is 0-σ-complete [15]. The converse assertion does not hold as the following example shows.

Example 1.6.([15]) Let $\mathcal{X} = [0, 1) \cap \mathbb{Q}$ and $\sigma : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be defined by

$$\sigma(x, y) = \begin{cases} 2x, & \text{if } \ x = y, \\ \max\{x, y\}, & \text{otherwise}, \end{cases}$$

for all $x, y \in \mathcal{X}$. Then $(\mathcal{X}, \sigma)$ is a metric-like space. It is easy to see that $(\mathcal{X}, \sigma)$ is a 0-σ-complete metric-like space, while it is not a σ-complete metric-like space.

In the paper [18], Wardowski introduced a new type of contractions which he called F-contractions. Several authors proved various variants of fixed point results using such contractions. In Wardowski's approach, the following set of functions is used:

Definition 1.7. Denote by $\mathfrak{F}$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ with the following properties:

(F1) F is strictly increasing;

(F2) for each sequence $\{t_n\}$ of positive numbers,

$$\lim_{n\to\infty} t_n = 0 \text{ if and only if } \lim_{n\to\infty} F(t_n) = -\infty.$$

(F3) There exists $k \in (0,1)$ such that $\lim_{t\to 0^+} t^k F(t) = 0$.

Example 1.8.([18]) Let $F_i : \mathbb{R}^+ \to \mathbb{R}$ ($i \in \{1,2,3,4\}$) be defined by $F_1(t) = \ln t$, $F_2(t) = t + \ln t$, $F_3(t) = -1/\sqrt{t}$, $F_4(t) = \ln(t^2 + t)$. Then each F_i satisfies the properties (F1)–(F3).

Definition 1.9.([18]) Let $(\mathcal{X}, d)$ be a metric space. A self-mapping f on $\mathcal{X}$ is called an *F-contraction* if there exist $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that

$$\tau + F(d(fx, fy)) \leq F(d(x, y)), \tag{1.1}$$

for all $x, y \in \mathcal{X}$ with $d(fx, fy) > 0$.

Note that, taking $F \in \mathfrak{F}$ defined by $F(t) = \ln t$, the relation (1.1) reduces to

$$d(fx, fy) \leq e^{-\tau} d(x, y), \quad x \neq y,$$

i.e., such mapping is a (Banach-type) contraction. However, other functions $F \in \mathfrak{F}$ may produce new concepts (see the respective examples in [18]). Hence, the notion of F-contraction is more general than the notion of contraction.

Contraction-type mappings have been also generalized in other directions. In the series of generalizations, Samet et al. [13] introduced the concept of α-admissible maps and gave the concept of α-ψ-contractive mapping, thus generalizing BCP. Hussain et al. [7] mixed both these concepts and introduced α-GF-contraction and obtained some interesting fixed point results. Following this direction of research, Gopal et al. [5] introduced the concept of α-type F-contractive mappings and gave relevance to fixed point and periodic point theorems, as well as an application to nonlinear fractional differential equations. Some other related work can be seen in [1, 8].

With the above discussion in mind, we introduce in this paper the notion of α-type rational F-contractive mapping in a metric-like space and derive some fixed and common fixed point results. Further, we give some examples and counterexamples to illustrate the applicability and effectiveness of the results compared with existing results in metric spaces. In the next section, we present an application to a certain class of integral equations. We consider also an integral equation arising in the theory of nonlinear fractional differential equations and verify it numerically.

2. Discussion on Fixed Point Results

Recently, Sintunavarat [16] introduced the notion of weakly α-admissible map and discussed respective fixed point results in metric spaces.

Definition 2.1. For a nonempty set $\mathcal{X}$, let $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ and $f : \mathcal{X} \to \mathcal{X}$ be two mappings. Then f is said to be:

(1) [13] *α-admissible* if:

$$x, y \in \mathcal{X} \text{ with } \alpha(x, y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1.$$

(2) [16] *weakly α-admissible* if:

$$x \in \mathcal{X} \text{ with } \alpha(x, fx) \geq 1 \Rightarrow \alpha(fx, ffx) \geq 1.$$

Example 2.2. Let $\mathcal{X} = [0, \infty)$. Define mappings $f : \mathcal{X} \to \mathcal{X}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} e^{x+y}, & x, y \in [0,1] \\ \ln(2x + y), & \text{otherwise,} \end{cases} \text{ and } f(x) = \begin{cases} 2\tanh\left(\frac{3x}{2}\right), & \text{if } x \in [0,1] \\ \ln 3x, & \text{otherwise.} \end{cases}$$

It is easy to see that f is not an α-admissible mapping, however, it is weakly α-admissible.

In what follows, we use the following terminology from the paper [17]. For a nonempty set $\mathcal{X}$ and a mapping $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$, we use $\mathcal{A}(\mathcal{X}, \alpha)$ and $\mathcal{WA}(\mathcal{X}, \alpha)$ to denote the collection of all α-admissible mappings on $\mathcal{X}$ and the collection of all weakly α-admissible mappings on $\mathcal{X}$, respectively. Obviously,

$$\mathcal{A}(\mathcal{X}, \alpha) \subset \mathcal{WA}(\mathcal{X}, \alpha)$$

and, by the previous example, the inclusion can be strict.

We introduce now the notion of α-type rational F-contraction in a metric-like space as follows.

Definition 2.3. Let $(\mathcal{X}, \sigma)$ be a metric-like space and $\alpha \colon \mathcal{X} \times \mathcal{X} \to [0, \infty)$. A self-mapping f on $\mathcal{X}$ is called an *α-type rational F-contraction*, if there exist $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that

$$\begin{aligned} &u, v \in \mathcal{X} \text{ with } \alpha(u, v) \geq 1 \text{ and } \sigma(fu, fv) > 0 \text{ implies} \\ &\qquad \tau + F(\sigma(fu, fv)) \leq F\left(\Delta_f(u, v)\right), \end{aligned} \tag{2.1}$$

where

$$\Delta_f(u, v) = \max\left\{ \begin{array}{c} \sigma(u, v), \sigma(u, fu), \sigma(v, fv), \frac{\sigma(u,fv)+\sigma(v,fu)}{4}, \\ \frac{\sigma(u,fu)\sigma(v,fv)}{1+\sigma(u,v)}, \frac{\sigma(u,fu)\sigma(v,fv)}{1+\sigma(fu,fv)} \end{array} \right\}. \tag{2.2}$$

We denote by $\Upsilon(\mathcal{X}, \alpha, \mathfrak{F})$ the collection of all α-type rational F-contractive mappings on $(\mathcal{X}, \sigma)$.

If we take $F(t) = \ln t$ and $\tau = \ln(\frac{1}{\lambda})$, where $\lambda = (0, 1)$, we see that every α-type rational contraction is also an α-type rational F-contraction in a metric-like space. However, for other functions $F \in \mathfrak{F}$, new conditions can be obtained (see further Remark 2.7).

We are equipped now to state our first main result.

Theorem 2.4. *Let $(\mathcal{X}, \sigma)$ be a 0-σ-complete metric-like space and let $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ and $f : \mathcal{X} \to \mathcal{X}$ be given mappings. Suppose that the following conditions hold:*

$(AF1)$ $f \in \Upsilon(\mathcal{X}, \alpha, \mathfrak{F}) \cap \mathcal{WA}(\mathcal{X}, \alpha)$;

$(AF2)$ *there exists* $u_0 \in \mathcal{X}$ *such that* $\alpha(u_0, fu_0) \geq 1$;

$(AF3)$ α *has a transitive property, that is, for* $u, v, w \in \mathcal{X}$,

$$\alpha(u, v) \geq 1 \text{ and } \alpha(v, w) \geq 1 \Rightarrow \alpha(u, w) \geq 1;$$

$(AF4)$ f *is* σ*-continuous.*

Then f *has a fixed point* $u^* \in \mathcal{X}$ *such that* $\sigma(u^*, u^*) = 0$.

Proof. Starting from the given $u_0 \in \mathcal{X}$ satisfying $\alpha(u_0, fu_0) \geq 1$, define a sequence $\{u_n\}$ in $\mathcal{X}$ by $u_{n+1} = fu_n$ for $n \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$. If there exists $n_0 \in \mathbb{N}^*$ such that $u_{n_0} = u_{n_0+1}$, then $u_{n_0} \in Fix(f)$ and hence the proof is completed. Hence, we will assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}^*$ and let $\varrho_n = \sigma(u_n, u_{n+1})$ for $n \in \mathbb{N}^*$. Then $\varrho_n > 0$ for all $n \in \mathbb{N}^*$. We will prove that $\lim_{n\to\infty} \varrho_n = 0$.

