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SOLUTIONS OF HIGHER ORDER INHOMOGENEOUS PERIODIC EVOLUTIONARY PROCESS

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Abstract. Let $\{U(t,s)\}_{t\geq s}$ be a periodic evolutionary process with period $\tau > 0$ on a Banach space X. Also, let L be the generator of the evolution semigroup associated with $\{U(t,s)\}_{t\geq s}$ on the phase space $P_\tau(X)$ of all τ-periodic continuous X-valued functions. Some kind of variation-of-constants formula for the solution u of the equation $(\alpha I - L)^n u = f$ will be given together with the conditions on $f \in P_\tau(X)$ for the existence of coefficients in the formula involving the monodromy operator $U(0, -\tau)$. Also, examples of ODEs and PDEs are presented as its application.

1. Introduction

Let X be a Banach space and $P_\tau(X)$ the phase space of all τ-periodic continuous X-valued functions. Let $\{U(t,s)\}_{t\geq s}$ be a periodic evolutionary process with period $\tau > 0$ on X. We denote by L the generator of the evolution semigroup $\{T^h\}_{h\geq 0}$ (see Section 2, and [3], [6], [10], etc.) on the phase space $P_\tau(X)$ associated with $\{U(t,s)\}_{t\geq s}$ on X. The representation of the solution $u \in P_\tau(X)$ to the homogeneous linear equation $(\alpha I - L)^n u = 0$ is exactly given in [9].

The purpose of this paper is to solve the inhomogeneous linear equation for $u \in P_\tau(X)$ of the form

$$(\alpha I - L)^n u = f, \tag{1.1}$$

where $f \in P_\tau(X)$. In other words, we obtain the range $(\alpha I - L)^n(P_\tau(X))$. Setting $U_\alpha(t,s) = e^{-\alpha(t-s)}U(t,s)$ and $V_\alpha(0) = U_\alpha(\tau, 0)$, we show that a solution

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$u \in \mathcal{D}(L^n) \subset P_\tau(X)$ of the equation (1.1) is represented as

$$u(t) = U_\alpha(t,0)\sum_{j=0}^{n-1}\frac{t^j}{j!}w_j + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}U_\alpha(t,s)f(s)ds, \quad t \geq 0 \tag{1.2}$$

with a solution $[w_0, w_1, \ldots, w_{n-1}]$ of the equations

$$(I - V_\alpha(0))w_{n-k} = V_\alpha(0)\sum_{j=1}^{k-1}\frac{\tau^j}{j!}w_{n-k+j} + \int_0^\tau \frac{(\tau-s)^{k-1}}{(k-1)!}U_\alpha(\tau,s)f(s)ds \qquad (k = 1, 2, \ldots, n) \tag{1.3}$$

(see Theorem 2.6) in Section 2. This result is proved by transforming the equation (1.1) to the equivalent system of the first order equations. The representation (1.2) is regarded as the variation-of-constants formula to the equation (1.1). The equation (1.3) is called the coefficient equation for the solution $u(t)$. In Section 3 we show that $w_1, \ldots, w_{n-1}$ in the solution (1.2) are determined by a solution w_0 satisfying the equation

$$\begin{aligned}(I - V_\alpha(0))^n w_0 = &\sum_{i=0}^{n-1}(-1)^i i!(I - V_\alpha(0))^{n-1-i} \\ &\times \sum_{j=i}^{n-1}\left\{ {j+1 \atop i+1} \right\}\frac{(-\tau)^j}{j!}\int_0^\tau \frac{(\tau-s)^{n-j-1}}{(n-j-1)!}U_\alpha(\tau,s)f(s)ds,\end{aligned}$$

where $\left\{ {a \atop b} \right\}$ stands for the Stirling number of the second kind (see Theorem 3.4). In this manner we obtain a representation of solutions to the equation (1.1) by substituting these values $w_1, \ldots, w_{n-1}$ in (1.2) (see Theorem 3.5). A necessary and sufficient condition (see Corollary 3.6) for f to be in the range of $(\alpha I - L)^n$ will be derived from the procedure of solving the equation (1.1). However, its condition is obtained by using a linear operator of a complicated form. So, we discuss sufficient conditions (see Corollaries 3.7 and 3.8) on f, which is simpler than the original condition. In addition, we shall illustrate the above-mentioned results by the simple examples for ODEs and PDEs in Section 4.

