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Regularity of a Particular Subsemigroup of the Semigroup of Transformations Preserving an Equivalence

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Abstract. In this paper, we use the notion of characters of transformations provided in [8] by Purisang and Rakbud to define a notion of weak regularity of transformations on an arbitrarily fixed set X. The regularity of a semigroup of weakly regular transformations on a set X is also investigated.

1. Introduction

For any semigroup $\mathcal{S}$, we call an element a of $\mathcal{S}$ a *regular element* of $\mathcal{S}$ if there exists an element b of $\mathcal{S}$ such that $aba = a$. It is well-known that an element a of a semigroup $\mathcal{S}$ is regular if and only if there is $c \in \mathcal{S}$ such that $aca = a$ and $cac = c$. We denote the set of all regular elements of a semigroup $\mathcal{S}$ by $R(\mathcal{S})$. A semigroup $\mathcal{S}$ is said to be *regular* if every element of $\mathcal{S}$ is regular, that is, if $R(\mathcal{S}) = \mathcal{S}$.

The notion of regularity plays an important role in semigroup theory. Over the years, there have been many people studying the regularity of subsemigroups of the regular semigroup $T(X)$ of functions on a nonempty set X under the composition, called a *full transformation semigroup*. The following are two simple subsemigroups of $T(X)$ which have widely been investigated or used as bases for building up some other subsemigroups of $T(X)$:

$$T(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}$$

and

$$\overline{T}(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\},$$

where Y is a fixed nonempty subset of X (see [2, 3, 7, 9, 10, 11] for some references). Here are some results on the regularity of $T(X,Y)$ and $\overline{T}(X,Y)$ provided in [7] by

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Nenthein, Youngkhong and Kemprasit, in [3] by Honyam and Sanwong, and in [10] by Sanwong and Sommanee.

Theorem 1.1.([7, Theorem 2.1]) *Let X be a nonempty set, and let Y be a nonempty subset of X. Then for any $\alpha \in T(X,Y)$, the following statements are equivalent:*

(1) $\alpha \in R(T(X,Y))$;

(2) $X\alpha = Y\alpha$;

(3) $Y \cap (x\alpha)\alpha^{-1} \neq \emptyset$ *for all* $x \in X$;

(4) $Y \cap x\alpha^{-1} \neq \emptyset$ *for all* $x \in X\alpha$.

By Thorem 1.1, the following corollary was deduced.

Corollary 1.2.([7, Corollary 2.2]) *Let X be a nonempty set, and let Y be a nonempty subset of X. Then $T(X,Y)$ is regular if and only if $Y = X$ or $|Y| = 1$.*

Remark 1.3. In the proof of the implication (4) $\Rightarrow$ (1) in Theorem 1.1, the authors defined a function β in $T(X,Y)$ under the assumption that $Y \cap x\alpha^{-1} \neq \emptyset$ for all $x \in X\alpha$ to make α regular by $x\beta = y_x$ if $x \in X\alpha$ and $x\beta = c$ otherwise, where c is a fixed element of Y and for each $x \in X\alpha$, y_x is a fixed element of $Y \cap x\alpha^{-1}$.

Theorem 1.4.([7, Theorem 2.3]) *Let X be a nonempty set, and let Y be a nonempty subset of X. Then for any $\alpha \in \overline{T}(X,Y)$, the following statements are equivalent:*

(1) $\alpha \in R\left(\overline{T}(X,Y)\right)$;

(2) $X\alpha \cap Y = Y\alpha$;

(3) $Y \cap (x\alpha)\alpha^{-1} \neq \emptyset$ *for all* $x \in Y\alpha^{-1}$;

(4) $Y \cap x\alpha^{-1} \neq \emptyset$ *for all* $x \in X\alpha \cap Y$.

By Thorem 1.4, the following corollary was obtained.

