JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **31**, No. 1, February 2018
http://dx.doi.org/10.14403/jcms.2018.31.1.53

THE EXISTENCE OF THE SOLUTION OF ELLIPTIC SYSTEM APPLYING TWO CRITICAL POINT THEOREM

Hyewon Nam*

Abstract. This paper deals with the study of solutions for the elliptic system with jumping nonlineartity and growth nonlinearity and Dirichlet boundary condition. We apply the two critical point theorem when proving the existence of nontrivial solutions for the elliptic system. We define the energy functional associated to the elliptic system and prove that the functional has two critical values.

1. Introduction

In this paper, we consider the existence of nontrivial solutions to the elliptic system

$$(1.1)\quad \begin{cases} -\triangle u = au + bv + (u^+)^{p_1} - (u^-)^{q_1} + f_1(x,u,v) & \text{in } \Omega, \\ \triangle v = bu + cv + (v^+)^{p_2} - \eta(v^-) + f_2(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

where $u^+ = \max\{0, u(x)\}$, $u^- = -\min\{0, u(x)\}$ and $\Omega \subset R^N$ is a smooth bounded domain with $N \geq 2$.

The nonlinearities will be assumed to be both superlinear and subcritical, that is, $1 < q_1 < p_1 < 2^* - 1$ and $1 < p_2 < 2^* - 1$, where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ if $N = 2$.

There exists a function $F : \bar{\Omega} \times R^2 \to R$ such that $\frac{\partial F}{\partial u} = f_1$ and $\frac{\partial F}{\partial v} = f_2$ without loss of generality, and we set

$$F(x,u,v) = \int_{(0,0)}^{(u,v)} f_1(x,u,v)du + f_2(x,u,v)dv.$$

Then $F \in C^1(\bar{\Omega} \times R^2, R)$.

We consider the following assumptions.

Received November 30, 2017; Accepted December 01, 2017.
2010 Mathematics Subject Classification: Primary 35J55; Secondary 49J35.
Key words and phrases: elliptic system, critical point theorem.

(F1) There exist $M > 0$ and $\alpha > 2$ such that

$$0 < \alpha F(x,u,v) \leq uF_u(x,u,v) + vF_v(x,u,v)$$

for all $(x,u,v) \in \bar{\Omega} \times R^2$ with $u^2 + v^2 > M^2$.

(F2) There exist constants $a_1 > 0$ and $a_2 > 0$ such that

$$|F_u(x,u,v)| + |F_v(x,u,v)| \leq a_1 + a_2(|u|^r + |v|^r)$$

where $1 \leq r < \frac{(N+2)}{(N-2)}$ if $N > 2$ and $1 \leq r < \infty$ if $N = 2$.

(F3) For $(u,0) \to (0,0)$,

$$\frac{F(x,u,0)}{u^2} \to 0.$$

(F4) For every $u \in H$,

$$F(u,0) \geq 0.$$

Remark 1.1. The condition (F1) shows that there exist constants $b_1 > 0$ and b_2 such that (cf. [1])

$$F(x,u,v) \geq b_1(|u|^\alpha + |v|^\alpha) - b_2.$$

The results of our study are as follows.

Theorem 1.2. *Assume F satisfies (F1), (F2), (F3) and (F4) with $\alpha = r + 1$. If a, b, and c are positive with $a < \lambda_1$ and $c + \eta < \lambda_1$ then system (1.1) has at least one nontrivial solutions.*

Presently there are many significant results with respect to the nonlinear elliptic equation and system with Dirichlet boundary condition [2, 6, 8, 9]. Many authors also investigated the nonlinear elliptic equation and system with jumping nonlinearity and subcritical growth nonlinearity and Dirichlet boundary condition [4, 5, 7]. We are interested in the two critical point theorem as a way of solving the elliptic system.

In this paper we prove the existence of two nontrivial solutions for the elliptic system with jumping nonlinearity and growth nonlinearity and Dirichlet boundary condition. In Section 2, we use a variational approach to look for critical points of the functional I on a Hilbert space H. In Section 3, we prove the Palais Smale star condition for the two critical point theorem. And we prove the Lemmas in order to apply the two critical point theorem, so we prove Theorem 1.2.