This paper is a sequel to the previous work [9] in which only the solutions of the homogeneous equation $(\alpha I - L)^n u = 0$ are obtained. In the case $f = 0$, the formulae (1.2) and (1.3) are reduced to the corresponding formulae in that previous results. The original motivation to solve the homogeneous equation $(\alpha I - L)^n u = 0$ is to show that, if 1 is a normal eigenvalue of $V(0)$, then 0 is also a normal eigenvalue of L. This result leads to the existence of the right inverse of $-L$ which is employed to discuss the perturbation of the degenerate periodic solution of the nonlinear oscillation problem $dx/dt = A(t)x + \epsilon f(t, x, \epsilon)$ (see [8] for the detail).

2. Coefficient equation for the solution $u(t)$

In this section we give conditions on the existence of solutions and a representation of solutions of the equation $(\alpha I - L)^n u = f \in P_\tau(X)$. A family of bounded linear operators $\{U(t,s)\}_{t\geq s}, (t, s \in \mathbb{R})$ from a Banach space X to itself is called τ-periodic (strongly continuous) evolutionary process ([3], [6], etc.) if the following conditions are satisfied

(1) $U(t,t) = I$ for all $t \in \mathbb{R}$, I the identity operator.
(2) $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r$,
(3) The map $(t,s) \mapsto U(t,s)x$ is continuous for every fixed $x \in X$,
(4) $U(t+\tau, s+\tau) = U(t,s)$ for all $t \geq s$,
(5) $\|U(t,s)\| \leq M_w e^{w(t-s)}$ for some $M_w \geq 1$ and $w \in \mathbb{R}$ which are independent of $t \geq s$.

The family of operators defined by

$$V(t) = U(t, t-\tau), \quad t \in \mathbb{R},$$

is called monodromy operator, period map, or Poincaré map ([6]). Associated with $\{U(t,s)\}_{t\geq s}$, we can define a C_0-semigroup $\{T^h\}_{h\geq 0}$ on $P_\tau(X)$ by

$$(T^h u)(t) = U(t, t-h)u(t-h) \quad \text{for} \quad u \in P_\tau(X), \quad t \in \mathbb{R}, \quad h \geq 0.$$

Let L be its generator. Then $u \in \mathcal{D}(L)$ and $-Lu = f$ if and only if $u, f \in P_\tau(X)$ and

$$u(t) = U(t,s)u(s) + \int_s^t U(t,r)f(r)dr$$

for any $t, s \in \mathbb{R}$ with $t \geq s$, that is, u is a mild solution.

For $\alpha \in \mathbb{C}$, we define the bounded operators $U_\alpha(t,s)$ by

$$U_\alpha(t,s) = e^{-\alpha(t-s)}U(t,s) \ (t \geq s), \tag{2.1}$$

and $V_\alpha(t)$ by

$$V_\alpha(t) = U_\alpha(t, t-\tau).$$

Note that $U_0(t,s) = U(t,s)$ and $V_0(t) = V(t)$. Then it is easy to verify that $\{U_\alpha(t,s)\}_{t\geq s}$ is a τ-periodic evolutionary process. Denote by L_α the generator of the evolution semigroup $\{T_\alpha^h\}_{h\geq 0}$ defined by

$$(T_\alpha^h u)(t) = U_\alpha(t, t-h)u(t-h), \ u \in P_\tau(X), \ t \in \mathbb{R}, \ h \geq 0.$$

Then $\mathcal{D}(L_\alpha) = \mathcal{D}(L)$ and $L_\alpha = L - \alpha I$, cf. [8, Lemma 3.9].

Now we introduce an integral operator M_α defined by

$$(M_\alpha f)(t) = \int_0^t U_\alpha(t,s)f(s)ds.$$

The following result is found in several literatures, e.g., [4], [9, Lemma 2.1 and Corollary 2.3], etc.