Using that $f \in \mathcal{WA}(\mathcal{X}, \alpha)$ and $\alpha(u_0, fu_0) \geq 1$, we have

$$\alpha(u_1, u_2) = \alpha(fu_0, ffu_0) \geq 1.$$

Repeating this process, we obtain

$$\alpha(u_{n+1}, u_{n+2}) \geq 1, \quad \forall n \in \mathbb{N}^*.$$

It follows from $f \in \Upsilon(\mathcal{X}, \alpha, \mathfrak{F})$ that

(2.3)

$$\begin{aligned}
\tau + F(\varrho_n) &= \tau + F(\sigma(x_{n+1}, x_n)) = \tau + F(\sigma(fx_n, fx_{n-1})) \\
&\leq F\left(\max\left\{\begin{array}{c} \sigma(x_n, x_{n-1}), \sigma(x_n, fx_n), \sigma(x_{n-1}, fx_{n-1}), \\ \frac{\sigma(x_n, fx_{n-1}) + \sigma(x_{n-1}, fx_n)}{4}, \frac{\sigma(x_n, fx_n)\sigma(x_{n-1}, fx_{n-1})}{1+\sigma(x_n, x_{n-1})}, \\ \frac{\sigma(x_n, fx_n)\sigma(x_{n-1}, fx_{n-1})}{1+\sigma(fx_n, fx_{n-1})} \end{array}\right\}\right) \\
&= F\left(\max\left\{\begin{array}{c} \sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1}), \sigma(x_{n-1}, x_n), \\ \frac{\sigma(x_n, x_n) + \sigma(x_{n-1}, x_{n+1})}{4}, \frac{\sigma(x_n, x_{n+1})\sigma(x_{n-1}, x_n)}{1+\sigma(x_n, x_{n-1})}, \\ \frac{\sigma(x_n, x_{n+1})\sigma(x_{n-1}, x_n)}{1+\sigma(x_{n+1}, x_n)} \end{array}\right\}\right) \\
&\leq F\Big(\max\Big\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1}), \frac{3\sigma(x_n, x_{n+1}) + \sigma(x_{n-1}, x_n)}{4}\Big\}\Big) \\
&\leq F\big(\max\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1})\}\big) \\
&\leq F\big(\max\{\varrho_{n-1}, \varrho_n\}\big).
\end{aligned}$$

If $\varrho_{n-1} \leq \varrho_n$ for some $n \in \mathbb{N}$, then from (2.3) we have $\tau + F(\varrho_n) \leq F(\varrho_n)$, which is a contradiction since $\tau > 0$. Thus $\varrho_{n-1} > \varrho_n$ for all $n \in \mathbb{N}$ and so from (2.3) we have

$$F(\varrho_n) \leq F(\varrho_{n-1}) - \tau.$$

Therefore we derive

$$F(\varrho_n) \leq F(\varrho_{n-1}) - \tau \leq F(\varrho_{n-2}) - 2\tau \leq \cdots \leq F(\varrho_0) - n\tau \text{ for all } n \in \mathbb{N},$$

that is,

$$F(\varrho_n) \leq F(\varrho_0) - n\tau \text{ for all } n \in \mathbb{N}. \tag{2.4}$$

From (2.4), we get $F(\varrho_n) \to -\infty$ as $n \to \infty$. Thus, from (F2), we have

$$\lim_{n\to\infty} \varrho_n = 0. \tag{2.5}$$

Now by the property (F3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (\varrho_n)^k F(\varrho_n) = 0. \tag{2.6}$$

By (2.6), the following holds for all $n \in \mathbb{N}$:

$$(\varrho_n)^k F(\varrho_n) - (\varrho_n)^k F(\varrho_0) \leq (\varrho_n)^k(-n\tau) \leq 0. \tag{2.7}$$

Passing to the limit as $n \to \infty$ in (2.7), and using (2.5)–(2.6) we obtain

$$\lim_{n\to\infty} n(\varrho_n)^k = 0. \tag{2.8}$$

From (2.8), there exits $n_1 \in \mathbb{N}$ such that $n(\varrho_n)^k \leq 1$ for all $n \geq n_1$. So, we have, for all $n \geq n_1$

$$\varrho_n \leq \frac{1}{n^{1/k}}. \tag{2.9}$$

In order to show that $\{x_n\}$ is a 0-σ-Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the property (σ3) and (2.9), we have

$$\begin{aligned}\sigma(x_n, x_m) &\leq \sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+2}) + \cdots + \sigma(x_{m-1}, x_m)\\ &= \varrho_n + \varrho_{n+1} + \cdots + \varrho_{m-1}\\ &= \sum_{i=n}^{m-1} \varrho_i \leq \sum_{i=n}^{\infty} \varrho_i \leq \sum_{i=n}^{\infty} \frac{1}{n^{1/k}}.\end{aligned}$$

By the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^{1/k}}$, passing to the limit as $n \to \infty$, we get $\sigma(x_n, x_m) \to 0$ and hence $\{x_n\}$ is a 0-σ-Cauchy sequence in $(\mathcal{X}, \sigma)$. Since $\mathcal{X}$ is a 0-σ-complete metric-like space, there exists a $u^* \in \mathcal{X}$ such that

$$\lim_{n,m\to\infty} \sigma(x_n, x_m) = \lim_{n\to\infty} \sigma(x_n, u^*) = \sigma(u^*, u^*) = 0. \tag{2.10}$$

Now, since f is σ-continuous, we obtain from (2.10) that

$$\lim_{n,m\to\infty} \sigma(fx_n, fu^*) = \lim_{n\to\infty} \sigma(x_{n+1}, fu^*) = \sigma(u^*, fu^*) = 0.$$

This proves that u^* is a fixed point of f such that $\sigma(u^*, u^*) = 0$. □

We note that the previous result can still be valid for f not necessarily σ-continuous. We have the following result.

Theorem 2.5. *Let $(\mathcal{X},\sigma)$ be a 0-σ-complete metric-like space, let $\alpha:\mathcal{X}\times\mathcal{X}\to[0,\infty)$, and $f:\mathcal{X}\to\mathcal{X}$ be given mappings. Suppose that the assumptions $(AF1)$–$(AF3)$ of Theorem 2.4 hold, as well as:*

$(AF4')$ *F is continuous;*

$(AF5)$ *$\mathcal{X}$ is α-regular, i.e., if $\{u_n\}$ is a sequence in $\mathcal{X}$ with $\alpha(u_n,u_{n+1})\ge 1$ for $n\in\mathbb{N}$ and $u_n\to u^*$ as $n\to\infty$, then $\alpha(u_n,u^*)\ge 1$ for $n\in\mathbb{N}$.*

Then f has a fixed point $u^*\in\mathcal{X}$ such that $\sigma(u^*,u^*)=0$.

Proof. Following the proof of Theorem 2.4, we obtain a 0-σ-Cauchy sequence $\{u_n\}$ in the 0-σ-complete metric-like space $(\mathcal{X},\sigma)$. Hence, there exists $u^*\in\mathcal{X}$ such that

$$\lim_{n\to\infty}\sigma(u_n,u^*)=\sigma(u^*,u^*)=0.$$

We have to prove that $fu^*=u^*$. Assume, to the contrary, that $u^*\ne fu^*$. In this case, there exist an $n_0\in\mathbb{N}$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\sigma(fu_{n_k},fu^*)>0$ for all $n_k\ge n_0$. (Otherwise, there exists $n_1\in\mathbb{N}$ such that $u_n=fu^*$ for all $n\ge n_1$, which implies that $u_n\to fu^*$. This is a contradiction, since $u^*\ne fu^*$.) By α-regularity of $\mathcal{X}$, we have $\alpha(u_n,u^*)\ge 1$ for all $n\in\mathbb{N}$. Since $\sigma(fu_{n_k},fu^*)>0$ for all $n_k\ge n_0$, then from (2.1), we have

$$\begin{aligned}
&\tau+F(\sigma(u_{n_k+1},fu^*))\\
&\quad=\tau+F(\sigma(fu_{n_k},fu^*))\\
&\quad\le F\left(\max\left\{\begin{array}{c}\sigma(u_{n_k},u^*),\sigma(u_{n_k},fu_{n_k}),\sigma(u^*,fu^*),\frac{\sigma(u_{n_k},fu^*)+\sigma(u^*,fu_{n_k})}{4},\\ \frac{\sigma(u_{n_k},fu_{n_k})\sigma(u^*,fu^*)}{1+\sigma(u_{n_k},u^*)},\frac{\sigma(u_{n_k},fu_{n_k})\sigma(u^*,fu^*)}{1+\sigma(fu_{n_k},fu^*)}\end{array}\right\}\right)\\
&\quad=F\left(\max\left\{\begin{array}{c}\sigma(u_{n_k},u^*),\sigma(u_{n_k},u_{n_k+1}),\sigma(u^*,fu^*),\frac{\sigma(u_{n_k},fu^*)+\sigma(u^*,u_{n_k+1})}{4},\\ \frac{\sigma(u_{n_k},u_{n_k+1})\sigma(u^*,fu^*)}{1+\sigma(u_{n_k},u^*)},\frac{\sigma(u_{n_k},u_{n_k+1})\sigma(u^*,fu^*)}{1+\sigma(u_{n_{k+1}},fu^*)}\end{array}\right\}\right).
\end{aligned}$$

Passing to the limit as $k\to\infty$ and using the continuity of F we have $\tau+F(\sigma(u^*,fu^*))\le F(\sigma(u^*,fu^*))$, a contradiction. Therefore the claim is true, that is $u^*=fu^*$ with $\sigma(u^*,u^*)=0$. □

To ensure the uniqueness of the fixed point, we will consider the following hypothesis.

(H0): For all $x,y\in Fix(f)$, $\alpha(x,y)\ge 1$.