2. Preliminaries

This section introduces the critical point theorem used to prove the existence of the nontrivial solutions for the elliptic system.

For any subspace Y of a Hilbert space H, consider

$$B_\rho(Y) := \{u \in Y \mid \|u\| \le \rho\}$$

and denote by $\partial B_\rho(Y)$ the boundary of $B_\rho(Y)$ relative to Y. Furthermore define, for any $e \in H$,

$$Q_R(Y, e) := \{u + ae \in Y \oplus [e] \mid u \in Y, a \ge 0, \|u + ad\| \le R\}$$

and denote by $\partial Q_R(Y, e)$ its boundary relative to $Y \oplus [e]$.

Let V be a C^2 complete connected Finsler manifold. Suppose $H = H_1 \oplus H_2$ and let $H_n = H_{1n} \oplus H_{2n}$ be a sequence of closed subspaces of H such that

$$H_{in} \subset H_i, \ \ 1 \le \dim H_{in} < +\infty \ \text{ for each } \ i = 1, 2 \quad \text{and} \quad n \in N.$$

Moreover suppose that there exist $e_1 \in \cap_{n=1}^{\infty} H_{1n}$, and $e_2 \in \cap_{n=1}^{\infty} H_{2n}$, with $\|e_1\| = \|e_2\| = 1$.

We recall the two critical points theorem in [3].

THEOREM 2.1. ([3] Theorem 2.1) *Suppose that f satisfies the $(PS)^*$ condition with respect to H_n. In addition assume that there exist ρ, R, such that $0 < \rho < R$ and*

$$\sup_{\partial Q_R(H_2, e_1) \times V} f < \inf_{\partial B_\rho(H_1) \times V} f,$$
$$\sup_{Q_R(H_2, e_1) \times V} f < +\infty, \quad \text{and} \quad \inf_{B_\rho(H_1) \times V} f < -\infty.$$

Then there exist at least 2 critical levels of f. Moreover the critical levels satisfy the following inequalities

$$\inf_{B_\rho(H_1) \times V} f \le c_1 \le \sup_{\partial Q_R(H_2, e_1) \times V} f < \inf_{\partial B_\rho(H_1) \times V} f \le c_2 \le \sup_{Q_R(H_2, e_1) \times V} f,$$

and exist at least $2 + 2$ cuplength(V) critical points of f.

3. Main result

We prove the Palais Smale star condition for the two critical point theorem. And we prove the Lemmas in order to apply the two critical point theorem, so we prove the existence of nontrivial solutions by using two critical points theorem.

3.1. The variational structure

Throughout the paper, we will denote by λ_k the eigenvalues and by e_k the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem

$$-\Delta u = \lambda u \quad \text{in } \Omega$$

with Dirichlet boundary condition, where each eigenvalue λ_k is respected as often as its multiplicity. We recall that

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots, \lambda_i \to +\infty$$

and that $e_1 > 0$ for all $x \in \Omega$.

Then $H = \text{span}\{e_i | i \in N\}$, where $H = W_0^{1,p}(\Omega)$, the usual Sobolev space with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$.

Let $e_i^1 = (e_i, 0)$ and $e_i^2 = (0, e_i)$. We define $H_j = \text{span}\{e_i^j | i \in N\}$ for $j = 1, 2$ and $E = H_1 \oplus H_2$ with the norm $\|(u,v)\|_E^2 = \|u\|^2 + \|v\|^2$.

We define the energy functional associated to (1.1) as

$$\begin{aligned} I(u,v) &= \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - |\nabla v|^2\right) dx - \frac{1}{2}\int_\Omega \left(au^2 + 2buv + cv^2\right) dx \\ &\quad - \int_\Omega \left(\frac{1}{p_1+1}(u^+)^{p_1+1} + \frac{1}{p_2+1}(v^+)^{p_2+1}\right) dx \\ &\quad + \int_\Omega \left(\frac{1}{q_1+1}(u^-)^{q_1+1} + \frac{\eta}{2}(v^-)^2\right) dx - \int_\Omega F(x,u,v)dx. \end{aligned} \tag{3.1}$$

