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Preservers of Gershgorin Set of Jordan Product of Matrices

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Abstract. For $A, B \in M_2(\mathbb{C})$, let the Jordan product be $AB + BA$ and $G(A)$ the eigenvalue inclusion set, the Gershgorin set of A. Characterization is obtained for maps $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ satisfying

$$G[\phi(A)\phi(B) + \phi(B)\phi(A)] = G(AB + BA)$$

for all matrices A and B. In fact, it is shown that such a map has the form $\phi(A) = \pm(PD)A(PD)^{-1}$, where P is a permutation matrix and D is a unitary diagonal matrix in $M_2(\mathbb{C})$.

1. Introduction

Eigenvalues of matrices play central role in linear algebra and its applications. When the order of the matrix is high, there is no efficient way to compute eigenvalues unless the matrix is of very special type. At times knowing the location of eigenvalues will be sufficient. Hence eigenvalue inclusion sets (i.e. sets containing eigenvalues) are of interest to the researchers.

The important types of eigenvalue inclusion sets of matrix A are Gershgorin set

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$G(A)$, Ostrowski set $O_\varepsilon(A)$, $\varepsilon \in [0,1]$ and Brauer's set $C(A)$ [3, 4]. It is known that $O_1(A) = G(A)$.

The study of maps on matrix spaces which preserve a particular property, the so called preservers has been an area of active research for the past several years.

Let M_n denote the set of $n \times n$ matrices over $\mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers and for $A \in M_n, S(A)$ be an eigenvalue inclusion set mentioned above.

In [2] authors have characterized maps $\phi : M_n \to M_n$ satisfying

$$S(\phi(A) - \phi(B)) = S(A - B)$$

and in [1] maps $\phi : M_n \to M_n$ satisfying

$$S(\phi(A)\phi(B)) = S(AB)$$

have been characterized. Among other things, authors proved (see [1] theorem (2.1)):

A mapping $\phi : M_n \to M_n$ satisfies $O_\varepsilon[\phi(A)\phi(B)] = O_\varepsilon(AB)$ for all $A, B \in M_n$ if and only if there exist $c = \pm 1$, a permutation matrix P, and an invertible diagonal matrix D, where D is unitary matrix unless $(n, \varepsilon) = (2, \frac{1}{2})$, such that $\phi(A) = c(DP)A(DP)^{-1}$ for all $A \in M_n$.

Characterization of maps $\phi : M_n \to M_n$ that satisfy

$$S(\phi(A) \circ \phi(B)) = S(A \circ B),$$

where $\circ$ is a binary operation on matrices such as the Jordan product $AB + BA$, the lie product $AB - BA$, is still open.

The aim of this paper is to characterize the map $\phi : M_2 \to M_2$ which preserves the Gershgorin set of Jordan product in the sense that

$$G[(\phi(A)\phi(B) + \phi(B)\phi(A)] = G(AB + BA).$$

It is noteworthy that these maps admit the same characterization as in theorem stated above. The material of this paper has been organized in to two sections one on preliminaries and the other on the main result.

2. Preliminaries

The following result from matrix theory [3], (theorem 3.2.4.2) will be needed in the sequel.

Proposition 2.1. *Suppose $A \in M_n$ has n distinct eigenvalues and $B \in M_n$ satisfies $AB = BA$. Then there is a complex polynomial $p(z)$ of degree at most $n-1$ such that $B = p(A)$.*

Definition 2.2.(Jordan product of matrices) If A and B are two matrices of order n, then *Jordan product* of these matrices is defined as $AB + BA$.

Definition 2.3.(Gershgorin Set) Given a matrix $A = [a_{ij}] \in M_n$, define $R_k = R_k(A) = \sum_{j=1, j\neq k}^{n} |a_{kj}|$. The *Gershgorin set* of A is defined by

$$G(A) = \cup_{k=1}^{n} G_k(A),$$

where $G_k(A) = \{z \in \mathbb{C} : |z - a_{kk}| \leq R_k\}$. The set $G_k(A)$ is called a *Gershgorin disk* of A.

It is well known that the Gershgorin set contains all the eigenvalues of A [3, 4].