Lemma 2.1. *Let $f \in P_\tau(X)$. Then the solution $u \in \mathcal{D}(L) \subset P_\tau(X)$ of the equation*

$$(\alpha I - L)u = f \tag{2.2}$$

is given by

$$u(t) = U_\alpha(t,0)w + M_\alpha f(t), \quad t \geq 0 \tag{2.3}$$

with a solution w of the equation

$$(I - V_\alpha(0))w = M_\alpha f(\tau). \tag{2.4}$$

If $f \in P_\tau(X)$ in the equation (2.2) is given by $f(t) = U_\alpha(t,0)g(t),\ t \geq 0$, by a continuous function $g : [0,\infty) \to X$, then the equation (2.3) is reduced to the equation

$$u(t) = U_\alpha(t,0)w + U_\alpha(t,0)\int_0^t g(r)dr, \quad t \geq 0,$$

and the equation (2.4) to the equation

$$(I - V_\alpha(0))w = V_\alpha(0)\int_0^\tau g(r)dr. \tag{2.5}$$

The following lemma is almost trivial.

Lemma 2.2. *If u is a solution of* (1.1)*, then $[u_0, u_1, \ldots, u_{n-1}]$ defined as $u_j = (\alpha I - L)^j u, j = 0, \ldots, n-1$ is a solution of the system*

$$\begin{cases} (\alpha I - L)u_0 = u_1 \\ (\alpha I - L)u_1 = u_2 \\ \cdots \\ (\alpha I - L)u_{n-2} = u_{n-1} \\ (\alpha I - L)u_{n-1} = f \end{cases} \tag{2.6}$$

and vice versa.

To solve the system, we prepare the following lemma. For simplicity, we use the notation $M_\alpha^k f(t)$ instead of $(M_\alpha^k f)(t), k = 0, 1, \ldots$, which is defined by

$$M_\alpha^k f(t) = M_\alpha(M_\alpha^{k-1}f)(t) = \int_0^t U_\alpha(t,s)(M_\alpha^{k-1}f)(s)ds,\ k \geq 1,$$
$$M_\alpha^1 f(t) = M_\alpha f(t), \text{ and } M_\alpha^0 f(t) = f(t).$$

Lemma 2.3.

$$M_\alpha^k f(t) = \int_0^t \frac{(t-s)^{k-1}}{(k-1)!}U_\alpha(t,s)f(s)ds \quad (k = 1, 2, \ldots). \tag{2.7}$$

Proof. We can prove this lemma by induction on $k = 1, 2, \ldots$. □

We promise hereafter that

$$\sum_{k=i+1}^{i} a_k = 0,\ a_k \in X. \tag{2.8}$$

Proposition 2.4. *The solution* $[u_0, u_1, \ldots, u_{n-1}]$ *of the system* (2.6) *is given by*

$$u_{n-k}(t) = U_\alpha(t,0)\sum_{j=0}^{k-1}\frac{t^j}{j!}w_{n-k+j} + M_\alpha^k f(t) \text{ for } t \geq 0 \quad (k = 1, 2, \ldots, n) \tag{2.9}$$

with a solution $[w_0, w_1, \ldots, w_{n-1}]$ *of*

$$(I - V_\alpha(0))w_{n-k} = V_\alpha(0)\sum_{j=1}^{k-1}\frac{\tau^j}{j!}w_{n-k+j} + M_\alpha^k f(\tau) \qquad (k = 1, 2, \ldots, n). \tag{2.10}$$

Proof. Consider the system

$$EL(k) \ : \ \begin{cases} (\alpha I - L)u_{n-k} = u_{n-k+1} \\ \cdots \\ (\alpha I - L)u_{n-2} = u_{n-1} \\ (\alpha I - L)u_{n-1} = f \end{cases}$$

and the system

$$EV(k) \ : \ (I - V_\alpha(0))w_{n-i} = V_\alpha(0)\sum_{j=1}^{i-1}\frac{\tau^j}{j!}w_{n-i+j} + M_\alpha^i f(\tau) \ (i = 1, 2, \ldots, k)$$

for all $k, 1 \leq k \leq n$. By induction we can prove that the following statement $ST(k)$ holds for all $k, 1 \leq k \leq n$:
$ST(k)$: The solution $[u_{n-k}, u_{n-k+1}, \ldots, u_{n-1}]$ of $EL(k)$ is given by

$$u_{n-i}(t) = U_\alpha(t,0)\sum_{j=0}^{i-1}\frac{t^j}{j!}w_{n-i+j} + M_\alpha^i f(t), \quad t \geq 0 \quad (i = 1, \ldots, k) \tag{2.11}$$

with a solution $[w_{n-k}, w_{n-k+1}, \ldots, w_{n-1}]$ of $EV(k)$. Proposition 2.4 follows from this result, and we omit the details since the argument is standard. □