Theorem 2.6. *Adding condition $(H0)$ to the hypotheses of Theorem 2.4 (respectively, Theorem 2.5) then $fu^*=u^*$, $fv^*=v^*$ and $\sigma(u^*,u^*)=\sigma(v^*,v^*)=0$ imply that $u^*=v^*$.*

Proof. Suppose that $fu^*=u^*$, $fv^*=v^*$, $\sigma(u^*,u^*)=\sigma(v^*,v^*)=0$, and, to the contrary, $u^* \neq v^*$, hence $\sigma(u^*, v^*) > 0$. By the assumption, we can replace u by u^* and v by v^* in the condition (2.1), and we get

$$\begin{aligned}
&\tau + F(\sigma(u^*, v^*)) = \tau + F(\sigma(fu^*, fv^*)) \\
&\quad \leq F\left(\max\left\{\begin{array}{c} \sigma(u^*, v^*), \sigma(u^*, fu^*), \sigma(v^*, fv^*), \frac{\sigma(u^*, fv^*)+\sigma(v^*, fu^*)}{4}, \\ \frac{\sigma(u^*, fu^*)\sigma(v^*, fv^*)}{1+\sigma(u^*, v^*)}, \frac{\sigma(u^*, fu^*)\sigma(v^*, fv^*)}{1+\sigma(fu^*, fv^*)} \end{array}\right\}\right) \\
&\quad \leq F\left(\max\left\{\sigma(u^*, v^*), \frac{\sigma(u^*, v^*)}{2}\right\}\right) \\
&\quad = F(\sigma(u^*, v^*))
\end{aligned}$$

a contradiction, which implies that $u^* = v^*$. □

Remark 2.7. Taking various concrete functions $F \in \mathfrak{F}$ in the condition (2.1) of Theorems 2.4–2.6, we can get various α-type rational F-contractive conditions. We state just a few examples (recall that $\Delta_f(u, v)$ is defined in (2.2)).

(I) Taking $F(t) = \ln t$ $(t > 0)$ and $\tau = \ln(\frac{1}{\lambda})$, where $\lambda \in (0, 1)$, we have the condition

$$u, v \in \mathfrak{X} \text{ with } \alpha(u, v) \geq 1 \text{ and } \sigma(fu, fv) > 0 \text{ implies} \\ \sigma(fu, fv) \leq \lambda \Delta_f(u, v). \tag{2.11}$$

(II) Taking $F(t) = \ln t + t$ $(t > 0)$ and $\tau = \ln(\frac{1}{\lambda})$, where $\lambda \in (0, 1)$, we have the condition

$$u, v \in \mathfrak{X} \text{ with } \alpha(u, v) \geq 1 \text{ and } \sigma(fu, fv) > 0 \text{ implies} \\ \sigma(fu, fv) e^{\sigma(fu, fv) - \Delta_f(u, v)} \leq \lambda \Delta_f(u, v). \tag{2.12}$$

(III) Taking $F(t) = -\frac{1}{\sqrt{t}}$ $(t > 0)$, the condition is

$$u, v \in \mathfrak{X} \text{ with } \alpha(u, v) \geq 1 \text{ and } \sigma(fu, fv) > 0 \text{ implies} \\ \sigma(fu, fv) \leq \frac{1}{(1 + \tau\sqrt{\Delta_f(u, v)})^2} \Delta_f(u, v), \tag{2.13}$$

(IV) Taking $F(t) = \ln(t^2 + t)$ $(t > 0)$ and $\tau = \ln(\frac{1}{\lambda})$, where $\lambda > 0$, we have

$$u, v \in \mathfrak{X} \text{ with } \alpha(u, v) \geq 1 \text{ and } \sigma(fu, fv) > 0 \text{ implies} \\ \sigma(fu, fv)[\sigma(fu, fv) + 1] \leq \lambda \Delta_f(u, v)[\Delta_f(u, v) + 1].$$

The following examples can be used to illustrate the usage of Theorems 2.4–2.6.

Example 2.8. (This example demonstrates the use of rational terms in the contractive condition.)

Let $\mathcal{X} = \{a, b, c\}$ and $\sigma : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ be given by

$$\begin{gathered}\sigma(a,a)=0, \qquad \sigma(b,b)=3, \qquad \sigma(c,c)=9,\\ \sigma(a,b)=\sigma(b,a)=9, \quad \sigma(a,c)=\sigma(c,a)=5, \quad \sigma(b,c)=\sigma(c,b)=8.\end{gathered}$$

Then $(\mathcal{X}, \sigma)$ is a 0-σ-complete metric-like space. Define $\alpha : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ and $f : \mathcal{X} \to \mathcal{X}$ by

$$\alpha(u,v)=1 \text{ for all } u,v\in\mathcal{X}, \quad f=\begin{pmatrix} a & b & c\\ a & c & a\end{pmatrix}.$$

Moreover, take $\tau = \ln(8/5)$ and $F \in \mathfrak{F}$ defined by $F(t) = \ln t$. Then it is easy to see that all the conditions of Theorem 2.6 are fulfilled—just the condition $f \in \Upsilon(\mathcal{X}, \alpha, \mathfrak{F})$ needs to be checked. In this case it reduces to

$$\sigma(fu,fv)\le\frac{5}{8}\Delta_f(u,v) \tag{2.14}$$

for all $u, v \in \mathcal{X}$ with $\sigma(fu, fv) > 0$, where $\Delta_f(u, v)$ is defined by (2.2). Only the following three cases have to be considered:
Case 1: $u = a$, $v = b$. Then $\sigma(fu, fv) = \sigma(a, c) = 5$ and

$$\begin{aligned}\Delta_f(u,v)&=\max\left\{\begin{array}{c}\sigma(a,b),\sigma(a,a),\sigma(b,c),\frac{\sigma(a,c)+\sigma(b,a)}{4},\\ \frac{\sigma(a,a)\sigma(b,c)}{1+\sigma(a,b)},\frac{\sigma(a,a)\sigma(b,c)}{1+\sigma(a,c)}\end{array}\right\}\\ &=\max\{9,0,8,\tfrac{7}{2},0,0\}=9.\end{aligned}$$

Hence, (2.14) reduces to $5 \le \frac{5}{8} \cdot 9$.
Case 2: $u = b$, $v = b$. Then $\sigma(fu, fv) = \sigma(c, c) = 9$ and

$$\begin{aligned}\Delta_f(u,v)&=\max\left\{\begin{array}{c}\sigma(b,b),\sigma(b,c),\sigma(b,c),\frac{\sigma(b,c)+\sigma(b,c)}{4},\\ \frac{\sigma(b,c)\sigma(b,c)}{1+\sigma(b,b)},\frac{\sigma(b,c)\sigma(b,c)}{1+\sigma(c,c)}\end{array}\right\}\\ &=\max\{3,8,8,4,16,\tfrac{32}{5}\}=16.\end{aligned}$$

Hence, (2.14) reduces to $9 \le \frac{5}{8} \cdot 16$. (Note that, in this case, the condition would not hold without the rational terms in Δ_f.)
Case 3: $u = b$, $v = c$. Then $\sigma(fu, fv) = \sigma(c, a) = 5$ and

$$\begin{aligned}\Delta_f(u,v)&=\max\left\{\begin{array}{c}\sigma(b,c),\sigma(b,c),\sigma(c,a),\frac{\sigma(b,a)+\sigma(c,c)}{4},\\ \frac{\sigma(b,c)\sigma(c,a)}{1+\sigma(b,c)},\frac{\sigma(b,c)\sigma(c,a)}{1+\sigma(c,a)}\end{array}\right\}\\ &=\max\{8,8,5,\tfrac{9}{2},\tfrac{40}{9},\tfrac{20}{3}\}=8.\end{aligned}$$

Hence, (2.14) reduces to $5 \le \frac{5}{8} \cdot 8$.

Thus, all the conditions of Theorem 2.6 are satisfied and the mapping f has a unique fixed point (which is $u^* = a$).

Example 2.9. (This example demonstrates the advantage of using a metric-like instead of a standard metric; it is inspired by [11, Example 6]. Also, it shows the advantage of using α-type conditions, since fewer cases need to be checked.)

Let $\mathcal{X} = \{0, 1, 2, 3\}$ be equipped with $\sigma : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ given by $\sigma(u, v) = u + v$. Then $(\mathcal{X}, \sigma)$ is a 0-σ-complete metric-like space. Define $\alpha : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ and $f : \mathcal{X} \to \mathcal{X}$ by

$$\alpha(u,v) = \begin{cases} 1, & (u,v) \in \{(0,0),(1,1),(2,2),(3,3),(0,1),(1,2),(0,2)\} \\ 0, & \text{otherwise}, \end{cases}$$

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Moreover, take $\tau = \ln(3/2)$ and $F \in \mathfrak{F}$ defined by $F(t) = \ln t$. Then it is easy to see that all the conditions of Theorem 2.6 are fulfilled—just the condition $f \in \Upsilon(\mathcal{X}, \alpha, \mathfrak{F})$ needs to be checked. In this case it reduces to

$$\sigma(fu, fv) \le \frac{2}{3}\Delta_f(u, v) \tag{2.15}$$

for all $u, v \in \mathcal{X}$ with $\sigma(fu, fv) > 0$, where $\Delta_f(u, v)$ is defined by (2.2). Only the following four cases have to be considered:

Case 1: $u = 0$, $v = 2$. Then $\sigma(fu, fv) = \sigma(0, 1) = 1$ and

$$\begin{aligned} \Delta_f(u,v) &= \max\left\{ \begin{array}{c} \sigma(0,2), \sigma(0,0), \sigma(2,1), \frac{\sigma(0,1)+\sigma(2,0)}{4}, \\ \frac{\sigma(0,0)\sigma(2,1)}{1+\sigma(0,2)}, \frac{\sigma(0,0)\sigma(2,1)}{1+\sigma(0,1)} \end{array} \right\} \\ &= \max\{2, 0, 3, \tfrac{3}{4}, 0, 0\} = 3. \end{aligned}$$

Hence, (2.15) reduces to $1 \le \frac{2}{3} \cdot 3$.