Remarks 2.4.

(i) A is the zero matrix if and only if $G(A) = \{0\}$.

(ii) A is a diagonal matrix if and only if $G(A)$ is the set of diagonal entries of A.

(iii) For $A \in M_n, G(A) = G(PAP^t)$, where P is a permutation matrix of order n and P^t is the transpose of the matrix P.

3. Main Result

Theorem 3.1. *A mapping $\phi : M_2 \to M_2$ satisfies*

$$G[\phi(A)\phi(B) + \phi(B)\phi(A)] = G(AB + BA)$$

for all $A, B \in M_2$ if and only if there exist $c = \pm 1$, a permutation matrix P and an unitary diagonal matrix D such that $\phi(A) = c(PD)A(PD)^{-1}$ for all $A \in M_2$.

The following propositions will be used in our proof.

Proposition 3.2. *Let $D \in M_2$ be an invertible unitary diagonal matrix, $P \in M_2$ be a permutation matrix. Then for all $A \in M_2$,*

$$G\left[(PD)A(PD)^{-1}\right] = G(A).$$

Proof. Let $A \in M_2, D = diag\{\alpha_1, \alpha_2\}$. If D is an invertible diagonal unitary matrix then $|\alpha_i| = 1$ for $i = 1, 2$. If $P = I$, then

$$(PD)A(PD)^{-1} = (D)A(D)^{-1} = \begin{bmatrix} a_{11} & \alpha_2^{-1}\alpha_1 a_{12} \\ \alpha_1^{-1}\alpha_2 a_{21} & a_{22} \end{bmatrix}$$
$$\Rightarrow \quad G\left[(PD)A(PD)^{-1}\right] = G(A) \text{ as } |\alpha_i| = 1 \text{ for } i = 1, 2.$$

If $P = P_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then

$$(PD)A(PD)^{-1} = \begin{bmatrix} a_{22} & \alpha_2^{-1}\alpha_1 a_{21} \\ \alpha_1^{-1}\alpha_2 a_{12} & a_{11} \end{bmatrix}$$
$$\Rightarrow \quad G(PD)A(PD)^{-1} = G(A).$$

□

Proposition 3.3. *If a mapping $\phi : M_2 \to M_2$ satisfies*

$$G\left[\phi\left(A\right)\phi\left(B\right)+\phi\left(B\right)\phi\left(A\right)\right]=G\left(AB+BA\right)$$

for all $A, B \in M_2$, then there exist permutation matrix P and $c_{ij} \in \mathbb{C}$ with $|c_{ij}| = 1$ and $c_{ij}c_{ji} = 1$ such that $\phi\left(E_{ij}\right) = c_{ij}PE_{ij}P^t$ for $i, j \in \{1, 2\}$.

Proof. Suppose $i = j = 1$ and $A = B = E_{11}$, then

$$\begin{aligned} & G\left(2\phi\left(E_{11}\right)^2\right) = G\left(2E_{11}^2\right) = \{2, 0\} \\ \Rightarrow \quad & \phi\left(E_{11}\right)^2 \text{ must be } E_{11}^2 \text{ or } E_{22}^2. \end{aligned}$$

That is $\phi\left(E_{11}\right)^2$ is a diagonal matrix with diagonal entries 0 and 1. Since $\phi\left(E_{11}\right)^2$ has distinct eigenvalues and commutes with $\phi\left(E_{11}\right)$, by proposition (2.1), $\phi\left(E_{11}\right)$ is a polynomial in $\phi\left(E_{11}\right)^2$. It follows $\phi\left(E_{11}\right)$ is a diagonal matrix with diagonal entries 0 and 1 or -1 as $\phi\left(E_{11}\right)^2 = E_{11}^2$ or E_{22}^2. Then $\phi(E_{11}) = c_{11}E_{11}$ or $\phi(E_{11}) = c_{11}E_{22}$, where $c_{11} = \pm 1$.