Corollary 2.5. *The system* (2.6) *has a solution* $[u_0, \ldots, u_{n-1}]$ *if and only if the system* (2.10) *has a solution* $[w_0, \ldots, w_{n-1}]$.

Combining Lemma 2.2 and Proposition 2.4, we obtain the following result.

Theorem 2.6. *The solution* u *of the equation* (1.1) *is represented as*

$$u(t) = U_\alpha(t,0)\sum_{j=0}^{n-1}\frac{t^j}{j!}w_j + \int_0^t \frac{(t-r)^{n-1}}{(n-1)!}U_\alpha(t,r)f(r)dr, \quad t \geq 0 \tag{2.12}$$

with a solution $[w_0, w_1, \ldots, w_{n-1}]$ *of the equations* (2.10).

3. Representation of solutions to the equation (1.1)

From Theorem 2.6 we will solve the equation (2.10) to obtain the solution (2.12) of the equation (1.1). As a result, any solution $[w_0, w_1, \dots, w_{n-1}]$ of the equation (2.10) will be represented by w_0 such that

$$(I - V_\alpha(0))^n w_0 = \sum_{i=0}^{n-1} (-1)^i i! (I - V_\alpha(0))^{n-1-i} \sum_{j=i}^{n-1} \begin{Bmatrix} j+1 \\ i+1 \end{Bmatrix} \frac{(-\tau)^j}{j!} M_\alpha^{n-j} f(\tau).$$

To deal with the equation (2.10), we introduce the following notations:

$$J = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$W = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{n-2} \\ w_{n-1} \end{bmatrix} \quad \text{and} \quad \mathbf{M}_\alpha f(\tau) = \begin{bmatrix} M_\alpha^n f(\tau) \\ M_\alpha^{n-1} f(\tau) \\ \vdots \\ M_\alpha^2 f(\tau) \\ M_\alpha^1 f(\tau) \end{bmatrix}.$$

Then

$$e^{\tau J} = \begin{bmatrix} 1 & \tau & \frac{\tau^2}{2!} & \cdots & \frac{\tau^{n-3}}{(n-3)!} & \frac{\tau^{n-2}}{(n-2)!} & \frac{\tau^{n-1}}{(n-1)!} \\ 0 & 1 & \tau & \cdots & \frac{\tau^{n-4}}{(n-4)!} & \frac{\tau^{n-3}}{(n-3)!} & \frac{\tau^{n-2}}{(n-2)!} \\ 0 & 0 & 1 & \cdots & \frac{\tau^{n-5}}{(n-5)!} & \frac{\tau^{n-4}}{(n-4)!} & \frac{\tau^{n-3}}{(n-3)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & \tau & \frac{\tau^2}{2!} \\ 0 & 0 & 0 & \cdots & 0 & 1 & \tau \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

If $H : X \to X$ is a linear operator, we write

$$[H]W = \begin{bmatrix} Hw_0 \\ Hw_1 \\ \vdots \\ Hw_{n-1} \end{bmatrix} \quad \text{for} \quad W = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{n-1} \end{bmatrix}, \tag{3.1}$$

that is, $[H]W = \mathrm{diagonal}(H, H, \dots, H)W$ by the standard notation.