Case 2: $u = 1$, $v = 2$. Then $\sigma(fu, fv) = \sigma(0, 1) = 1$ and

$$\begin{aligned} \Delta_f(u,v) &= \max\left\{ \begin{array}{c} \sigma(1,2), \sigma(1,0), \sigma(2,1), \frac{\sigma(1,1)+\sigma(0,2)}{4}, \\ \frac{\sigma(1,0)\sigma(2,1)}{1+\sigma(1,2)}, \frac{\sigma(1,0)\sigma(2,1)}{1+\sigma(0,1)} \end{array} \right\} \\ &= \max\{3, 1, 3, 1, \tfrac{3}{4}, \tfrac{3}{2}\} = 3. \end{aligned}$$

Hence, (2.15) reduces to $1 \le \frac{2}{3} \cdot 3$.

Case 3: $u = 2$, $v = 2$. Then $\sigma(fu, fv) = \sigma(1, 1) = 2$ and

$$\begin{aligned} \Delta_f(u,v) &= \max\left\{ \begin{array}{c} \sigma(2,2), \sigma(2,1), \sigma(2,1), \frac{\sigma(2,1)+\sigma(1,2)}{4}, \\ \frac{\sigma(2,1)\sigma(2,1)}{1+\sigma(2,2)}, \frac{\sigma(2,1)\sigma(2,1)}{1+\sigma(1,1)} \end{array} \right\} \\ &= \max\{4, 3, 3, \tfrac{3}{2}, \tfrac{9}{5}, 3\} = 4. \end{aligned}$$

Hence, (2.15) reduces to $2 \le \frac{2}{3} \cdot 4$.

Case 4: $u=3$, $v=3$. Then $\sigma(fu,fv)=\sigma(2,2)=4$ and

$$\begin{aligned}\Delta_f(u,v)&=\max\left\{\begin{array}{c}\sigma(3,3),\sigma(3,2),\sigma(3,2),\frac{\sigma(3,2)+\sigma(2,3)}{4},\\ \frac{\sigma(3,2)\sigma(3,2)}{1+\sigma(3,3)},\frac{\sigma(3,2)\sigma(3,2)}{1+\sigma(2,2)}\end{array}\right\}\\ &=\max\{6,5,5,\tfrac{5}{2},\tfrac{25}{7},5\}=6.\end{aligned}$$

Hence, (2.15) reduces to $4\le\frac{2}{3}\cdot 6$.

Thus, all the conditions of Theorem 2.6 are satisfied and the mapping f has a unique fixed point (which is $u^*=0$).

Consider now the same example, but using the standard metric $d(u,v)=|u-v|$ instead of σ on $\mathcal{X}$. Then, e.g., for $u=1$, $v=2$ we get

$$\begin{aligned}d(f1,f2)=d(0,1)=1&\not\le\lambda=\lambda\max\{1,1,1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\}\\ &=\lambda\max\left\{\begin{array}{c}d(1,2),d(1,0),d(2,1),\frac{d(1,1)+d(0,2)}{4},\\ \frac{d(1,0)d(2,1)}{1+d(1,2)},\frac{d(1,0)d(2,1)}{1+d(0,1)}\end{array}\right\},\end{aligned}$$

whatever $\lambda<1$ is chosen.

Example 2.10. (This example, inspired by [18, Example 2.5] and [1, Example 22], shows the reason for using various functions $F\in\mathfrak{F}$—it cannot be treated by the simplest example $F(t)=\ln t$.)

Let $\mathcal{X}=\{x_0,x_1,\dots,x_n\dots\}$, where $x_0=0$ and $x_n=\sum_{k=1}^n k$ for $n\in\mathbb{N}$. Define a metric-like σ on $\mathcal{X}$ by

$$\sigma(u,v)=\max\{u,v\}+|u-v|.$$

Then $(\mathcal{X},\sigma)$ is a 0-σ-complete metric-like space. Consider the mappings $\alpha:\mathcal{X}\times\mathcal{X}\to[0,+\infty)$ and $f:\mathcal{X}\to\mathcal{X}$ given by

$$\begin{gathered}\alpha(u,v)=1 \text{ for all } u,v\in\mathcal{X},\\ fx_0=x_0,\quad fx_n=x_{n-1} \text{ for } n\in\mathbb{N}.\end{gathered}$$

Take $\tau=e^{-1}$ and $F\in\mathfrak{F}$ given by $F(t)=t+\ln t$. Then the conditions of Theorem 2.6 are fulfilled—just the contractive condition has to be checked. In this case, it takes the form (see (2.12))

$$\frac{\sigma(fu,fv)}{\Delta_f(u,v)}\,e^{\sigma(fu,fv)-\Delta_f(u,v)}\le e^{-1},$$

for $u,v\in\mathcal{X}$ with $\sigma(fu,fv)>0$, where $\Delta_f(u,v)$ is defined in (2.2). Since $\Delta_f(u,v)\ge\sigma(u,v)$, this is a consequence of

$$\frac{\sigma(fu,fv)}{\sigma(u,v)}\,e^{\sigma(fu,fv)-\sigma(u,v)}\le e^{-1}, \tag{2.16}$$

for $\sigma(fu,fv)>0$. In order to prove (2.16), consider the following two cases.

Case 1: $u = x_0$, $v = x_n$, $n \geq 2$. Then $fu = x_0$, $fv = x_{n-1}$ and

$$\begin{aligned}\frac{\sigma(fu,fv)}{\sigma(u,v)}\, e^{\sigma(fu,fv)-\sigma(u,v)} &= \frac{\sigma(x_0,x_{n-1})}{\sigma(x_0,x_n)}\, e^{\sigma(x_0,x_{n-1})-\sigma(x_0,x_n)} \\ &= \frac{x_{n-1}}{x_n}\, e^{2(x_{n-1}-x_n)} < e^{-2n} < e^{-1}.\end{aligned}$$

Case 2: $u = x_m$, $v = x_n$, $1 \leq m \leq n$. Then

$$\begin{aligned}\frac{\sigma(fu,fv)}{\sigma(u,v)}\, e^{\sigma(fu,fv)-\sigma(u,v)} &= \frac{\sigma(x_{m-1},x_{n-1})}{\sigma(x_m,x_n)}\, e^{\sigma(x_{m-1},x_{n-1})-\sigma(x_m,x_n)} \\ &= \frac{x_{n-1}+(x_{n-1}-x_{m-1})}{x_n+(x_n-x_m)}\, e^{x_{n-1}+(x_{n-1}-x_{m-1})-x_n-(x_n-x_m)}.\end{aligned}$$

Now,

$$\begin{aligned}x_n + (x_n - x_m) &= \sum_{k=1}^{n} k + [(m+1)+(m+2)+\cdots+n] \\ &> \sum_{k=1}^{n-1} k + n + [m+(m+1)+\cdots+(n-1)] \\ &= x_{n-1} + (x_{n-1}-x_{m-1}) + n.\end{aligned}$$

Hence,

$$\frac{\sigma(fu,fv)}{\sigma(u,v)}\, e^{\sigma(fu,fv)-\sigma(u,v)} < e^{-n} < e^{-1}.$$

Therefore, all the conditions of Theorem 2.6 are satisfied, and f has a unique fixed point (which is $u^* = 0$).

The same conclusion cannot be obtained if the simplest function $F(t) = \ln t$ from $\mathfrak{F}$ is used, because the Banach-type condition (2.11) is not fulfilled. Indeed, putting $u = x_0 = 0$ and $v = x_n$, $n \geq 2$ in (2.11), we obtain $\sigma(0, x_{n-1}) \leq \lambda\sigma(0, x_n)$, which immediately reduces to

$$\frac{(n-1)n}{2} \leq \lambda\, \frac{n(n+1)}{2},$$

i.e., $\frac{n-1}{n+1} \leq \lambda$, for each $n \geq 2$. Letting $n \to \infty$, this implies $\lambda \geq 1$, hence (2.11) cannot hold for any $\lambda < 1$.

3. Discussion on Common Fixed Point Results

In order to complete the results, we first need the following notion.

Definition 3.1.([3]) For a nonempty set $\mathcal{X}$, let $f, g : \mathcal{X} \to \mathcal{X}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be mappings. We say that (f, g) is a *generalized α-admissible pair* if for all $u, v \in \mathcal{X}$, we have

$$\alpha(u,v) \geq 1 \Rightarrow \alpha(fu,gv) \geq 1 \text{ and } \alpha(gfu,fgv) \geq 1. \tag{3.1}$$

We use $\mathcal{GA}(\mathcal{X}, \alpha)$ to denote the collection of all generalized α-admissible pairs (f, g).

Remark 3.2. If the operator f is invertible such that $f = f^{-1}$ with $g = f$ in Definition 3.1, we get Definition 2.1.(i). So the class of mappings given in Definition 3.1 is not empty.

If f is α-admissible, it is obvious that (f, f) is a generalized α-admissible pair.

Now we introduce the notion of rational α-type F-contractive pair in a metric-like space.