If $\phi(E_{11}) = c_{11}E_{11}$, then the permutation matrix $P = I$ satisfies
$$\phi(E_{11}) = c_{11}PE_{11}P^t.$$

Without loss of generality we may assume that $c_{11} = 1$ otherwise replace ϕ by $-\phi$. So we assume $\phi(E_{11}) = E_{11}$. Similarly $\phi(E_{22}) = c_{22}E_{11}$ or $c_{22}E_{22}$, where $c_{22} = \pm 1$. Suppose $\phi(E_{22}) = c_{22}E_{11}$. If $A = E_{11}, B = E_{22}$

$$G[\phi(E_{11})\phi(E_{22}) + \phi(E_{22})\phi(E_{11})] = G(E_{11}E_{22} + E_{22}E_{11}) = \{0\}.$$

But,

$$G[\phi(E_{11})\phi(E_{22}) + \phi(E_{22})\phi(E_{11})] = G(c_{22}E_{11}E_{11} + c_{22}E_{11}E_{11}) = \{2c_{22}, 0\}.$$

As $c_{22} = \pm 1$, this is a contradiction. Hence $\phi(E_{22}) = c_{22}E_{22}$, where $c_{22} = \pm 1$.

Suppose $i \neq j$ and $\phi(E_{21}) = [b_{ij}] \in M_2$. Then

$$G(2\phi(E_{21})^2) = G(2E_{21}^2) = \{0\}.$$

But,

$$G(2\phi(E_{21})^2) = G\left(2\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}^2\right) = G\left(2\begin{bmatrix} b_{11}^2 + b_{12}b_{21} & b_{11}b_{12} + b_{12}b_{22} \\ b_{21}b_{11} + b_{22}b_{21} & b_{21}b_{12} + b_{22}^2 \end{bmatrix}\right)$$

Therefore,

$$\left\{ \begin{aligned} & b_{11}^2 + b_{12}b_{21} = 0, \\ & b_{11}b_{12} + b_{12}b_{22} = 0, \\ & b_{21}b_{11} + b_{22}b_{21} = 0, \\ & b_{21}b_{12} + b_{22}^2 = 0 \end{aligned} \right. \tag{3.1}$$

Taking $A = E_{11}, B = E_{21}$, we have

$$G[\phi(E_{11})\phi(E_{21}) + \phi(E_{21})\phi(E_{11})] = G(E_{11}E_{21} + E_{21}E_{11}) = G(E_{21})$$

Note that $G(E_{21})$ is a disk with center 0 and radius 1. Hence

$$G[\phi(E_{11})\phi(E_{21}) + \phi(E_{21})\phi(E_{11})] = G\left(\begin{bmatrix} 2b_{11} & b_{12} \\ b_{21} & 0 \end{bmatrix}\right)$$

is a disk with centre 0 and radius 1. It follows that $|2b_{11}| < 1$ and $|b_{21}| \leq 1$. Two cases arise (i) $|b_{21}| = 1$ or (ii) $|b_{21}| < 1$, in which case $b_{11} = 0$ and $|b_{12}| = 1$.
Case (i) : If $|b_{21}| = 1$, take $A = E_{22}, B = E_{21}$. Then

$$G[\phi(E_{22})\phi(E_{21}) + \phi(E_{21})\phi(E_{22})] = G(E_{22}E_{21} + E_{21}E_{22}) = G(E_{21})$$

which is a disk with centre 0 and radius 1. But

$$G[\phi(E_{22})\phi(E_{21}) + \phi(E_{21})\phi(E_{22})] = G\left(\begin{bmatrix} 0 & c_{22}b_{12} \\ c_{22}b_{21} & 2c_{22}b_{22} \end{bmatrix}\right)$$

As $|b_{21}| = 1$ and $c_{22} = \pm 1$, $b_{22} = 0$. Otherwise this disk will not be identical with the above disk.
Now from the set of equations (1) we have $b_{11} = 0$ and $b_{12} = 0$ and hence $\phi(E_{21}) = b_{21}E_{21}$, where $|b_{21}| = 1$

$$\phi(E_{21}) = c_{21}PE_{21}P^t, \text{ where } c_{21} = b_{21} \text{ and } P = I.$$