By these notations, the system (2.10) is rewritten as

$$[(I - V_\alpha(0))]W = (e^{\tau J} - I)[V_\alpha(0)]W + \mathbf{M}_\alpha f(\tau). \tag{3.2}$$

This is equivalent to the system $W = e^{\tau J}[V_\alpha(0)]W + \mathbf{M}_\alpha f(\tau)$, that is,

$$[V_\alpha(0)]W = e^{-\tau J} W - e^{-\tau J} \mathbf{M}_\alpha f(\tau).$$

Then (3.2) becomes

$$[(I - V_\alpha(0))]W = (I - e^{-\tau J})W + e^{-\tau J}\mathbf{M}_\alpha f(\tau), \tag{3.3}$$

in which

$$I - e^{-\tau J} = \begin{bmatrix} 0 & \tau & -\frac{(-\tau)^2}{2!} & \cdots & -\frac{(-\tau)^{n-3}}{(n-3)!} & -\frac{(-\tau)^{n-2}}{(n-2)!} & -\frac{(-\tau)^{n-1}}{(n-1)!} \\ 0 & 0 & \tau & \cdots & -\frac{(-\tau)^{n-4}}{(n-4)!} & -\frac{(-\tau)^{n-3}}{(n-3)!} & -\frac{(-\tau)^{n-2}}{(n-2)!} \\ 0 & 0 & 0 & \cdots & -\frac{(-\tau)^{n-5}}{(n-5)!} & -\frac{(-\tau)^{n-4}}{(n-4)!} & -\frac{(-\tau)^{n-3}}{(n-3)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \tau & -\frac{(-\tau)^2}{2!} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \tau \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

To solve the system (2.10), we need the Stirling numbers. We introduce the factorial function $(x)_k$ of degree k defined by

$$(x)_k = x(x-1)\cdots(x-k+1) \quad \text{for} \quad k \geq 1,$$

and by $(x)_0 = 1$. Then $(x)_0, (x)_1, \ldots, (x)_n$ is a basis of the space of polynomials of degree $\leq n$. The Stirling numbers are defined as coefficients in the transform between this basis and the basis $1, x, \ldots, x^n$ as follows. The Stirling numbers of the first kind, denoted by $\left[{n \atop k}\right]$ for $k = 0, \ldots, n$, are defined as

$$(x)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \tag{3.4}$$

and the Stirling numbers of the second kind, denoted by $\left\{{n \atop k}\right\}$ for $k = 0, \ldots, n$, are defined as

$$x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (x)_k.$$

The following formulae are well known.

$$\sum_{i=k}^{n} \begin{Bmatrix} n \\ i \end{Bmatrix} \begin{bmatrix} i \\ k \end{bmatrix} = \delta_{nk}, \quad \sum_{i=k}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \begin{Bmatrix} i \\ k \end{Bmatrix} = \delta_{nk}, \tag{3.5}$$

$$\begin{Bmatrix} n+1 \\ m \end{Bmatrix} = \begin{Bmatrix} n \\ m-1 \end{Bmatrix} + m \begin{Bmatrix} n \\ m \end{Bmatrix}, \tag{3.6}$$

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{Bmatrix} n \\ 0 \end{Bmatrix} = 0 \quad (n \geq 1). \tag{3.7}$$

The following lemma is found in [1, Sec. 2.1, Chap. 2].

Lemma 3.1.

$$\frac{(e^z - 1)^k}{k!} = \sum_{n=k}^{\infty} \begin{Bmatrix} n \\ k \end{Bmatrix} \frac{z^n}{n!}.$$

The following lemma is prepared in [9, Lemmas 4.1 and 4.3].

Lemma 3.2.

$$\sum_{k=m}^{\ell-1} \begin{bmatrix} k \\ m \end{bmatrix} (k+1) \begin{Bmatrix} \ell \\ k+1 \end{Bmatrix} = \sum_{k=m}^{\ell} \begin{bmatrix} k \\ m \end{bmatrix} \begin{Bmatrix} \ell+1 \\ k+1 \end{Bmatrix} = \binom{\ell}{m} \quad (0 \le m \le \ell). \tag{3.8}$$

Also, the formula (3.5) yields the inversion formula of (3.8).