Definition 3.3. Let $(\mathcal{X}, \sigma)$ be a metric-like space and $\alpha\colon \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a mapping. Two self-mappings f, g on $\mathcal{X}$ are said to form a *rational α-type F-contractive pair*, if there exist $F \in \mathfrak{F}$, $\tau \in \mathbb{R}^+$ such that

$$u, v \in \mathcal{X} \text{ with } \alpha(u, v) \geq 1 \text{ and } \sigma(fu, gv) > 0 \text{ implies} \tag{3.2}$$

$$\tau + F(\sigma(fu, gv)) \leq F\left(\max\left\{\begin{array}{c} \sigma(u, v), \sigma(u, fu), \sigma(v, gv), \frac{\sigma(u,gv)+\sigma(v,fu)}{4}, \\ \frac{\sigma(u,fu)\sigma(v,gv)}{1+\sigma(u,v)}, \frac{\sigma(u,fu)\sigma(v,gv)}{1+\sigma(fu,gv)} \end{array}\right\}\right).$$

We denote by $\Gamma(\mathcal{X}, \alpha, \mathfrak{F})$ the collection of all rational α-type F-contractive pairs on $(\mathcal{X}, \sigma)$.

We are equipped now to state the first result of this section.

Theorem 3.4. *Let $(\mathcal{X}, \sigma)$ be a 0-σ-complete metric-like space and let $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$, and $f, g : \mathcal{X} \to \mathcal{X}$ be given mappings. Suppose that the following conditions hold:*

$(F1)$ $(f, g) \in \Gamma(\mathcal{X}, \alpha, \mathfrak{F}) \cap \mathcal{GA}(\mathcal{X}, \alpha)$;

$(F2)$ *there exists $u_0 \in \mathcal{X}$ such that $\alpha(u_0, fu_0) \geq 1$;*

$(F3)$ *$\alpha(gfu, gu) \geq 1$ for all $u \in \mathcal{X}$;*

$(F4)$ *f and g are σ-continuous.*

Then there exists $u^* \in \mathcal{X}$ such that

$$\sigma(u^*, fu^*) = \sigma(fu^*, fu^*),\ \sigma(u^*, gu^*) = \sigma(gu^*, gu^*) \text{ and } \sigma(u^*, u^*) = 0. \tag{3.3}$$

Proof. By the given condition $(F2)$, there exists $u_0 \in \mathcal{X}$ such that $\alpha(u_0, fu_0) \geq 1$. Define a sequence $\{u_n\} \in \mathcal{X}$ by

$$u_{2n} = gu_{2n-1} \text{ and } u_{2n+1} = fu_{2n},\ \forall\, n \in \mathbb{N}. \tag{3.4}$$

If there exists $n_0 \in \mathbb{N}^*$ such that $\sigma(u_{n_0}, fu_{n_0}) = 0$ or $\sigma(u_{n_0}, gu_{n_0}) = 0$, then the proof is finished. So without loss of generality we can suppose that the successive terms of $\{u_n\}$ are different and let $\varrho_n = \sigma(u_{n+1}, u_n)$ for $n \in \mathbb{N}^*$. Then $\varrho_n > 0$ for all $n \in \mathbb{N}^*$. First we prove

$$\lim_{n\to\infty} \varrho_n = 0. \tag{3.5}$$

Since $(f,g) \in \mathcal{GA}(\mathcal{X},\alpha)$ and $\alpha(u_0, fu_0) \geq 1$, we have

$$\alpha(u_1,u_2) = \alpha(fu_0, gu_1) \geq 1 \text{ and } \alpha(u_2,u_3) = \alpha(gu_1, fu_2) = \alpha(gfu_0, fgu_1) \geq 1.$$

Repeating this process, we obtain

$$\alpha(u_n, u_{n+1}) \geq 1,\ \forall\ n \in \mathbb{N}^*.$$

Again from $(F3)$, we have

$$\alpha(u_2,u_1) = \alpha(gfu_0, fu_0) \geq 1 \text{ and } \alpha(u_4,u_3) = \alpha(gfu_2, fu_2) \geq 1.$$

Repeating this process, we obtain

$$\alpha(u_{2n}, u_{2n-1}) \geq 1,\ \forall\ n \in \mathbb{N}.$$

Therefore $(f,g) \in \Upsilon(\mathcal{X},\alpha,\mathfrak{F})$ implies that

(3.6)

$$\begin{aligned}
\tau + F(\varrho_{2n-1}) &= \tau + F(\sigma(u_{2n-1},u_{2n})) = \tau + F(\sigma(fu_{2n-2}, gu_{2n-1}))\\
&\leq F\left(\max\left\{\begin{array}{c} \sigma(u_{2n-2},u_{2n-1}), \sigma(u_{2n-2}, fu_{2n-2}),\\ \sigma(u_{2n-1}, gu_{2n-1}), \frac{\sigma(u_{2n-2},gu_{2n-1})+\sigma(u_{2n-1},fu_{2n-2})}{4},\\ \frac{\sigma(u_{2n-2},fu_{2n-2})\sigma(u_{2n-1},gu_{2n-1})}{1+\sigma(u_{2n-2},u_{2n-1})}\\ \frac{\sigma(u_{2n-2},fu_{2n-2})\sigma(u_{2n-1},gu_{2n-1})}{1+\sigma(fu_{2n-2},gu_{2n-1})}\end{array}\right\}\right)\\
&= F\left(\max\left\{\begin{array}{c} \sigma(u_{2n-2},u_{2n-1}), \sigma(u_{2n-2}, u_{2n-1}),\\ \sigma(u_{2n-1}, u_{2n}), \frac{\sigma(u_{2n-2},u_{2n})+\sigma(u_{2n-1},u_{2n-1})}{4},\\ \frac{\sigma(u_{2n-2},u_{2n-1})\sigma(u_{2n-1},u_{2n})}{1+\sigma(u_{2n-2},u_{2n-1})}\\ \frac{\sigma(u_{2n-2},u_{2n-1})\sigma(u_{2n-1},u_{2n})}{1+\sigma(u_{2n-1},u_{2n})}\end{array}\right\}\right)\\
&\leq F\left(\max\left\{\begin{array}{c} \sigma(u_{2n-2},u_{2n-1}), \sigma(u_{2n-1}, u_{2n}),\\ \frac{\sigma(u_{2n-2},u_{2n-1})+3\sigma(u_{2n-1},u_{2n})}{4},\end{array}\right\}\right)\\
&\leq F(\max\{\sigma(u_{2n-2},u_{2n-1}), \sigma(u_{2n-1},u_{2n})\}).
\end{aligned}$$

If $\varrho_{2n-2} \leq \varrho_{2n-1}$ for some $n \in \mathbb{N}$, then from (3.6) we have $\tau + F(\varrho_{2n-1}) \leq F(\varrho_{2n-1})$, which is a contradiction since $\tau > 0$. Thus $\varrho_{2n-2} > \varrho_{2n-1}$ for all $n \in \mathbb{N}$ and so from (3.6) we have

$$F(\varrho_{2n-1}) \leq F(\varrho_{2n-2}) - \tau. \tag{3.7}$$

Again using $(f,g) \in \Gamma(\mathcal{X}, \alpha, \mathfrak{F})$, we get

$$\begin{aligned}
\tau + F(\varrho_{2n}) &= \tau + F(\sigma(u_{2n+1}, u_{2n})) = \tau + F(\sigma(fu_{2n}, gu_{2n-1})) \\
&\leq F\left(\max\left\{\begin{array}{c} \sigma(u_{2n}, u_{2n-1}), \sigma(u_{2n}, fu_{2n}), \sigma(u_{2n-1}, gu_{2n-1}), \\ \frac{\sigma(u_{2n}, gu_{2n-1}) + \sigma(u_{2n-1}, fu_{2n})}{4}, \\ \frac{\sigma(u_{2n}, fu_{2n})\sigma(u_{2n-1}, gu_{2n-1})}{1+\sigma(u_{2n}, u_{2n-1})}, \\ \frac{\sigma(u_{2n}, fu_{2n})\sigma(u_{2n-1}, gu_{2n-1})}{1+\sigma(fu_{2n}, gu_{2n-1})} \end{array}\right\}\right) \\
&\leq F\left(\max\left\{\begin{array}{c} \sigma(u_{2n}, u_{2n-1}), \sigma(u_{2n}, u_{2n+1}), \sigma(u_{2n-1}, u_{2n}), \\ \frac{\sigma(u_{2n}, u_{2n}) + \sigma(u_{2n-1}, u_{2n+1})}{4}, \\ \frac{\sigma(u_{2n}, u_{2n+1})\sigma(u_{2n-1}, u_{2n})}{1+\sigma(u_{2n}, u_{2n-1})}, \frac{\sigma(u_{2n}, u_{2n+1})\sigma(u_{2n-1}, u_{2n})}{1+\sigma(u_{2n+1}, u_{2n})} \end{array}\right\}\right) \\
&\leq F\left(\max\left\{\begin{array}{c} \sigma(u_{2n}, u_{2n-1}), \sigma(u_{2n}, u_{2n+1}), \\ \frac{3\sigma(u_{2n}, u_{2n+1}) + \sigma(u_{2n-1}, u_{2n})}{4} \end{array}\right\}\right) \\
&\leq F(\max\{\sigma(u_{2n}, u_{2n-1}), \sigma(u_{2n}, u_{2n+1})\}).
\end{aligned}$$

With similar arguments, we get

$$F(\varrho_{2n}) \leq F(\varrho_{2n-1}) - \tau. \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$F(\varrho_n) \leq F(\varrho_{n-1}) - \tau, \text{ for all } n \in \mathbb{N}.$$

Using the arguments of proof of Theorem 2.4, we reach to (3.5) and show that $\{u_n\}$ is a 0-Cauchy sequence in $(\mathcal{X}, \sigma)$. Since $\mathcal{X}$ is a 0-complete metric-like space, there exists an $u^* \in \mathcal{X}$ such that $\lim_{n\to\infty} u_n = u^*$; moreover,

$$\lim_{n,m\to\infty} \sigma(u_n, u_m) = \lim_{n\to\infty} \sigma(u_n, u^*) = \sigma(u^*, u^*) = 0.$$

This implies that

$$\lim_{n\to\infty} \sigma(u_{2n+1}, u^*) = \lim_{n\to\infty} \sigma(u_{2n+2}, u^*) = 0.$$

Using (3.4), we convert to

$$\lim_{n\to\infty} \sigma(fu_{2n}, u^*) = \lim_{n\to\infty} \sigma(gu_{2n+1}, u^*) = 0.$$

Further we shall prove (3.3).