Corollary 1.5.([7, Corollary 2.4]) *Let X be a nonempty set, and let Y be a nonempty subset of X. Then $\overline{T}(X,Y)$ is regular if and only if $Y = X$ or $|Y| = 1$.*

Theorem 1.6.([10, Theorem 2.4]) *Let X be a nonempty set, and let Y be a nonempty subset of X. Then $R(T(X,Y))$ is the largest regular subsemigroup of $T(X,Y)$.*

Theorem 1.7.([3, Lemma 1]) *Let X be a nonempty set, and let Y be a nonempty subset of X. Then $R\left(\overline{T}(X,Y)\right)$ is a subsemigroup of $\overline{T}(X,Y)$ if and only if $Y = X$ or $|Y| = 1$. In this trivial situation, $R\left(\overline{T}(X,Y)\right) = \overline{T}(X,Y)$ is regular.*

In this paper, by a *partition* of a nonempty set X, we mean a family $\mathscr{F} = \{Y_i : i \in I\}$ of nonempty subsets of X such that $X = \bigcup_{i\in I} Y_i$ and $Y_i \neq Y_j$ for all $i, j \in I$ with $i \neq j$. Each of the two partitions $\{X\}$ and $\{\{x\} : x \in X\}$ is called a *trivial partition* of X.

Throughout the remainder of this paper, let X be a nonempty set, and let $\mathscr{F} = \{Y_i : i \in I\}$ be a partition of X, which are arbitrarily fixed. Let

$$T_{\mathscr{F}}(X) = \{\alpha \in T(X) : \forall i \in I \exists j \in I, Y_i\alpha \subseteq Y_j\}.$$

It is clear that $T_{\mathscr{F}}(X)$ is a subsemigroup of the full transformation semigroup $T(X)$. Note that $T_{\mathscr{F}}(X)$ is exactly the semigroup of transformations preserving the equivalence $\mathcal{E}$ induced by the partition $\mathscr{F}$ (see [4] for more detials). There have been several works on the semigroup of transformations preserving an equivalence (see [1, 5, 6] for some references). For each $\alpha \in T_{\mathscr{F}}(X)$, let $\chi^{(\alpha)} : I \to I$ be defined by $i\chi^{(\alpha)} = j$ if and only if $Y_i\alpha \subseteq Y_j$. By the defintion of a partition, we see that $\chi^{(\alpha)}$ is well-defined, that is, $\chi^{(\alpha)} \in T(I)$. For each $\alpha \in T_{\mathscr{F}}(X)$, we call the function $\chi^{(\alpha)}$ the *character* of α with respect to $\mathscr{F}$. In addition to the set X and the partition $\mathscr{F}$ of X, let J be an arbitrarily fixed nonempty subset of I. Let

$$T_{\mathscr{F}}^{(J)}(X) = \left\{\alpha \in T(X) : \chi^{(\alpha)} \in T(I,J)\right\}.$$

It is clear that

$$T_{\mathscr{F}}^{(J)}(X) = \{\alpha \in T(X) : \forall i \in I \exists j \in J, Y_i\alpha \subseteq Y_j\}.$$

The set $T_{\mathscr{F}}^{(J)}(X)$, which is indeed a subsemigroup of $T_{\mathscr{F}}(X)$, as well as the notion of character were first introduced in [8] by Purisang and Rakbud. In that paper, the authors studied the regularity of the semigroup $T_{\mathscr{F}}^{(J)}(X)$ and some other semigroups defined via the notion of character. We summarize some of their results as follows.

Proposition 1.8.([8, Proposition 2.2]) *Let $Y = \bigcup_{j\in J} Y_j$. Then the following statements hold:*

(1) $T_{\mathscr{F}}^{(J)}(X) = T(X,Y)$ *if and only if* $|J| = 1$ *or* $\mathscr{F} = \{\{x\} : x \in X\}$.