Case (ii): If $|b_{21}| < 1$, in which case $b_{11} = 0$ and $|b_{12}| = 1$, from the set of equations (1) we have $b_{21} = 0$ and $b_{22} = 0$ and hence $\phi(E_{21}) = b_{12}E_{12}$, where $|b_{12}| = 1$.
This implies $\phi(E_{21}) = c_{21}PE_{21}P^t$, where $c_{21} = b_{12}$ and $P = P_{12}$.
Similarly, $\phi(E_{12}) = c_{12}PE_{12}P^t$.
Thus $\phi(E_{ij}) = c_{ij}PE_{ij}P^t$. If $P \neq I_2$, then we replace the map ϕ by $P^t\phi P$ and get $\phi(E_{ij}) = c_{ij}E_{ij}$.
Claim : $c_{ij}c_{ji} = 1$
If $i = j$, then $c_{ij}c_{ji} = 1$. If $i \neq j$, take $A = E_{12}$ and $B = E_{21}$, then

$$\begin{aligned} G[\phi(E_{12})\phi(E_{21}) + \phi(E_{21})\phi(E_{12})] &= G(E_{12}E_{21} + E_{21}E_{12}) \\ &= G(E_{11} + E_{22}) \\ &= \{1\} \end{aligned}$$

But,

$$\begin{aligned} G[\phi(E_{12})\phi(E_{21}) + \phi(E_{21})\phi(E_{12})] &= G(c_{12}E_{12}c_{21}E_{21} + c_{21}E_{21}c_{12}E_{12}) \\ &= G(c_{12}c_{21}E_{11} + c_{21}c_{12}E_{22}) \\ &= \{c_{12}c_{21}\}. \end{aligned}$$

Therefore, $c_{12}c_{21} = 1$. □

Proof of theorem 3.1

Sufficiency : For $A, B \in M_2$ let $\phi(A) = c(PD)A(PD)^{-1}, \phi(B) = c(PD)B(PD)^{-1}$, where $c = \pm 1, P$ is permutation matrix and D is a unitary diagonal matrix. Then

$$\begin{aligned} & \phi(A)\phi(B) + \phi(B)\phi(A) && = (PD)(AB+BA)(PD)^{-1} (\text{ as } c^2 = 1) \\ \Rightarrow \quad & G[\phi(A)\phi(B) + \phi(B)\phi(A)] && = G[(PD)(AB+BA)(PD)^{-1}] \\ & && = G(AB+BA) (\text{ from proposition (2.2) }) \end{aligned}$$

Necessity : Suppose $\phi : M_2 \to M_2$ satisfies

$$G[(\phi(A)\phi(B) + \phi(B)\phi(A)] = G(AB+BA)$$

Let $A = [a_{ij}]$ in M_2. First we claim $\phi(A) = [c_{ij}a_{ij}]$, with $|c_{ij}| = 1$. Proof of this claim is divided into four steps. Let $\phi(A) = [d_{ij}]$

Step 1: To show that $d_{ij} = c_{ij}a_{ij}$ for $i \neq j$ with $|c_{ij}| = 1$.

Step 2: $\phi(I) = I$, where I is the identify matrix of order two.

Step 3: $\phi \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Step 4: Using the results of steps 1, 2 and 3 we prove $d_{ij} = a_{ij}$ for $i = j$.

In view of proposition 3.3, we have $\phi(E_{ij}) = c_{ij}E_{ij}$, where $|c_{ij}| = 1$, $c_{22} = \pm 1$ and $c_{11} = 1$.

Step 1: $d_{ij} = c_{ij}a_{ij}$ for $i \neq j$.

Now, take $B = E_{21}$, then

$$\begin{aligned} & G[\phi(A)\phi(E_{21}) + \phi(E_{21})\phi(A)] = G(AE_{21} + E_{21}A) \\ \Rightarrow \quad & G\left(\begin{bmatrix} c_{21}d_{12} & 0 \\ c_{21}(d_{11}+d_{22}) & c_{21}d_{12} \end{bmatrix}\right) = G\left(\begin{bmatrix} a_{12} & 0 \\ a_{11}+a_{22} & a_{12} \end{bmatrix}\right) \end{aligned}$$

Since both the sides are single disks comparing the centers we get ,

$$c_{21}d_{12} = a_{12} \Rightarrow d_{12} = c_{12}a_{12} (\text{as } c_{21}^{-1} = c_{12})$$

Similarly, $d_{21} = c_{21}a_{21}$. Hence $d_{ij} = c_{ij}a_{ij}$ for $i \neq j$.