Using $u_{2n+1} \to u^* \in (\mathcal{X}, \sigma)$ and $\sigma(u^*, u^*) = 0$ in Lemma 1.5.(b), we have

$$\lim_{n\to\infty} \sigma(u_{2n+1}, fu^*) = \sigma(u^*, fu^*). \tag{3.9}$$

By the σ-continuity of f and $u_{2n} \to u^*$ in $(\mathcal{X}, \sigma)$, we have

$$\lim_{n\to\infty} \sigma(u_{2n+1}, fu^*) = \lim_{n\to\infty} \sigma(fu_{2n}, fu^*) = \sigma(fu^*, fu^*). \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$\sigma(fu^*, fu^*) = \sigma(u^*, fu^*).$$

Similarly using σ-continuity of g and the above arguments, we get

$$\sigma(gu^*, gu^*) = \sigma(u^*, gu^*).$$

Thus we have reached (3.3). □

Example 3.5. Let $\mathcal{X} = [0, 3]$ be endowed with the σ-complete metric-like defined by

$$\sigma(u, v) = \max\{u, v\} \quad \text{for all} \quad u, v \in \mathcal{X}.$$

Consider the mappings $f, g : \mathcal{X} \to \mathcal{X}$ given by

$$fu = \begin{cases} \frac{2}{3}u, & \text{if } u \in [0, 2], \\ u - \frac{2}{3}, & \text{if } u \in (2, 3] \end{cases} \quad \text{and} \quad gu = \begin{cases} \frac{3}{4}u, & \text{if } u \in [0, 2], \\ \frac{1}{2}u + \frac{1}{2}, & \text{if } u \in (2, 3] \end{cases}$$

and $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\alpha(u, v) = \begin{cases} 2, & \text{if } 0 \le u \le v \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, take $\tau = \ln(\frac{4}{3})$ and $F \in \mathfrak{F}$ defined by $F(t) = \ln t$. In this case the contraction condition (3.2) reduces to

$$\sigma(fu, gv) \le \frac{3}{4}\Delta_{f,g}(u, v), \tag{3.11}$$

where

$$\Delta_{f,g}(u, v) = \max\left\{ \begin{array}{c} \sigma(u, v), \sigma(u, fu), \sigma(v, gv), \frac{\sigma(u,gv)+\sigma(v,fu)}{4}, \\ \frac{\sigma(u,fu)\sigma(v,gv)}{1+\sigma(u,v)}, \frac{\sigma(u,fu)\sigma(v,gv)}{1+\sigma(fu,gv)} \end{array} \right\}.$$

In order to show $(f, g) \in \Gamma(\mathcal{X}, \alpha, \mathfrak{F})$, suppose that $\alpha(u, v) \ge 1$, i.e., $0 \le u \le v \le 2$. Then

$$\sigma(fu, gv) = \max\left\{\frac{2}{3}u, \frac{3}{4}v\right\} = \frac{3}{4}v,$$

$$\begin{aligned} \Delta_{f,g}(u, v) &= \max\left\{ \begin{array}{c} \sigma(u, v), \sigma(u, \frac{2}{3}u), \sigma(v, \frac{3}{4}v), \frac{\sigma(u,\frac{3}{4}v)+\sigma(v,\frac{2}{3}u)}{4}, \\ \frac{\sigma(u,\frac{2}{3}u)\sigma(v,\frac{3}{4}v)}{1+\sigma(u,v)}, \frac{\sigma(u,\frac{2}{3}u)\sigma(v,\frac{3}{4}v)}{1+\sigma(\frac{2}{3}u,\frac{3}{4}v)} \end{array} \right\} \\ &= \max\left\{ v, u, v, \frac{\max\{u,\frac{3}{4}v\}+v}{4}, \frac{uv}{1+v}, \frac{uv}{1+\frac{3}{4}v} \right\} = v. \end{aligned}$$

Hence, (3.11) reduces to $\frac{3}{4}v \le \frac{3}{4}v$. Thus $(f, g) \in \Gamma(\mathcal{X}, \alpha, \mathfrak{F})$.

Next we prove $(f,g) \in \mathcal{GA}(\mathcal{X},\alpha)$. Let $u,v \in \mathcal{X}$ be such that $\alpha(u,v) \geq 1$. By definition of α, this implies that $u,v \in [0,2]$. Thus,

$$\alpha(fu,gv) = \alpha\left(\frac{2}{3}u, \frac{3}{4}v\right) = 2 > 1,$$

and

$$\alpha(gfu,fgv) = \alpha\left(g\left(\frac{2}{3}u\right), f\left(\frac{3}{4}v\right)\right) = \alpha\left(\frac{1}{2}u, \frac{1}{2}v\right) = 2 > 1.$$

Then, $(f,g) \in \mathcal{GA}(\mathcal{X},\alpha)$.

We divide in three parts the proof that $\alpha(gfu,gu) \geq 1$, for all $u \in \mathcal{X}$.

1° For $u \in [0,2]$, we have

$$\alpha(gfu,gu) = \alpha\left(\frac{1}{2}u, \frac{3}{4}u\right) = 2 > 1.$$

2° For $u \in (2, \frac{8}{3}]$, we have $fu = u - \frac{2}{3} \in [0,2]$. Therefore

$$\alpha(gfu,gu) = \alpha\left(g\left(u-\frac{2}{3}\right), \frac{1}{2}u+\frac{1}{2}\right) = 2 > 1.$$

3° For $u \in (\frac{8}{3}, 3]$, we have $fu = u - \frac{2}{3} \in (2,3]$. Therefore

$$\alpha(gfu,fu) = \alpha\left(g\left(u-\frac{2}{3}\right), gu\right) = \alpha\left(\frac{1}{2}\left(u+\frac{1}{3}\right), \frac{1}{2}(u+1)\right) = 2 > 1.$$

Thus,

$$\alpha(gfu,gu) = 2 > 1, \text{ for all } u \in \mathcal{X}.$$

Moreover, the mappings f and g are σ-continuous on $(\mathcal{X},\sigma)$ and there exists $u_0 = 0$ such that $\alpha(u_0, fu_0) = \alpha(0,0) = 2$. Thus, all the hypotheses of Theorem 3.4 are verified, so there exists a common fixed point of f and g (which is $u^* = 0$).

We note that the previous result can be sharpened when continuity of f and g is replaced by the following condition.

(H1): If $\{u_n\}$ is a sequence in $\mathcal{X}$ such that if $\alpha(u_n,u_{n+1}) \geq 1$ and $\alpha(u_{n+1},u_n) \geq 1$ for all n and $u_n \to u \in X$ as $n \to \infty$, then there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $\alpha(u_{n(k)},u) \geq 1$ and $\alpha(u,u_{n(k)}) \geq 1$ for all k.

Now we have the following result.

Theorem 3.6. *Let all the conditions of Theorem 3.4 be satisfied, apart from the hypothesis $(F4)$ which is replaced by $(H1)$ and F is continuous. Then there exists a common fixed point of f and g in $\mathcal{X}$, that is, $u^* = fu^* = gu^*$.*

Proof. Following the proof of Theorem 3.4, we obtain a 0-σ-Cauchy sequence $\{u_n\}$ in the 0-σ-complete metric-like space $(\mathcal{X},\sigma)$. Hence, there exists $u^* \in \mathcal{X}$ such that

$$\lim_{n\to\infty} \sigma(u_n,u^*) = \sigma(u^*,u^*) = 0,$$

that is, $u_n \to u^*$ as $n \to \infty$. By the property $(H1)$ of $\mathcal{X}$, there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $\alpha(u_{2nk}, u^*) \geq 1$ and $\alpha(u^*, u_{2nk-1}) \geq 1$ for all k. Using $(F1)$ for $u = u_{2n(k)}$ and $v = u^*$ we have

$$\tag{3.12} \begin{aligned} \tau + F(\sigma(u_{2n(k)+1}, gu^*)) &= \tau + F(\sigma(fu_{2n(k)}, gu^*)) \\ &\leq F\left(\max\left\{\begin{array}{c} \sigma(u_{2n(k)}, u^*), \sigma(u_{2n(k)}, fu_{2n(k)}), \\ \sigma(u^*, gu^*), \frac{\sigma(u_{2n(k)}, gu^*)+\sigma(u^*, fu_{2n(k)})}{4}, \\ \frac{\sigma(u_{2n(k)}, fu_{2n(k)})\sigma(u^*, gu^*)}{1+\sigma(u_{2n(k)}, u^*)}, \frac{\sigma(u_{2n(k)}, fu_{2n(k)})\sigma(u^*, gu^*)}{1+\sigma(fu_{2n(k)}, gu^*)} \end{array}\right\}\right) \\ &= F\left(\max\left\{\begin{array}{c} \sigma(u_{2n(k)}, u^*), \sigma(u_{2n(k)}, u_{2n(k)+1}), \\ \sigma(u^*, gu^*), \frac{\sigma(u_{2n(k)}, gu^*)+\sigma(u^*, u_{2n(k)+1})}{4}, \\ \frac{\sigma(u_{2n(k)}, u_{2n(k)+1})\sigma(u^*, gu^*)}{1+\sigma(u_{2n(k)}, u^*)}, \frac{\sigma(u_{2n(k)}, u_{2n(k)+1})\sigma(u^*, gu^*)}{1+\sigma(u_{2n(k)+1}, gu^*)} \end{array}\right\}\right). \end{aligned}$$

Applying the limit as $n \to \infty$ to (3.12) and the continuity of F, we get

$$\tau + F(\sigma(u^*, gu^*)) \leq F(\sigma(u^*, gu^*))$$

a contradiction. Therefore we have $u^* = gu^*$.