(2) $T_{\mathscr{F}}^{(J)}(X) = T(X)$ *if and only if* $J = I$ *or* $\mathscr{F}$ *is trivial.*

Lemma 1.9.([8, Lemma 2.3]) *For every $\alpha, \beta \in T_{\mathscr{F}}^{(J)}(X)$, $\chi^{(\alpha\beta)} = \chi^{(\alpha)}\chi^{(\beta)}$.*

By using the notion of character, the authors defined two congruence relations χ and $\widetilde{\chi}$ on $T_{\mathscr{F}}^{(J)}(X)$ as follows:

$$(\alpha,\beta) \in \chi \Leftrightarrow \chi^{(\alpha)} = \chi^{(\beta)},$$

$$(\alpha,\beta) \in \widetilde{\chi} \Leftrightarrow \chi^{(\alpha)}|_J = \chi^{(\beta)}|_J.$$

And then they studied the regularity of the quotient semigroups $T_{\mathscr{F}}^{(J)}(X)/\chi$ and $T_{\mathscr{F}}^{(J)}(X)/\widetilde{\chi}$. The following are what they obtained.

Theorem 1.10.([8, Theorem 2.4]) *For each $\alpha \in T_{\mathscr{F}}^{(J)}(X)$, let $[\alpha]$ and $\widetilde{[\alpha]}$ be the equivalence classes of α under the equivalence relations χ and $\widetilde{\chi}$ respectively. Then the following statements hold:*

(1) $T_{\mathscr{F}}^{(J)}(X)/\chi \cong T(I,J)$ *by the isomorphism* $[\alpha] \mapsto \chi^{(\alpha)}$.

(2) $T_{\mathscr{F}}^{(J)}(X)/\widetilde{\chi} \cong T(J)$ *by the isomorphism* $\widetilde{[\alpha]} \mapsto \chi^{(\alpha)}|_J$.

By Corollary 1.2 and Theorem 1.10, the following corollary was obtained.

Corollary 1.11.([8, Corollary 2.5]) *The following statements hold:*

(1) *The three statements* (a), (b) *and* (c) *below are all equivalent:*

- (a) *the quotient semigroup* $T_{\mathscr{F}}^{(J)}(X)/\chi$ *is regular;*
- (b) *the semigroup* $T(I,J)$ *is regular;*
- (c) $J = I$ *or* $|J| = 1$.

(2) *The quotient semigroup* $T_{\mathscr{F}}(X)/\chi$, *which is exactly* $T_{\mathscr{F}}^{(I)}(X)/\chi$, *is regular.*

(3) *The quotient semigroup* $T_{\mathscr{F}}^{(J)}(X)/\widetilde{\chi}$ *is regular.*

In [8], the regularity of the semigroup $T_{\mathscr{F}}^{(J)}(X)$ was obtained as follows.

Theorem 1.12.([8, Theorem 2.6]) *The semigroup* $T_{\mathscr{F}}^{(J)}(X)$ *is regular if and only if* $\left|T_{\mathscr{F}}^{(J)}(X)\right| = 1$ *or* $T_{\mathscr{F}}^{(J)}(X) = T(X)$.

Note that, from Theorem 1.12, we immediately have that $T_{\mathscr{F}}(X)$ is regular if and only if $\mathscr{F}$ is trivial. This can also be deduced from Proposition 2.4 of Huisheng [5].

It is clear that for each $\alpha \in T_{\mathscr{F}}(X)$, the equivalence class $[\alpha]$ of α under the equivalence relation χ is a subsemigroup of $T_{\mathscr{F}}(X)$ if and only if $\chi^{(\alpha)}$ is an idempotent element of the full transformation semigroup $T(I)$. The regularity of the semigroup $[\alpha]$, in the case where α is an idempotent element of $T(I)$, was also studied in [8]. In [8] as well, some other subsemigroups of $T_{\mathscr{F}}(X)$ were defined by using the notion of character as follows: Let $I_{\mathscr{F}}(X)$, $S_{\mathscr{F}}(X)$ and $B_{\mathscr{F}}(X)$ be the sets of all elements of $T_{\mathscr{F}}(X)$ whose characters are injective, surjective and bijective respectively. The regularity of each of these three semigroups was also studied.