It is easy to see that $I \in C^1(E, R)$ and thus it makes sense to look for solutions to (1.1) in weak sense as critical points for I *i.e.*, $(u,v) \in E$ such that $I'(u,v) = 0$, where

$$\begin{aligned} I'(u,v)\cdot(\phi,\psi) &= \int_\Omega (\nabla u \nabla \phi - \nabla v \nabla \psi)\, dx \\ &\quad - \int_\Omega (au\phi + bv\phi + bu\psi + cv\psi)\, dx \\ &\quad - \int_\Omega \left((u^+)^{p_1}\phi + (v^+)^{p_2}\psi\right) dx \\ &\quad + \int_\Omega \left((u^-)^{q_1}\phi + \eta(v^-)\psi\right) dx \\ &\quad - \int_\Omega (f_1(x,u,v)\phi + f_2(x,u,v)\psi)\, dx. \end{aligned}$$

3.2. The Palais Smale star condition

In this section we will prove the $(PS)_c^*$ condition which was required for the application of Theorem 2.1. In the following, we consider the following sequence of subspaces of E :

$$E_n = \mathrm{span}\{e_i^j | i = 1, \cdots, n \quad \text{and} \quad j = 1, 2\}, \quad \text{for } n \geq 1.$$

LEMMA 3.1. *Assume F satisfies (F1) and (F2) with $\alpha = r + 1$. If $a < \lambda_1$ and $c + 1 < \lambda_1$, then any $(PS)_c^*$ sequence is bounded.*

Proof. Let $\{(u_n, v_n)\} \subset E$ be a sequence such that

$$(u_n, v_n) \in E_n, \quad I(u_n, v_n) \to C, \quad I'(u_n, v_n) \to 0 \quad \text{as} \quad n \to \infty.$$

To show the contradiction, we assume that $\{(u_n, v_n)\}$ is not bounded *i.e.*, $\|(u_n, v_n)\|_E \to \infty$.

In the following we denote different constants by C_1, C_2 and etc.

$$\begin{aligned}
C_1 \quad &+ \quad \frac{1}{2} o(1) \left(\|u_n\| + \|v_n\| \right) \\
&\geq \quad I(u_n, v_n) - \frac{1}{2} I'(u_n, v_n) \cdot (u_n, v_n) \\
&= \quad \int_\Omega \left(\frac{p_1 - 1}{2(p_1 + 1)} ({u_n}^+)^{p_1+1} + \frac{p_2 - 1}{2(p_2 + 1)} ({v_n}^+)^{p_2+1} \right) dx \\
&\quad - \int_\Omega \left(\frac{q_1 - 1}{2(q_1 + 1)} ({u_n}^-)^{q_1+1} \right) dx \\
&\quad + \frac{1}{2} \int_\Omega (u_n f_1 + v_n f_2) dx - \int_\Omega F(x, u_n, v_n) dx \qquad (3.2) \\
&\geq \quad \frac{q_1 - 1}{2(q_1 + 1)} \int_\Omega \left(({u_n}^+)^{p_1+1} - ({u_n}^-)^{q_1+1} \right) dx \\
&\quad + \frac{p_2 - 1}{2(p_2 + 1)} \int_\Omega \left(({v_n}^+)^{p_2+1} \right) dx \\
&\quad + \frac{1}{2} \int_\Omega (u_n f_1 + v_n f_2) dx - \int_\Omega F(x, u_n, v_n) dx.
\end{aligned}$$

(F1) and Remark imply that

$$
\begin{aligned}
\frac{1}{2}\int_\Omega (u_n f_1 + v_n f_2)dx \quad &- \quad \int_\Omega F(x,u_n,v_n)dx \\
&\geq \left(\frac{\alpha}{2}-1\right)\int_\Omega F(x,u_n,v_n)dx \\
&\geq \left(\frac{\alpha}{2}-1\right) b_1 \int_\Omega (|u_n|^\alpha + |v_n|^\alpha)dx - C_2 \\
&\geq \left(\frac{\alpha}{2}-1\right) b_1(\|u_n\|^\alpha_{L^\alpha} + \|v_n\|^\alpha_{L^\alpha}) - C_2.
\end{aligned}
\tag{3.3}
$$