Step 2: $\phi(I) = I$

Let $\phi(I) = [b_{ij}]$. Taking $A = E_{11}, B = I$, we have

$$\begin{aligned} & G[\phi(E_{11})\phi(I) + \phi(I)\phi(E_{11})] = G(E_{11}I + IE_{11}) \\ \Rightarrow \quad & G[E_{11}[b_{ij}] + [b_{ij}]E_{11}] = G(2E_{11}) \\ \Rightarrow \quad & G\left(\begin{bmatrix} 2b_{11} & b_{12} \\ b_{21} & 0 \end{bmatrix}\right) = \{2, 0\} \\ \Rightarrow \quad & b_{11} = 1 \text{ and } b_{12} = b_{21} = 0 \\ \Rightarrow \quad & \phi(I) = \left(\begin{bmatrix} 1 & 0 \\ 0 & b_{22} \end{bmatrix}\right) \end{aligned}$$

Now, take $A = E_{22}, B = I$. Then

$$
\begin{aligned}
& G[\phi(E_{22})\phi(I) + \phi(I)\phi(E_{22})] = G(E_{22}I + IE_{22}) \\
\Rightarrow\ & G\left(\begin{bmatrix} 0 & 0 \\ 0 & 2c_{22}b_{22} \end{bmatrix}\right) = G(2E_{22}) = \{2, 0\} \\
\Rightarrow\ & c_{22}b_{22} = 1 \\
\Rightarrow\ & b_{22} = c_{22} \text{ as } c_{22} = \pm 1.
\end{aligned}
$$

Therefore $\phi(I) = \begin{bmatrix} 1 & 0 \\ 0 & c_{22} \end{bmatrix}$.

Taking $A = E_{12}, B = I$, we get

$$
\begin{aligned}
& G[\phi(E_{12})\phi(I) + \phi(I)\phi(E_{12})] = G(E_{12}I + IE_{12}) \\
\Rightarrow\ & G\left(\begin{bmatrix} 0 & c_{12}(1 + c_{22}) \\ 0 & 0 \end{bmatrix}\right) = G(2E_{12})
\end{aligned}
$$

But, $G(2E_{12})$ is a disk with center 0 and radius 2. This gives

$$
\begin{aligned}
& |c_{12}(1 + c_{22})| = 2 \\
\Rightarrow\ & |(1 + c_{22})| = 2 \text{ as } |c_{12}| = 1 \\
\Rightarrow\ & c_{22} = 1 \text{ as } c_{22} = \pm 1
\end{aligned}
$$

Thus $\phi(I) = I$.

Step 3 : $\phi \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Take $A = I$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then

$$
\begin{aligned}
& G[\phi(I)\phi(B) + \phi(B)\phi(I)] = G(IB + BI) \\
\Rightarrow\ & G[2\phi(B)] = G(2B) = \{2, -2\} \\
\Rightarrow\ & \phi(B) \text{ is a diagonal matrix with diagonal entries 1 and } -1 \\
\Rightarrow\ & \phi(B) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

If $\phi(B) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, taking $A = E_{11}$, we get

$$
\begin{aligned}
& G[\phi(E_{11})\phi(B) + \phi(B)\phi(E_{11})] = G(E_{11}I + IE_{11}) \\
\Rightarrow\ & G\left(\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}\right) = G\left(\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\right) \\
\Rightarrow\ & \{-2, 0\} = \{2, 0\}
\end{aligned}
$$

Which is not possible and our supposition is wrong. Therefore

$$
\phi \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
$$

Step 4: $d_{11} = a_{11}$and $d_{22} = a_{22}$. For $A = [a_{ij}]$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, we have

$$\begin{aligned} & G[(\phi(A)\phi(B) + \phi(B)\phi(A)] = G(AB + BA) \\ \Rightarrow \quad & G\begin{bmatrix} 2d_{11} & 0 \\ 0 & -2d_{22} \end{bmatrix} = G\begin{bmatrix} 2a_{11} & 0 \\ 0 & -2a_{22} \end{bmatrix} \end{aligned}$$