Observe that the semigroups $T_{\mathscr{F}}^{(J)}(X)$, $[\alpha]$ when $\chi^{(\alpha)}$ is idempotent, $I_{\mathscr{F}}(X)$, $S_{\mathscr{F}}(X)$ and $B_{\mathscr{F}}(X)$ can simultaneously be generalized by making use of the notion of character as follows: For every subsemigroup $\mathcal{S}$ of $T(I)$, let

$$T_{\mathscr{F}}^{(\mathcal{S})}(X) = \left\{\alpha \in T_{\mathscr{F}}(X) : \chi^{(\alpha)} \in \mathcal{S}\right\}.$$

By Lemma 1.9, we see that $T_{\mathscr{F}}^{(\mathcal{S})}(X)$ is a subsemigroup of $T_{\mathscr{F}}(X)$. And, furthermore, Lemma 1.9 also implies that for every subsemigroup $\mathcal{H}$ of $T_{\mathscr{F}}(X)$, $\mathcal{H}$ is necessarily of the form $T_{\mathscr{F}}^{(\mathcal{S})}(X)$ for some subsemigroup $\mathcal{S}$ of $T(I)$, in fact, $\mathcal{S} = \{\chi^{(\alpha)} : \alpha \in \mathcal{H}\}$. We state this pleasant result in the following theorem.

Theorem 1.13. *For every $\mathcal{H} \subseteq T_{\mathscr{F}}(X)$, $\mathcal{H}$ is a subsemigroup of $T_{\mathscr{F}}(X)$ if and only if there is a subsemigroup $\mathcal{S}$ of $T(I)$ such that $\mathcal{H} = T_{\mathscr{F}}^{(\mathcal{S})}(X)$. In this situation, $\mathcal{S} = \{\chi^{(\alpha)} : \alpha \in \mathcal{H}\}$.*

Let $\mathcal{S}$ be a subsemigroup of $T(I)$. Then by considering the congruence relation χ on $T_{\mathscr{F}}(X)$ restricted to $T_{\mathscr{F}}^{(\mathcal{S})}(X)$, we have the quotient semigroup $T_{\mathscr{F}}^{(\mathcal{S})}(X)/\chi$. It is clear that $T_{\mathscr{F}}^{(\mathcal{S})}(X)/\chi = \left\{[\alpha] : \alpha \in T_{\mathscr{F}}^{(\mathcal{S})}(X)\right\}$, and that $T_{\mathscr{F}}^{(\mathcal{S})}(X)/\chi$ is a subsemigroup of $T_{\mathscr{F}}(X)/\chi$. Analogously to Theorem 1.10(1), the following result is obtained.

Theorem 1.14. *$T_{\mathscr{F}}^{(\mathcal{S})}(X)/\chi \cong \mathcal{S}$ by the isomorpism $[\alpha] \mapsto \chi^{(\alpha)}$.*

Immediately from Theorem 1.14, we have the following corollary.

Corollary 1.15. *The quotient semigroup $T_{\mathscr{F}}^{(\mathcal{S})}(X)/\chi$ is regular if and only if the semigroup $\mathcal{S}$ is regular.*

We note here that the regularity of $\mathcal{S}$ may not imply $T_{\mathscr{F}}^{(\mathcal{S})}(X)$ to be regular. For example, when $\mathcal{S}$ is exactly the regular semigroup $T(I)$, we have that $T_{\mathscr{F}}^{(\mathcal{S})}(X) = T_{\mathscr{F}}(X)$ which is regular only when $\mathscr{F}$ is trivial. By making use of the notion of character, we can define a subset of $T_{\mathscr{F}}^{(\mathcal{S})}(X)$ as follows: Let

$$R_w\left(T_{\mathscr{F}}^{(\mathcal{S})}(X)\right) = \left\{\alpha \in T_{\mathscr{F}}^{(\mathcal{S})}(X) : \chi^{(\alpha)} \in R(\mathcal{S})\right\}.$$