Combining (3.2), (3.3), we obtain

$$
\begin{aligned}
C_1 \quad &+ \quad \frac{1}{2}o(1)\left(\|u_n\| + \|v_n\|\right) \\
&\geq \quad \frac{q_1-1}{2(q_1+1)}\int_\Omega \left((u_n{}^+)^{p_1+1} - (u_n{}^-)^{q_1+1}\right)dx \\
&\quad +\frac{p_2-1}{2(p_2+1)}\int_\Omega \left((v_n{}^+)^{p_2+1}\right)dx \\
&\quad +\left(\frac{\alpha}{2}-1\right) b_1(\|u_n\|^\alpha_{L^\alpha} + \|v_n\|^\alpha_{L^\alpha}) - C_2.
\end{aligned}
\tag{3.4}
$$

Since $\alpha > 2$ and $b_1 > 0$, we get

$$
\begin{aligned}
C_3 + \frac{1}{2}o(1)\left(\|u_n\| + \|v_n\|\right) \quad \geq \quad & \frac{q_1-1}{2(q_1+1)}\int_\Omega \left((u_n{}^+)^{p_1+1} - (u_n{}^-)^{q_1+1}\right)dx \\
& +\frac{p_2-1}{2(p_2+1)}\int_\Omega \left((v_n{}^+)^{p_2+1}-\right)dx.
\end{aligned}
$$

By observing that each term in the expression above is nonnegative, we conclude that the estimate from above holds for each of them, and then

$$
\frac{1}{\|(u_n,v_n)\|_E}\int_\Omega \left((u_n{}^+)^{p_1+1} - (u_n{}^-)^{q_1+1}\right)dx \to 0, \tag{3.5}
$$

$$
\frac{1}{\|(u_n,v_n)\|_E}\int_\Omega (v_n{}^+)^{p_2+1}dx \to 0. \tag{3.6}
$$

On the other hand

$$
\begin{aligned}
o(1)\|u_n\| &\geq I'(u_n, v_n)\cdot(u_n, 0)\\
&= \|u_n\|^2 - \int_\Omega (a{u_n}^2 + b u_n v_n)dx \qquad (3.7)\\
&\quad - \int_\Omega \left(({u_n}^+)^{p_1+1} - ({u_n}^-)^{q_1+1}\right)dx - \int_\Omega u_n f_1 dx,\\
-o(1)\|v_n\| &\leq I'(u_n, v_n)\cdot(0, v_n)\\
&= -\|v_n\|^2 - \int_\Omega (b u_n v_n + c{v_n}^2)dx \qquad (3.8)\\
&\quad - \int_\Omega \left(({v_n}^+)^{p_2+1} - \eta({v_n}^-)^2\right)dx - \int_\Omega v_n f_2 dx.
\end{aligned}
$$

Combining (3.7) and (3.8),

$$
\begin{aligned}
\|u_n\|^2 + \|v_n\|^2 &\leq o(1)(\|u_n\| + \|v_n\|) + \int_\Omega (a{u_n}^2 - c{v_n}^2)dx\\
&\quad + \int_\Omega \left(({u_n}^+)^{p_1+1} - ({u_n}^-)^{q_1+1}\right)dx\\
&\quad - \int_\Omega \left(({v_n}^+)^{p_2+1} - \eta({v_n}^-)^2\right)dx \qquad (3.9)\\
&\quad + \int_\Omega (u_n f_1 - v_n f_2)dx.
\end{aligned}
$$

By the continuous embedding of H in L^2, we get

$$\int_\Omega (u^-)^2 dx \leq \int_\Omega u^2 dx \leq \frac{1}{\lambda_1}\|u\|^2$$

for any $u \in H$. Hence

$$
\begin{aligned}
\int_\Omega (a{u_n}^2 - c{v_n}^2)dx &\leq \int_\Omega (a{u_n}^2 + c{v_n}^2)dx\\
&\leq \frac{a}{\lambda_1}\|u_n\|^2 + \frac{c}{\lambda_1}\|v_n\|^2. \qquad (3.10)
\end{aligned}
$$

Using (F2), we obtain

$$\int_\Omega (u_n f_1 - v_n f_2)dx \leq C_4 \int_\Omega (|u_n|^{r+1} + |v_n|^{r+1})dx + C_5. \qquad (3.11)$$