If $a_{11} = -a_{22}$ then the right hand side is one degenerated disk i.e. single point and therefore

$$\begin{aligned} & a_{11} = -a_{22} \\ \Rightarrow \quad & d_{11} = -d_{22} \\ \Rightarrow \quad & a_{11} = d_{11} \text{ and } a_{22} = d_{22}. \end{aligned}$$

If $a_{11} \neq -a_{22}$ then we get two degenerated disks on the right hand side and there are two cases

Case (i) $a_{11} = d_{11}$ and $a_{22} = d_{22}$, then there is nothing to prove.

Case (ii) $a_{11} = -d_{22}$ and $a_{22} = -d_{11}$.

Now, we claim this case does not arise. We have either $|a_{12}| = |a_{21}|$ or $|a_{12}| > |a_{21}|$ or$|a_{12}| < |a_{21}|$. Taking $B = E_{11}$, we have

$$\begin{aligned} & G[(\phi(A)\phi(E_{11}) + \phi(E_{11})\phi(A)] = G(AE_{11} + E_{11}A) \\ \text{Hence} \quad & G\left(\begin{bmatrix} 2d_{11} & d_{12} \\ d_{21} & 0 \end{bmatrix}\right) = G\left(\begin{bmatrix} 2a_{11} & a_{12} \\ a_{21} & 0 \end{bmatrix}\right) \end{aligned}$$

If they represent two disks then comparison of centers gives $a_{11} = d_{11}$ and then from case (i) $a_{22} = d_{22}$. Suppose they represent a single disk.

If $|a_{12}| = |a_{21}|$ then $|d_{12}| = |d_{21}|$.

By comparison of centers, we get $a_{11} = d_{11} = 0$ and then from case (i) $a_{22} = d_{22}$.

If $|a_{12}| > |a_{21}|$, then from step (i) $|d_{12}| > |d_{21}|$.

By comparison of centers, we get $a_{11} = d_{11}$ and hence $a_{22} = d_{22}$ from case (i).

If $|a_{12}| < |a_{21}|$, then $|d_{12}| < |d_{21}|$.

Taking $B = E_{22}$ we have

$$\begin{aligned} & G[\phi(A)\phi(E_{22}) + \phi(E_{22})\phi(B)] = G(AE_{22} + E_{22}A) \\ \Rightarrow \quad & G\left(\begin{bmatrix} 0 & c_{22}d_{12} \\ c_{22}d_{21} & 2c_{22}d_{22} \end{bmatrix}\right) = G\left(\begin{bmatrix} 0 & a_{12} \\ a_{21} & 2a_{22} \end{bmatrix}\right). \end{aligned}$$

As $|a_{12}| < |a_{21}|$, $|c_{22}d_{12}| < |c_{22}d_{21}|$, then by comparison of centers we have $a_{22} = d_{22}$ and hence $a_{11} = d_{11}$ from case (i).

Therefore case (ii) does not arise.

Thus $d_{ij} = c_{ij}a_{ij}$ with $|c_{ij}| = 1$, $c_{ij}c_{ji} = 1$ and $c_{ij} = 1$ for $i = j$. Thus

$$\phi(A) = [d_{ij}] = \left(\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & c_{12}a_{12} \\ c_{21}a_{21} & a_{22} \end{bmatrix}.$$

Let $D = \begin{bmatrix} 1 & 0 \\ 0 & c_{21} \end{bmatrix}$, then $D^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & c_{12} \end{bmatrix}$. Hence

$$\phi(A) = DAD^{-1}.$$

Recall that in proposition 3.3 we obtained $\phi(E_{ij}) = c_{ij}PE_{ij}P^t$. When $P \neq I$ we replaced the map ϕ by $P^t\phi P$ and assumed $\phi(E_{ij}) = c_{ij}E_{ij}$ and the rest of the proof was carried out under this assumption. Hence

$$\phi(A) = c(PD)A(PD)^{-1}$$

and this completes the proof. □

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