Similarly, by taking $u = u^*$ and $v = u_{2n(k)-1}$ in $(F1)$, we get $u^* = fu^*$. Hence f and g have a common fixed point u^* such that $\sigma(u^*, u^*) = 0$. □

To ensure the uniqueness of the common fixed point, we will consider the following hypothesis as a generalized form of $(H0)$.

(H2): For all $u, v \in CF(f, g)$ such that $\sigma(u, u) = \sigma(v, v) = 0$, we have $\alpha(u, v) \geq 1$, where $CF(f, g)$ is the set of common fixed points of f and g.

Theorem 3.7. *Adding condition $(H2)$ to the hypotheses of Theorem 3.6, the uniqueness of the common fixed point u^* of f and g such that $\sigma(u^*, u^*) = 0$ is obtained.*

Proof. Suppose that v^* is another common fixed point of f and g, such that $\sigma(v^*, v^*) = 0$ and, contrary to what is going to be proved, $\sigma(fu^*, gv^*) = \sigma(u^*, v^*) > 0$. By the assumption, we can replace u by u^* and v by v^* in the condition (3.1), and we get easily

$$\begin{aligned} \tau + F(\sigma(u^*, v^*)) &= \tau + F(\sigma(fu^*, gv^*)) \\ &\leq F\left(\max\left\{\begin{array}{c} \sigma(u^*, v^*), \sigma(u^*, fu^*), \sigma(v^*, gv^*), \frac{\sigma(u^*, gv^*)+\sigma(v^*, fu^*)}{4}, \\ \frac{\sigma(u^*, fu^*)\sigma(v^*, gv^*)}{1+\sigma(u^*, v^*)}, \frac{\sigma(u^*, fu^*)\sigma(v^*, gv^*)}{1+\sigma(fu^*, gv^*)} \end{array}\right\}\right) \\ &\leq F\left(\max\left\{\sigma(u^*, v^*), \frac{\sigma(u^*, v^*)}{2}\right\}\right) \\ &= F(\sigma(u^*, v^*)) \end{aligned}$$

a contradiction, which implies that $u^* = v^*$. □

4. Application to Integral Equations

This section is devoted to the existence of solutions of an integral equation as an application of Theorem 2.5.

Let $\mathcal{X} = C(I,\mathbb{R})$, where $I = [0,1]$, be endowed with the metric-like distance function

$$\sigma(u,v) = \max_{t\in I}\{|u(t)|,|v(t)|\}.$$

We will consider the integral equation

$$u(t) = \int_0^t K(t,s)f(s,u(s))\,ds, \tag{4.1}$$

where $f : I\times\mathbb{R}\to\mathbb{R}$ and $K : I\times I\to[0,\infty)$ are given functions. Let

$$\mathcal{J}u(t) = \int_0^t K(t,s)f(s,u(s))\,ds, \tag{4.2}$$

define the respective operator $\mathcal{J}\colon \mathcal{X}\to\mathcal{X}$.

Theorem 4.1. *Let $\xi : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a given function. Suppose the following assertions hold:*

(F1) *f is a continuous function, non-decreasing in the second variable;*

(F2) *there exists $u_0\in\mathcal{X}$ such that $\xi(u_0(t),\mathcal{J}u_0(t))\ge 0$ for all $t\in I$;*

(F3) *for all $u,v,w\in\mathcal{X}$, $\xi(u(t),v(t))\ge 0$ and $\xi(v(t),w(t))\ge 0$ for all $t\in I$ imply that $\xi(u(t),w(t))\ge 0$ for all $t\in I$;*

(F4) *if $\{u_n\}$ is a sequence in $\mathcal{X}$ such that $u_n\to u$ in $\mathcal{X}$ and $\xi(u_n(t),u_{n+1}(t))\ge 0$ for all $n\in\mathbb{N}$ and $t\in I$, then $\xi(u_n(t),u(t))\ge 0$ for all $n\in\mathbb{N}$ and $t\in I$;*

(F5) *$(u,v)\in\mathcal{X}^2$ and $\xi(u(t),\mathcal{J}u(t))\ge 0$ for all $t\in I$ implies that $\xi(\mathcal{J}u(t),\mathcal{J}\mathcal{J}u(t))\ge 0$ for all $t\in I$;*

(F6) *there exists $\tau>0$ such that for $u,v\in\mathcal{X}$ with $\xi(u,v)\ge 0$, and $t\in I$ we have*

$$\max\{|f(t,u(t))|,|f(t,v(t))|\}\le A^{-1}\,\frac{\Delta_1(u,v)(t)}{\left(1+\tau\sqrt{\max_{t\in I}\Delta_1(u,v)(t)}\,\right)^2},$$

where $A=\sup_{t\in I}\int_0^t K(t,r)\,dr$ and

$$\Delta_1(u,v)(t) = \max\left\{\begin{array}{c} \max\{|u(t)|,|v(t)|\},\max\{|u(t)|,|\mathcal{J}u(t)|\},\max\{|v(t)|,|\mathcal{J}v(t)|\},\\ \dfrac{\max\{|u(t)|,|\mathcal{J}v(t)|\}+\max\{|v(t)|,|\mathcal{J}u(t)|\}}{4},\\ \dfrac{(\max\{|u(t)|,|\mathcal{J}u(t)|\})(\max\{|v(t)|,|\mathcal{J}v(t)|\})}{1+\max_{t\in I}[\max\{|u(t)|,|v(t)|\}]},\\ \dfrac{(\max\{|u(t)|,|\mathcal{J}u(t)|\})(\max\{|v(t)|,|\mathcal{J}v(t)|\})}{1+\max_{t\in I}[\max\{|\mathcal{J}u(t)|,|\mathcal{J}v(t)|\}]} \end{array}\right\}$$

Then the equation (4.1) has at least one solution $u^*\in\mathcal{X}$.

Proof. Define a function $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{for } \xi(x(t), y(t)) \geq 0, \text{ for all } t \in I \\ \gamma, & \text{otherwise,} \end{cases}$$

where $\gamma \in (0,1)$. It is easy to see that the assumption (F2) readily implies the condition (AF2) of Theorem 2.4. Also, the assumption (F3) implies that α has the transitive property (AF3). Finally, (F4) implies the regularity condition (AF5) of Theorem 2.5 and the assumption (F5) easily implies that $\mathcal{J} \in \mathcal{WA}(\mathcal{X}, \alpha)$.

We are going to check that $\mathcal{J} \in \Upsilon(\mathcal{X}, \alpha, \mathfrak{F})$. For this, let $u, v \in \mathcal{X}$ be such that $\alpha(u,v) \geq 1$, i.e., $\xi(u(t), v(t)) \geq 0$ for all $t \in I$. For each $t \in I$, by the definition (4.2) of operator $\mathcal{J}$, we have

$$\begin{aligned} &\max_{t \in I}\{|\mathcal{J}u(t)|, |\mathcal{J}v(t)|\} \\ &\qquad = \max_{t\in I}\left\{\left|\int_0^t K(t,r) f(r,u(r))\,dr\right|, \left|\int_0^t K(t,r) f(r,v(r))\,dr\right|\right\} \\ &\qquad \leq \max_{t\in I}\left\{\int_0^t K(t,r)\,|f(r,u(r))|\,dr, \int_0^t |K(t,r)|\,|f(r,v(r))|\,dr\right\} \\ &\qquad = \max_{t\in I}\int_0^t K(t,r) \max_{t\in I}(|f(r,u(r))|, |f(r,v(r))|)\,dr. \end{aligned}$$

Now, using the assumption (F6), after routine calculations, we obtain

$$\sigma(\mathcal{J}u, \mathcal{J}v) = \max_{t\in I}\{|\mathcal{J}u(t)|, |\mathcal{J}v(t)|\} \leq \frac{\Delta_{\mathcal{J}}(u,v)}{\left(1 + \tau\sqrt{\Delta_{\mathcal{J}}(u,v)}\,\right)^2},$$

where $\Delta_{\mathcal{J}}(u,v)$ is given in (2.2).

Now, by considering $F \in \mathfrak{F}$ given by $F(t) = \frac{-1}{\sqrt{t}}$, $t > 0$ (see the condition (2.13) in Remark 2.7), we have the condition

$$u, v \in \mathcal{X} \text{ with } \alpha(u,v) \geq 1 \Rightarrow \tau + F(\sigma(\mathcal{J}u, \mathcal{J}v)) \leq F(\Delta_{\mathcal{J}}(u,v))$$

for all $u, v \in \mathcal{X}$ with $\sigma(\mathcal{J}u, \mathcal{J}v) > 0$. Thus $\mathcal{J} \in \Upsilon(\mathcal{X}, \alpha, \mathfrak{F})$. Therefore, all the hypotheses of Theorem 2.5 are satisfied and we conclude that there is a fixed point $u^* \in \mathcal{X}$ of the operator $\mathcal{J}$. It is well known that in this case u^* is also a solution of the integral equation (4.1). □

5. Application to Fractional Differential Equations

This section is devoted to the existence of solutions of a nonlinear fractional differential equation as an application of Theorem 2.5. It is inspired by the paper [4].