Apply (3.10) and (3.11) to (3.9), and we obtain the inequlity

$$\begin{aligned}\|u_n\|^2+\|v_n\|^2 \leq\ & o(1)(\|u_n\|+\|v_n\|)+\frac{a}{\lambda_1}\|u_n\|^2+\frac{c+\eta}{\lambda_1}\|v_n\|^2 \\ & +\int_\Omega \left(({u_n}^+)^{p_1+1}-({u_n}^-)^{q_1+1}\right)dx \\ & -\int_\Omega ({v_n}^+)^{p_2+1}dx+C_4\int_\Omega(|u_n|^{r+1}+|v_n|^{r+1})dx+C_5.\end{aligned}\tag{3.12}$$

(3.12) implies that if $a<\lambda_1$ and $c+\eta<\lambda_1$ then

$$\begin{aligned}\|u_n\|^2+\|v_n\|^2 \leq\ & o(1)C_6(\|u_n\|+\|v_n\|) \\ & +C_6\int_\Omega \left(({u_n}^+)^{p_1+1}-({u_n}^-)^{q_1+1}\right)dx \\ & -C_6\int_\Omega ({v_n}^+)^{p_2+1}dx+C_7\int_\Omega(|u_n|^{r+1}+|v_n|^{r+1})dx+C_8.\end{aligned}\tag{3.13}$$

Combining (3.4), (3.13) and using $\alpha=r+1$, one infers that

$$\begin{aligned}\|u_n\|^2+\|v_n\|^2 \leq\ & o(1)C_9(\|u_n\|+\|v_n\|)+C_{10} \\ & +C_{11}\int_\Omega \left(({u_n}^+)^{p_1+1}-({u_n}^-)^{q_1+1}\right)dx \\ & +C_{12}\int_\Omega ({v_n}^+)^{p_2+1}dx.\end{aligned}$$

We get

$$\begin{aligned}\|(u_n,v_n)\|_E \quad\leq\quad & \frac{o(1)C_9(\|u_n\|+\|v_n\|)+C_{10}}{\|(u_n,v_n)\|_E} \\ & +\frac{C_{11}}{\|(u_n,v_n)\|_E}\int_\Omega \left(({u_n}^+)^{p_1+1}-({u_n}^-)^{q_1+1}\right)dx \\ & +\frac{C_{12}}{\|(u_n,v_n)\|_E}\int_\Omega ({v_n}^+)^{p_2+1}dx \to 0\end{aligned}$$

which, by using (3.5) and (3.6), implies that $\|(u_n,v_n)\|_E\to 0$. This gives rise to a contradiction to the assumtion of $\{(u_n,v_n)\}$. We conclude that $\{(u_n,v_n)\}$ is bounded. □

LEMMA 3.2. *Assume F satisfies (F1) and (F2) with $\alpha=r+1$. If $a<\lambda_1$ and $c+\eta<\lambda_1$, then the functional I satisfies the $(PS)_c^*$ condition with respect to E_n.*

Proof. By Lemma 3.1, any $(PS)_c^*$ sequence $\{(u_n,v_n)\}$ in E is bounded and hence $\{(u_n,v_n)\}$ has a weakly convergent subsequence. That is, there exist a subsequence $\{(u_{n_j},v_{n_j})\}$ and $(u,v)\in E$, with $u_{n_j}\rightharpoonup u$ and

$v_{n_j} \rightharpoonup v$. Since $\{u_{n_j}\}$ and $\{v_{n_j}\}$ are bounded, by Remark of Rellich-Kondrachov compactness theorem [4], $u_{n_j} \to u$, $v_{n_j} \to v$ and thus I satisfies $(PS)_c^*$ condition. □

3.3. Proof of main theorem

LEMMA 3.3. *Assume F satisfies (F3). If $a < \lambda_1$, then there exists $\rho > 0$ such that*

$$I(u,0) \geq 0 \qquad \text{for} \qquad u \in H \quad \text{and} \quad \|u\| \leq \rho.$$

If $\|u\| = \rho$, then $I(u,0) > 0$.