Corollary 3.3.

$$\begin{Bmatrix} \ell+1 \\ i+1 \end{Bmatrix} = \sum_{m=i}^{\ell} \binom{\ell}{m} \begin{Bmatrix} m \\ i \end{Bmatrix} \quad (0 \le i \le \ell). \tag{3.9}$$

Proof. By Lemma 3.2 we have

$$\sum_{m=i}^{\ell} \begin{Bmatrix} m \\ i \end{Bmatrix} \binom{\ell}{m} = \sum_{m=i}^{\ell} \begin{Bmatrix} m \\ i \end{Bmatrix} \sum_{k=m}^{\ell} \begin{bmatrix} k \\ m \end{bmatrix} \begin{Bmatrix} \ell+1 \\ k+1 \end{Bmatrix}.$$

The right hand side becomes

$$\begin{aligned} \sum_{m=i}^{\ell} \begin{Bmatrix} m \\ i \end{Bmatrix} \sum_{k=m}^{\ell} \begin{bmatrix} k \\ m \end{bmatrix} \begin{Bmatrix} \ell+1 \\ k+1 \end{Bmatrix} &= \sum_{k=i}^{\ell} \left(\sum_{m=i}^{k} \begin{bmatrix} k \\ m \end{bmatrix} \begin{Bmatrix} m \\ i \end{Bmatrix} \right) \begin{Bmatrix} \ell+1 \\ k+1 \end{Bmatrix} \\ &= \sum_{k=i}^{\ell} \delta_{ki} \begin{Bmatrix} \ell+1 \\ k+1 \end{Bmatrix} = \begin{Bmatrix} \ell+1 \\ i+1 \end{Bmatrix}, \end{aligned}$$

which proves the corollary. □

The following result is a key lemma in this paper. It gives a necessary and sufficient condition in order that $[w_0, w_1, \ldots, w_{n-1}]$ is a solution of the equation (2.10), or (3.3). To describe the lemma, we introduce a notation such that

$$H_{\alpha,n}(k)f = \sum_{i=0}^{k-1} (-1)^i i! (I - V_\alpha(0))^{k-1-i} \sum_{j=i}^{n-1} \begin{Bmatrix} j+1 \\ i+1 \end{Bmatrix} \frac{(-\tau)^j}{j!} M_\alpha^{n-j} f(\tau) \tag{3.10}$$

for $k = 1, \ldots, n$.

Theorem 3.4. *$[w_0, w_1, \ldots, w_{n-1}]$ is a solution of the equation* (2.10) *if and only if*

$$(I - V_\alpha(0))^n w_0 = H_{\alpha,n}(n)f \tag{3.11}$$

and

(3.12)

$$w_m = \frac{m!}{(-\tau)^m} \sum_{k=m}^{n-1} \begin{bmatrix} k \\ m \end{bmatrix} \frac{(-1)^k}{k!} \left[(I - V_\alpha(0))^k w_0 - H_{\alpha,n}(k)f \right] \quad (m = 1, \ldots, n-1).$$

Proof. We will solve the equation (3.3). For the simplicity of notation we set $I - V_\alpha(0) = A, I - e^{-\tau J} = B$ and

$$e^{-\tau J} \mathbf{M}_\alpha f(\tau) = G(\tau) = {}^t[g_0(\tau), g_1(\tau), \ldots, g_{n-1}(\tau)],$$

that is,

$$g_\ell(\tau) = \sum_{j=\ell}^{n-1} \frac{(-\tau)^{j-\ell}}{(j-\ell)!} M_\alpha^{n-j} f(\tau) \quad (0 \le \ell \le n-1).$$

Then, the equation (3.3) is rewritten as

$$[A]W = BW + G(\tau).$$

Denote by b_{ij} the (i,j) component of the $n \times n$ matrix B temporary. Then for any $V = {}^t[v_1, \dots, v_n] \in X^n$, the i-th component of $[A]BV$ is equal to $A\sum_{j=1}^n b_{ij}v_j = \sum_{j=1}^n b_{ij}Av_j$, that is, we can write $[A](BV) = B[A]V$. Thus

$$\begin{aligned}[A^2]W &= [A][A]W = [A](BW + G(\tau)) = B[A]W + [A]G(\tau)\\ &= B(BW + G(\tau)) + [A]G(\tau) = B^2W + BG(\tau) + [A]G(\tau).\end{aligned}$$

In the same manner, we have the following result by induction:

$$[A^k]W = B^kW + \left(\sum_{i=0}^{k-1}[A^{k-1-i}]B^i\right)G(\tau) \quad (1 \le k \le n).$$