Recall that the Caputo derivative of fractional order β is defined as

$${}^cD^\beta(g(t)) = \frac{1}{\Gamma(n-\beta)}\int_0^t (t-s)^{n-\beta-1} g^{(n)}(s)\,ds \quad (n-1 < \beta < n,\ n = [\beta]+1),$$

where $g \in C^n([0,\infty))$, $[\beta]$ denotes the integer part of the positive real number β and Γ is the gamma function.

We consider the nonlinear fractional differential equation of the form

$$ {}^cD^\beta(x(t)) = f(t,x(t)) \quad (0 < t < 1,\ \ 1 < \beta \le 2) \tag{5.1}$$

with the integral boundary conditions

$$x(0) = 0, \quad x(1) = \int_0^\eta x(s)\,ds \quad (0 < \eta < 1), \tag{5.2}$$

where $I = [0,1]$, $x \in C(I,\mathbb{R})$, and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let the corresponding operator $\mathcal{J}\colon \mathcal{X} \to \mathcal{X}$ be defined by

(5.3)
$$\begin{aligned}\mathcal{J}u(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,u(s))\,ds - \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,u(s))\,ds \\ + \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \left(\int_0^s (s-z)^{\beta-1} f(z,u(z))\,dz\right) ds,\end{aligned}$$

for $u \in \mathcal{X}$, $t \in I$. This time we will use the metric-like distance function σ_2 on $\mathcal{X} = C(I,\mathbb{R})$ given by

$$\sigma_2(u,v) = \max_{t\in I}[|u(t)|^2 + |v(t)|^2].$$

Theorem 5.1. *Under the assumptions (F1)-(F5) of Theorem 4.1 and*

(F6') *there exists $\tau > 0$ such that for $u, v \in \mathcal{X}$ with $\xi(u,v) \ge 0$, and $t \in I$ we have*

$$|f(t,u(t))|^2 + |f(t,v(t))|^2 \le B\,\frac{\Delta_2(u,v)(t)}{\left(1+\tau\sqrt{\max_{t\in I}\Delta_2(u,v)(t)}\,\right)^2},$$

where $B = \frac{(2\beta-1)\Gamma(\beta)\Gamma(\beta+1)}{2(5\beta+2)}$ *and*

$$\Delta_2(u,v)(t) = \max\left\{\begin{array}{c} |u(t)|^2 + |v(t)|^2, |u(t)|^2 + |\mathcal{J}u(t)|^2, |v(t)|^2 + |\mathcal{J}v(t)|^2, \\ \dfrac{|u(t)|^2 + |\mathcal{J}v(t)|^2 + |v(t)|^2 + |\mathcal{J}u(t)|^2}{4}, \\ \dfrac{(|u(t)|^2 + |\mathcal{J}u(t)|^2)(|v(t)|^2 + |\mathcal{J}v(t)|^2)}{1+\max_{t\in I}[|u(t)|^2 + |v(t)|^2]}, \\ \dfrac{(|u(t)|^2 + |\mathcal{J}u(t)|^2)(|v(t)|^2 + |\mathcal{J}v(t)|^2)}{1+\max_{t\in I}[|\mathcal{J}u(t)|^2 + |\mathcal{J}v(t)|^2]} \end{array}\right\},$$

the problem (5.1)–(5.2) has at least one solution $u^* \in \mathcal{X}$.

Proof. Here we have to check the contraction condition (2.13) for $u, v \in \mathcal{X}$ (see Remark 2.7.(III)).

For each $t \in I$, by the definition (5.3) of operator $\mathcal{J}$, we have

$$\begin{aligned}
|\mathcal{J}u(t)|^2 + |\mathcal{J}v(t)|^2 = \bigg| &\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, u(s))\, ds \\
&- \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f(s, u(s))\, ds \\
&+ \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-z)^{\beta-1} f(z, u(z))\, dz \right) ds \bigg|^2 \\
+ \bigg| &\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, v(s))\, ds \\
&+ \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f(s, v(s))\, ds \\
&- \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-z)^{\beta-1} f(z, v(z))\, dz \right) ds \bigg|^2
\end{aligned}$$

$$\begin{aligned}
&\le \frac{2}{\Gamma^2(\beta)} \left\{ \left[\int_0^t (t-s)^{\beta-1} f(s, u(s))\, ds \right]^2 + \left[\int_0^t (t-s)^{\beta-1} f(s, v(s))\, ds \right]^2 \right\} \\
&\quad + \frac{8t^2}{(2-\eta^2)^2\Gamma^2(\beta)} \left\{ \left[\int_0^1 (1-s)^{\beta-1} f(s, u(s))\, ds \right]^2 \right. \\
&\qquad \left. + \left[\int_0^1 (1-s)^{\beta-1} f(s, v(s))\, ds \right]^2 \right\} \\
&\quad + \frac{8t^2}{(2-\eta^2)^2\Gamma^2(\beta)} \left\{ \left[\int_0^\eta \left(\int_0^s (s-z)^{\beta-1} f(z, u(z))\, dz \right) ds \right]^2 \right. \\
&\qquad \left. + \left[\int_0^\eta \left(\int_0^s (s-z)^{\beta-1} f(z, v(z))\, dz \right) ds \right]^2 \right\}.
\end{aligned}$$

Applying now Cauchy-Schwartz inequality, we get

$$\begin{aligned}
&|\mathcal{J}u(t)|^2 + |\mathcal{J}v(t)|^2 \\
&\quad \le \frac{2}{\Gamma^2(\beta)} \int_0^t (t-s)^{2\beta-2} ds \int_0^t [f^2(s, u(s)) + f^2(s, v(s))]\, ds \\
&\qquad + \frac{8t^2}{(2-\eta^2)^2\Gamma^2(\beta)} \int_0^1 (1-s)^{2\beta-2} ds \int_0^1 [f^2(s, u(s)) + f^2(s, v(s))]\, ds \\
&\qquad + \frac{8t^2}{(2-\eta^2)^2\Gamma^2(\beta)} \int_0^\eta \int_0^s (s-z)^{2\beta-2} dz\, ds \\
&\qquad\quad \times \int_0^\eta \int_0^s [f^2(z, u(z)) + f^2(z, v(z))]\, dz\, ds.
\end{aligned}$$

Now, using the assumption (F6'), after routine calculations, we obtain

$$|\mathcal{J}u(t)|^2 + |\mathcal{J}v(t)|^2 \le \frac{\Delta_{\mathcal{J}}(u,v)}{\left(1+\tau\sqrt{\Delta_{\mathcal{J}}(u,v)}\right)^2}.$$

This implies that

$$\sigma_2(\mathcal{J}u,\mathcal{J}v) = \max_{t\in I}(|(\mathcal{J}u)(t)|^2 + |(\mathcal{J}v)(t)|^2) \le \frac{\Delta_{\mathcal{J}}(u,v)}{\left(1+\tau\sqrt{\Delta_{\mathcal{J}}(u,v)}\right)^2},$$

where $\Delta_{\mathcal{J}}(u,v)$ is given in (2.2).

Now, by considering $F \in \mathfrak{F}$ given by $F(t) = \frac{-1}{\sqrt{t}}$, $t > 0$, we have the condition

$$u,v \in \mathcal{X} \text{ with } \alpha(u,v) \ge 1 \Rightarrow \tau + F(\sigma_2(\mathcal{J}u,\mathcal{J}v)) \le F(\Delta_{\mathcal{J}}(u,v))$$

for all $u,v \in \mathcal{X}$ with $\sigma_2(\mathcal{J}u,\mathcal{J}v) > 0$. Thus $\mathcal{J} \in \Upsilon(\mathcal{X},\alpha,\mathfrak{F})$. Therefore, all the hypotheses of Theorem 2.5 are satisfied and we conclude that there is a fixed point $u^* \in \mathcal{X}$ of the operator $\mathcal{J}$. It is well known (see, e.g., [4, Theorem 3.1]) that in this case u^* is also a solution of the integral equation (5.3) and the fractional differential equation (5.1) with the condition (5.2). □

Example 5.2. Consider the following nonlinear fractional differential equation

$$(5.4)\qquad {}^cD^{\frac{3}{2}}u(t) = \frac{1}{(t+3)^2}\frac{|u(t)|}{1+|u(t)|}, \quad t \in [0,1]$$

with the three-point integral boundary value condition

$$(5.5)\qquad u(0) = 0,\ u(1) = \int_0^{3/4} u(s)\,ds.$$

Here $\beta = \frac{3}{2}$, $\eta = \frac{3}{4} < 1$ and $f(t,u(t)) = \frac{1}{(t+3)^2}\frac{|u(t)|}{1+|u(t)|}$. Therefore, the considered system (5.4)–(5.5) is an example of the system (5.1)–(5.2). Further, $|f(t,u)| \le \frac{1}{9}$, $B = \frac{33}{266}$ and therefore, taking $\tau = \frac{9}{2}\sqrt{\frac{33}{133}} - \frac{1}{\sqrt{\Delta_1}} > 0$, we can apply Theorem 5.1. Hence, there exists a solution $u^* \in \mathcal{X}$ of the equation (5.4) with the conditions (5.5).

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