By Lemma 1.9, we have that $R\left(T_{\mathscr{F}}^{(\mathcal{S})}(X)\right) \subseteq R_w\left(T_{\mathscr{F}}^{(\mathcal{S})}(X)\right)$. And obviously, if $T_{\mathscr{F}}^{(\mathcal{S})}(X)$ is regular, then $R\left(T_{\mathscr{F}}^{(\mathcal{S})}(X)\right) = T_{\mathscr{F}}^{(\mathcal{S})}(X) = R_w\left(T_{\mathscr{F}}^{(\mathcal{S})}(X)\right)$. It is easy to see that the set $R_w\left(T_{\mathscr{F}}^{(\mathcal{S})}(X)\right)$ is a subsemigroup of $T_{\mathscr{F}}^{(\mathcal{S})}(X)$ if and only if $R(\mathcal{S})$ is a subsemigroup of $\mathcal{S}$. And in this situation, we have $R_w\left(T_{\mathscr{F}}^{(\mathcal{S})}(X)\right) = T_{\mathscr{F}}^{(R(\mathcal{S}))}(X)$.

From the inclusion $R\left(T_{\mathscr{F}}^{(\mathcal{S})}(X)\right) \subseteq R_w\left(T_{\mathscr{F}}^{(\mathcal{S})}(X)\right)$ and the defintion of the set $R_w\left(T_{\mathscr{F}}^{(\mathcal{S})}(X)\right)$, it makes perfect sense to call every $\alpha \in R_w\left(T_{\mathscr{F}}^{(\mathcal{S})}(X)\right)$ a *weakly regular transformation with respect to* $\mathcal{S}$, or simply an $\mathcal{S}$-*weakly-regular transformation.* By Theorem 1.14, we have for any $\alpha \in T_{\mathscr{F}}^{(\mathcal{S})}(X)$ that α is an $\mathcal{S}$-weakly-regular transformation if and only if the equivalence class $[\alpha]$ of α under the congruence relation χ is a regular element of the quotient semigroup $T_{\mathscr{F}}^{(\mathcal{S})}(X)/\chi$. This yileds, in the case where $R(\mathcal{S})$ is a subsemigroup of $\mathcal{S}$, that $R_w\left(T_{\mathscr{F}}^{(\mathcal{S})}(X)\right)/\chi = R\left(T_{\mathscr{F}}^{(\mathcal{S})}(X)/\chi\right)$. Note that for any semigroup $\mathcal{Z}$ and a subsemigroup $\mathcal{A}$ of $\mathcal{Z}$, if $R(\mathcal{Z}) \subseteq \mathcal{A}$, then $R(\mathcal{Z}) = R(\mathcal{A})$. From this elementary fact, we immediately obtain that if $R(\mathcal{S})$ is a subsemigroup of $\mathcal{S}$, then $R\left(R_w\left(T_{\mathscr{F}}^{(\mathcal{S})}(X)\right)\right) = R\left(T_{\mathscr{F}}^{(\mathcal{S})}(X)\right)$.

The aim of this paper is to investigate the regularity of $R_w\left(T_{\mathscr{F}}^{(\mathcal{S})}(X)\right)$ for a certain subsemigroup $\mathcal{S}$ of $T(I)$ with $R(\mathcal{S})$ a subsemigroup of $\mathcal{S}$.

2. Regularity of a Semigroup of Weakly Regular Transformations

By virtue of Theorem 1.6, we have that the set $R_w\left(T_{\mathscr{F}}^{(J)}(X)\right)$ is a subsemigroup of $T_{\mathscr{F}}^{(J)}(X)$. This section is devoted to studying the regualrity of this semigroup. From a result of Huisheng [5], the regularity of elements of the semigroup $T_{\mathscr{F}}(X)$ can immediately be deduced as follows.

Proposition 2.1. *An element α of the semigroup $T_{\mathscr{F}}(X)$ is regualr if and only if for every $i \in I$, there exists $j \in I$ such that $Y_i \cap X\alpha \subseteq Y_j\alpha$.*

The result in Proposition 2.1 can straightforwardly be generalized to the semigroup $T_{\mathscr{F}}^{(J)}(X)$ as follows.