Proof. By (F3), for any $\varepsilon > 0$, there exists $\rho_1 > 0$ such that

$$\|u\| < \rho_1 \quad \Rightarrow \quad |F(x,u,0)| \leq \varepsilon |u|^2.$$

Then

$$\left| \int_\Omega F(x,u,0)dx \right| \leq \int_\Omega |F(x,u,0)|dx \leq \int_\Omega \varepsilon |u|^2 dx \leq \frac{\varepsilon}{\lambda_1}\|u\|^2.$$

By the continuous embedding of H in L^{p_1+1}, we get

$$\int_\Omega \frac{(u^+)^{p_1+1}}{p_1+1}dx \leq \int_\Omega \frac{|u|^{p_1+1}}{p_1+1}dx \leq \beta \|u\|^{p_1+1},$$

where β is a positive constant.

And hence

$$\begin{aligned} I(u,0) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{a}{2}\int_\Omega u^2 dx - \frac{1}{p_1+1}\int_\Omega (u^+)^{p_1+1}dx \\ &\quad + \frac{1}{q_1+1}\int_\Omega (u^-)^{q_1+1}dx - \int_\Omega F(x,u,0)dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{a}{2\lambda_1}\|u\|^2 - \beta\|u\|^{p_1+1} - \frac{\varepsilon}{\lambda_1}\|u\|^2 \\ &\geq \frac{1}{2}\left(1 - \frac{a+2\varepsilon}{\lambda_1} - 2\beta(\rho_1)^{p_1-1}\right)\|u\|^2 \geq 0 \end{aligned}$$

which gives the result for sufficiently small ε and ρ_1. Therefore we can choose $0 < \rho < \rho_1$ such that $I(u,0) > 0$ for any $\|u\| = \rho$. □

LEMMA 3.4. *Assume F satisfies (F4). If $\eta < \lambda_1$, then*

$$\sup_{H_2} I \leq 0.$$

Proof. We know that

$$\int_\Omega (v^-)^2 dx \le \int_\Omega v^2 dx \le \frac{1}{\lambda_1}\|v\|^2$$

for any $v \in H$. By (F4) one has

$$\begin{aligned} I(0,v) &= -\frac{1}{2}\int_\Omega |\nabla v|^2 dx - \frac{c}{2}\int_\Omega v^2 dx - \frac{1}{p_2+1}\int_\Omega (v^+)^{p_2+1} dx \\ &\quad + \frac{\eta}{2}\int_\Omega (v^-)^2 dx - \int_\Omega F(x,0,v) dx \\ &\le \frac{\eta - \lambda_1}{2\lambda_1}\|v\|^2 \le 0. \end{aligned}$$

Hence the proof is complete. □

Furthermore define, for some R

$$W_R := \{(ke_1, v) | v \in H, k > 0, \|(ke_1, v)\|_E = R\}.$$

LEMMA 3.5. *Assume F satisfies (F1). If a, b and c are positive and $\eta < \lambda_1$, then there exists an $R > 0$ such that*

$$\sup_{W_R} I < 0.$$

Proof. In the following we denote other positive constants by C_1, C_2 etc. Remark 1.1 implies that

$$\begin{aligned} I(ke_1, v) &= \frac{(\lambda_1 - a)k^2}{2} - \frac{1}{2}\|v\|^2 - bk\int_\Omega e_1 v dx - \frac{c}{2}\int_\Omega v^2 dx \\ &\quad - \frac{1}{p_1+1}\int_\Omega (ke_1)^{p_1+1} dx - \frac{1}{p_2+1}\int_\Omega (v^+)^{p_2+1} dx \\ &\quad + \frac{\eta}{2}\int_\Omega (v^-)^2 dx - \int_\Omega F(x, ke_1, v) dx \\ &\le \frac{(\lambda_1 - a)k^2}{2} - \frac{1}{2}\|v\|^2 + \frac{\eta}{2}\int_\Omega (v^-)^2 dx - \int_\Omega F(x, ke_1, v) dx \\ &\le \frac{(\lambda_1 - a)k^2}{2} + \frac{\eta - \lambda_1}{2\lambda_1}\|v\|^2 - b_1\int_\Omega (|u|^\alpha + |\beta e_1|^\alpha) dx + C_1 \\ &\le -C_2 k^\alpha + C_3 + \frac{(\lambda_1 - a)k^2}{2} + \frac{\eta - \lambda_1}{2\lambda_1}\|v\|^2, \end{aligned}$$

for any $v \in H$ and any constant $k > 0$. Since $\alpha > 2$ and $\eta < \lambda_1$, $I(ke_1, v) \to -\infty$ for $k \to \infty$ or $\|v\| \to \infty$. Therefore we can choose $0 < R < \infty$ such that $I(ke_1, v) < 0$ for any $\|(ke_1, v)\|_E = R$. □