Since $J^\ell = 0$ for $\ell \ge n$, it follows from Lemma 3.1 that

$$B^i = (I - e^{-\tau J})^i = (-1)^i i! \sum_{\ell=i}^{n-1} \begin{Bmatrix} \ell \\ i \end{Bmatrix} \frac{1}{\ell!}(-\tau J)^\ell \quad (1 \le i \le n).$$

Thus

$$\begin{aligned}[A^k]W &= (-1)^k k! \sum_{\ell=k}^{n-1} \begin{Bmatrix} \ell \\ k \end{Bmatrix} \frac{1}{\ell!}(-\tau J)^\ell W\\ &\quad + \sum_{i=0}^{k-1}(-1)^i i! [A^{k-1-i}] \sum_{\ell=i}^{n-1} \begin{Bmatrix} \ell \\ i \end{Bmatrix} \frac{1}{\ell!}(-\tau J)^\ell G(\tau) \quad (1 \le k \le n).\end{aligned}$$

We observe the first component of this vector equation. Since the first component of the vector $J^\ell W$ is w_ℓ, it follows that

$$A^k w_0 = (-1)^k k! \sum_{\ell=k}^{n-1} \begin{Bmatrix} \ell \\ k \end{Bmatrix} \frac{(-\tau)^\ell}{\ell!} w_\ell + v(k),$$

where

$$v(k) = \sum_{i=0}^{k-1}(-1)^i i! A^{k-1-i} \sum_{\ell=i}^{n-1} \begin{Bmatrix} \ell \\ i \end{Bmatrix} \frac{(-\tau)^\ell}{\ell!} g_\ell(\tau).$$

From the definition of $g_\ell(\tau)$ we have

$$\sum_{\ell=i}^{n-1} \begin{Bmatrix} \ell \\ i \end{Bmatrix} \frac{(-\tau)^\ell}{\ell!} g_\ell(\tau) = \sum_{\ell=i}^{n-1} \begin{Bmatrix} \ell \\ i \end{Bmatrix} \frac{(-\tau)^\ell}{\ell!} \sum_{j=\ell}^{n-1} \frac{(-\tau)^{j-\ell}}{(j-\ell)!} M_\alpha^{n-j} f(\tau)$$

$$= \sum_{j=i}^{n-1} \sum_{\ell=i}^{j} \binom{j}{\ell} \left\{ {\ell \atop i} \right\} \frac{(-\tau)^j}{j!} M_\alpha^{n-j} f(\tau)$$

$$= \sum_{j=i}^{n-1} \left\{ {j+1 \atop i+1} \right\} \frac{(-\tau)^j}{j!} M_\alpha^{n-j} f(\tau)$$

by Corollary 3.3. Hence $v(k) = H_{\alpha,n}(k)f$ which is defined by (3.10), so that

$$A^k w_0 = (-1)^k k! \sum_{\ell=k}^{n-1} \left\{ {\ell \atop k} \right\} \frac{(-\tau)^\ell}{\ell!} w_\ell + H_{\alpha,n}(k)f \quad (k = 1, \dots, n). \tag{3.13}$$

We separate this relation into the two parts according to the value of k as follows.

$$A^n w_0 = H_{\alpha,n}(n)f, \tag{3.14}$$

$$\frac{(-1)^k}{k!}(A^k w_0 - H_{\alpha,n}(k)f) = \sum_{\ell=k}^{n-1} \left\{ {\ell \atop k} \right\} \frac{(-\tau)^\ell}{\ell!} w_\ell \quad (k = 1, \dots, n-1). \tag{3.15}$$

Obviously, (3.14) is (3.11). From (3.15) and the first equation in (3.5), it follows that

$$\begin{aligned} \sum_{k=m}^{n-1} \left[{k \atop m} \right] \frac{(-1)^k}{k!}(A^k w_0 - H_{\alpha,n}(k)f) &= \sum_{k=m}^{n-1} \left[{k \atop m} \right] \sum_{\ell=k}^{n-1} \left\{ {\ell \atop k} \right\} \frac{(-\tau)^\ell}{\ell!} w_\ell \\ &= \sum_{\ell=m}^{n-1} \sum_{k=m}^{\ell} \left[{k \atop m} \right] \left\{ {\ell \atop k} \right\} \frac{(-\tau)^\ell}{\ell!} w_\ell \\ &= \frac{(-\tau)^m}{m!} w_m \quad (m = 1, \dots, n-1) \end{aligned}$$

that is,

$$\frac{(-\tau)^m}{m!} w_m = \sum_{k=m}^{n-1} \left[{k \atop m} \right] \frac{(-1)^k}{k!}(A^k w_0 - H_{\alpha,n}(k)f), \tag{3.16}$$

so that (3.12) holds.