Theorem 2.2. *For every $\alpha \in T_{\mathscr{F}}^{(J)}(X)$, α is regualr if and only if for every $i \in I$, there exists $j \in J$ such that $Y_i \cap X\alpha \subseteq Y_j\alpha$.*

Proof. Let $\alpha \in T_{\mathscr{F}}^{(J)}(X)$. We are now going to prove the necessity. Suppose that the condition holds. Let $E = \{i \in I : Y_i \cap X\alpha \neq \emptyset\}$. It is clear that $E \neq \emptyset$. Let $i \in E$ be arbitrarily fixed. By the assumption, we fix $j_i \in J$ such that $Y_i \cap X\alpha \subseteq Y_{j_i}\alpha$. For each $x \in Y_i \cap X\alpha$, by the inclusion $Y_i \cap X\alpha \subseteq Y_{j_i}\alpha$, we fix $z_x^{(i)} \in Y_{j_i}$ such that $z_x^{(i)}\alpha = x$. Also, we fix $c_i \in Y_{j_i}$, and then define a function $\beta_i : Y_i \to X$ by $x\beta_i = z_x^{(i)}$ for all $x \in Y_i \cap X\alpha$ and $x\beta_i = c_i$ otherwise. It is clear that $Y_i\beta_i \subseteq Y_{j_i}$. Next, let $\beta : X \to X$ be defined by $\beta|_{Y_i} = \beta_i$ for all $i \in E$ and $x\beta = a$ for all $x \in \bigcup_{i \in I\setminus E} Y_i$, where a is a fixed element in $\bigcup_{j\in J} Y_j$. Then $\beta \in T_{\mathscr{F}}^{(J)}(X)$ and $\alpha\beta\alpha = \alpha$. To prove the sufficiency, suppose that α is regular. Then there is $\beta \in T_{\mathscr{F}}^{(J)}(X)$ such that $\alpha\beta\alpha = \alpha$. Let $i \in I$, and let $j = i\chi^{(\beta)}$. Then $j \in J$. We will show that $Y_i \cap X\alpha \subseteq Y_j\alpha$. To see this, let $x \in Y_i \cap X\alpha$. Then $x \in Y_i$ and there is $z \in X$ such that $z\alpha = x$. Since $x \in Y_i$ and $j = i\chi^{(\beta)}$, we have $x\beta \in Y_j$. Thus $x = z\alpha = z\alpha\beta\alpha = x\beta\alpha \in Y_j\alpha$. Therefore, we obtain the inclusion $Y_i \cap X\alpha \subseteq Y_j\alpha$ as desired. □

Previously in Section 1, we have seen that $R\left(T_{\mathscr{F}}^{(J)}(X)\right) \subseteq R_w\left(T_{\mathscr{F}}^{(J)}(X)\right)$, and that the regulairty of the semigroup $T_{\mathscr{F}}^{(J)}(X)$ suffices to obtain that $R\left(T_{\mathscr{F}}^{(J)}(X)\right) = R_w\left(T_{\mathscr{F}}^{(J)}(X)\right)$. Next, we give a necessary and sufficient condition for obtaining the equality $R\left(T_{\mathscr{F}}^{(J)}(X)\right) = R_w\left(T_{\mathscr{F}}^{(J)}(X)\right)$.

Theorem 2.3. $R\left(T_{\mathscr{F}}^{(J)}(X)\right) = R_w\left(T_{\mathscr{F}}^{(J)}(X)\right)$ *if and only if $|Y_j| = 1$ for all $j \in J$.*