LEMMA 3.6. *If $a < \lambda_1$, then*

$$\sup_{Q_R(H_2,e_1^1)} I < +\infty.$$

Proof. If $\|(ke_1, v)\|_E \le R$, then $k \le R$ and $\|v\| \le R$. Proof of Lemma 3.5 implies that

$$\begin{aligned} I(ke_1, v) &\le \frac{(\lambda_1 - a)k^2}{2} - \frac{1}{2}\|v\|^2 + \frac{\eta}{2}\int_\Omega (v^-)^2 dx - \int_\Omega F(x, ke_1, v)dx \\ &\le \frac{(\lambda_1 - a)k^2}{2} + \frac{\eta}{2}\int_\Omega (v^-)^2 dx \\ &\le \frac{(\lambda_1 - a)k^2}{2} + \frac{\eta}{2\lambda_1}\|v\|^2, \\ &\le \left(\frac{(\lambda_1 - a)}{2} + \frac{\eta}{2\lambda_1}\right) R^2 < +\infty. \end{aligned}$$

Hence the proof is complete. □

Proof of Theorem 1.2. By Lemma 3.3, 3.4, 3.5 and 3.6, there exists $0 < \rho < R$ such that

$$\sup_{\partial Q_R(H_2,e_1^1)} I \le 0 < \inf_{\partial B_\rho(H_1)} I,$$

and

$$\sup_{Q_R(H_2,e_1^1)} I < +\infty \qquad \text{and} \qquad \inf_{B_\rho(H_1)} I \ge 0 > -\infty.$$

By Theorem 2.1, $I(u, v)$ has at least two nonzero critical values c_1, c_2

$$\inf_{B_\rho(H_1)} I \le c_1 \le \sup_{\partial Q_R(H_2,e_1^1)} I < \inf_{\partial B_\rho(H_1)} I \le c_2 \le \sup_{Q_R(H_2,e_1^1)} I.$$

Since $\sup_{\partial Q_R(H_2,e_1^1)} I \le 0$ and $\inf_{B_\rho(H_1)} I \ge 0$, $c_1 = 0$. Therefore, (1) has at least one nontrivial solutions. □

References

[1] A. sSzulkin, *Critical point theory of Ljusterni-Schnirelmann type and applications to partial differential equations,* Sem. Math. Sup. **107**, 35-96, Presses Univ. Montreal, Montreal, QC, 1989.

[2] D. D. Hai, *On a class of semilinear elliptic systems,* J. Math. Anal. Appl. **285** (2003), 477-486.

[3] D. Lupo and A. M. Micheletti, *Two applications of a three critical points theorem,* J. Differential Equations, **132** (1996), no. 2, 222-238.

[4] E. Massa, *Multiplicity results for a subperlinear elliptic system with partial interference with the spectrum,* Nonlinear Anal. **67** (2007), 295-306.

[5] H. Nam, *Multiplicity results for the elliptic system using the minimax theorem,* Korean J. Math. **16** (2008), no. 4, 511-526.
[6] J. Yinghua, *Nontrivial solutions of nonlinear elliptic equations and elliptic systems,* Inha U. 2004.
[7] T. Jung and Q. H. Choi, *Multiple solutions result for the mixed type nonlinear elliptic problem,* Korean J. Math. **19** (2011), 423-436.
[8] Y. An, *Mountain pass solutions for the coupled systems of second and fourth order elliptic equations,* Nonlinear Anal. **63** (2005), 1034-1041.
[9] W. Zou, *Multiple solutions for asymptotically linear elliptic systems,* J. Math. Anal. Appl. **255** (2001), no. 1, 213-229.

*
Department of General Education
Namseoul University
Cheonan 31020, Republic of Korea
E-mail: hwnam@nsu.ac.kr