Proof. Suppose that $Y_j = \{z_j\}$ for all $j \in J$. We want to show that $R_w\left(T_{\mathscr{F}}^{(J)}(X)\right) \subseteq R\left(T_{\mathscr{F}}^{(J)}(X)\right)$. To see this, let $\alpha \in R_w\left(T_{\mathscr{F}}^{(J)}(X)\right)$. Then the character $\chi^{(\alpha)}$ of α is a regular element of $T(I, J)$. Thus, by Theorem 1.1, we have that $I\chi^{(\alpha)} = J\chi^{(\alpha)}$. Let $Y = \bigcup_{j\in J} Y_j$. Then $Y\alpha = \left\{z_{j\chi^{(\alpha)}} : j \in J\right\}$. Since $I\chi^{(\alpha)} = J\chi^{(\alpha)}$, we have for each $i \in I \setminus J$ that there is $j_i \in J$ such that $i\chi^{(\alpha)} = j_i\chi^{(\alpha)}$. This yields that $Y_i\alpha = \left\{z_{j_i\chi^{(\alpha)}}\right\}$ for all $i \in I \setminus J$. Therefore, $X\alpha = Y\alpha$, which implies by Theorem 1.1 agian that α is a regular element of the semigroup $T(X,Y)$, and that $Y \cap x\alpha^{-1} \neq \emptyset$ for all $x \in X\alpha$. As mentioned in Remark 1.3, we can define a function $\beta : X \to X$, under the condition that $Y \cap x\alpha^{-1} \neq \emptyset$ for all $x \in X\alpha$, which makes α regular in the semigroup $T(X,Y)$ as follows: Fix an element $c \in Y$, and for each $x \in X\alpha$, fix an element $y_x \in Y \cap x\alpha^{-1}$. And then define $\beta : X \to X$ by $x\beta = y_x$ for all $x \in X\alpha$ and $x\beta = c$ otherwise. In our situation here, we can easily see from the way of defining the function β that $\beta \in T_{\mathscr{F}}^{(J)}(X)$. Hence $\alpha \in R\left(T_{\mathscr{F}}^{(J)}(X)\right)$; and thus $R\left(T_{\mathscr{F}}^{(J)}(X)\right) = R_w\left(T_{\mathscr{F}}^{(J)}(X)\right)$. Conversely, suppose that there is $j \in J$ such that $|Y_j| \geq 2$. We want to show that $R_w\left(T_{\mathscr{F}}^{(J)}(X)\right) \nsubseteq R\left(T_{\mathscr{F}}^{(J)}(X)\right)$. To see this, we fix two distinct elements a and b of Y_j. And define $\alpha : X \to X$ by $x\alpha = a$ for all $x \in Y_j$ and $x\alpha = b$ otherwise. Since $Y_j\alpha = \{a\}$ and $Y_i\alpha = \{b\}$ for all $i \in I \setminus \{j\}$, it follows that $Y_j \cap X\alpha = \{a,b\} \nsubseteq Y_i\alpha$ for all $i \in J$. From this, we have by Theorem 2.2 that $\alpha \notin R\left(T_{\mathscr{F}}^{(J)}(X)\right)$. And since $\chi^{(\alpha)}$ is a constant function, we get that $\alpha \in R_w\left(T_{\mathscr{F}}^{(J)}(X)\right)$. Thus $R_w\left(T_{\mathscr{F}}^{(J)}(X)\right) \nsubseteq R\left(T_{\mathscr{F}}^{(J)}(X)\right)$. □

Since $R_w\left(T_{\mathscr{F}}^{(J)}(X)\right)$ is a subsemigroup of $T_{\mathscr{F}}^{(J)}(X)$, we have $R_w\left(T_{\mathscr{F}}^{(J)}(X)\right)/\chi$ is a subsemigroup of $T_{\mathscr{F}}^{(J)}(X)/\chi$. By Theorem 1.6 and Theorem 1.10(1), we obtain that $R\left(T_{\mathscr{F}}^{(J)}(X)/\chi\right)$ is a regular subsemigroup of $T_{\mathscr{F}}^{(J)}(X)/\chi$. Hecne the quotient semigroup $R_w\left(T_{\mathscr{F}}^{(J)}(X)\right)/\chi = R\left(T_{\mathscr{F}}^{(J)}(X)/\chi\right)$ is regular. Next, we investigate the regularity of the semigroup $R_w\left(T_{\mathscr{F}}^{(J)}(X)\right)$ itself. By Theorem 2.3 and the fact that $R\left(R_w\left(T_{\mathscr{F}}^{(J)}(X)\right)\right) = R\left(T_{\mathscr{F}}^{(J)}(X)\right)$, the following result on the regularity of $R_w\left(T_{\mathscr{F}}^{(J)}(X)\right)$ is immediately obtained.

Corollary 2.4. *The semigroup $R_w\left(T_{\mathscr{F}}^{(J)}(X)\right)$ is regular if and only if $|Y_j| = 1$ for all $j \in J$.*

By the definition of $T_{\mathscr{F}}^{(J)}(X) := T_{\mathscr{F}}^{(T(I,J))}(X)$, defined relatively to the semigroup $T(I,J)$, and the regularity of $R(T(I,J))$, we expect to have that $R\left(T_{\mathscr{F}}^{(J)}(X)\right)$ is a regular subsemigroup of $T_{\mathscr{F}}^{(J)}(X)$. But we find that, in general, the set $R\left(T_{\mathscr{F}}^{(J)}(X)\right)$ is not a subsemigroup of $T_{\mathscr{F}}^{(J)}(X)$. This is affirmed by the following proposition.

Proposition 2.5. *If $|J| > 1$ and there is $j \in J$ such that $|Y_j| > 1$, then the set $R\left(T_{\mathscr{F}}^{(J)}(X)\right)$ is not a subsemigroup of $T_{\mathscr{F}}^{(J)}(X)$.*

Proof. Suppose that $|J| > 1$ and there is $j \in J$ such that $|Y_j| > 1$. Fix two distinct elements a and b in Y_j, and define $\alpha : X \to X$ by $x\alpha = a$ for all $x \in Y_j$ and $x\alpha = b$

otherwise. Clearly, $\alpha \in T_{\mathscr{F}}^{(J)}(X)$. And, as explained in the proof of Theorem 2.3, we have that α is not regular. Next, let $k \in J$ be different from j, and fix an element $c \in Y_k$. Let $\beta : X \to X$ be defined by $x\beta = c$ for all $x \in Y_j$ and $x\beta = b$ otherwise. And let $\gamma : X \to X$ be defined by $b\gamma = b$ and $x\gamma = a$ for all $x \in X \setminus \{b\}$. Then β and α belong to $T_{\mathscr{F}}^{(J)}(X)$. Since $Y_j \cap X\beta = \{b\} = Y_k\beta$, $Y_k \cap X\beta = \{c\} = Y_j\beta$ and $Y_i \cap X\beta = \emptyset$ for all $i \in I \setminus \{j,k\}$, it follows from Theorem 2.2 that β is regular. Similarly, since $Y_j \cap X\gamma = \{a,b\} = Y_j\gamma$ and $Y_i \cap X\gamma = \emptyset$ for all $i \in I \setminus \{j\}$, we have that γ is regualr as well. Form the defintions of β and γ, it is easy to see that $x\beta\gamma = a$ for all $x \in Y_j$ and $x\beta\gamma = b$ otherwise. Hence $\beta\gamma = \alpha$. This yileds that the set $R\left(T_{\mathscr{F}}^{(J)}(X)\right)$ is not a subsemigroup of $T_{\mathscr{F}}^{(J)}(X)$. □

According to Proposition 2.5, the set $R\left(T_{\mathscr{F}}^{(J)}(X)\right)$ is not necessarily a subsemigroup of $T_{\mathscr{F}}^{(J)}(X)$. The following result tells us when $R\left(T_{\mathscr{F}}^{(J)}(X)\right)$ is a subsemigroup of $T_{\mathscr{F}}^{(J)}(X)$. It is immediately obtained from Theorem 1.6, Theorem 2.3 and Propostion 2.5.

Corollary 2.6. *The set $R\left(T_{\mathscr{F}}^{(J)}(X)\right)$ is a subsemigroup of $T_{\mathscr{F}}^{(J)}(X)$ if and only if $|J| = 1$ or $|Y_j| = 1$ for all $j \in J$. In this circumstance, $R\left(T_{\mathscr{F}}^{(J)}(X)\right) = R(T(X, Y_j))$ if $|J| = 1$ with $J = \{j\}$, and $R\left(T_{\mathscr{F}}^{(J)}(X)\right) = R_w\left(T_{\mathscr{F}}^{(J)}(X)\right)$ if $|Y_j| = 1$ for all $j \in J$. Furthermore, we have in each case that the semigroup $R\left(T_{\mathscr{F}}^{(J)}(X)\right)$ is regular.*

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