Conversely, assume that w_0 satisfies (3.11) or (3.14) and that $w_1, w_2, \dots, w_{n-1}$ are defined as in (3.12) by this w_0. Multiply the both side of (3.16) by $\left\{ {m \atop \ell} \right\}$, and take the sum for m from ℓ up to n. Then we obtain (3.15) by using the second formula in (3.5). Thus $w_0, w_1, w_2, \dots, w_{n-1}$ satisfy (3.13).

We will check that ${}^t[w_0, w_1, \dots, w_{n-1}]$ satisfies the equation (3.3). Since $\left\{ {\ell \atop 1} \right\} = 1$ for $\ell \geq 1$, the equation (3.13) for $k = 1$ or (3.11) becomes

$$Aw_0 = -\sum_{\ell=1}^{n-1} \left\{ {\ell \atop 1} \right\} \frac{(-\tau)^\ell}{\ell!} w_\ell + \sum_{j=0}^{n-1} \left\{ {j+1 \atop 1} \right\} \frac{(-\tau)^j}{j!} M_\alpha^{n-j} f(\tau)$$

$$= -\sum_{\ell=1}^{n-1} \frac{(-\tau)^\ell}{\ell!} w_\ell + \sum_{j=0}^{n-1} \frac{(-\tau)^j}{j!} M_\alpha^{n-j} f(\tau).$$

Thus the first components of the both sides in the equation (3.3) are the same.

We will check the components after the second in the equation (3.3). Let $1 \le m \le n-1$. Multiplying (3.12) or (3.16) by A, we have

$$\frac{(-\tau)^m}{m!} A w_m = \sum_{k=m}^{n-1} \begin{bmatrix} k \\ m \end{bmatrix} \frac{(-1)^k}{k!} (A^{k+1} w_0 - A H_{\alpha,n}(k) f).$$

From (3.13) and (3.10) it follows that

$$\begin{aligned} A^{k+1} w_0 - A H_{\alpha,n}(k) f = (-1)^{k+1}(k+1)! \sum_{\ell=k+1}^{n-1} \begin{Bmatrix} \ell \\ k+1 \end{Bmatrix} \frac{(-\tau)^\ell}{\ell!} w_\ell \\ + H_{\alpha,n}(k+1) f - A H_{\alpha,n}(k) f, \end{aligned}$$

and

$$\begin{aligned} H_{\alpha,n}(k+1) f - A H_{\alpha,n}(k) f &= \sum_{i=0}^{k} (-1)^i i! A^{k-i} \sum_{j=i}^{n-1} \begin{Bmatrix} j+1 \\ i+1 \end{Bmatrix} \frac{(-\tau)^j}{j!} M_\alpha^{n-j} f(\tau) \\ &\quad - \sum_{i=0}^{k-1} (-1)^i i! A^{k-i} \sum_{j=i}^{n-1} \begin{Bmatrix} j+1 \\ i+1 \end{Bmatrix} \frac{(-\tau)^j}{j!} M_\alpha^{n-j} f(\tau) \\ &= (-1)^k k! \sum_{j=k}^{n-1} \begin{Bmatrix} j+1 \\ k+1 \end{Bmatrix} \frac{(-\tau)^j}{j!} M_\alpha^{n-j} f(\tau). \end{aligned}$$

Thus

$$\begin{aligned} \frac{(-\tau)^m}{m!} A w_m &= \sum_{k=m}^{n-1} \begin{bmatrix} k \\ m \end{bmatrix} \frac{(-1)^k}{k!} \Bigg((-1)^{k+1}(k+1)! \sum_{\ell=k+1}^{n-1} \begin{Bmatrix} \ell \\ k+1 \end{Bmatrix} \frac{(-\tau)^\ell}{\ell!} w_\ell \\ &\qquad + (-1)^k k! \sum_{j=k}^{n-1} \begin{Bmatrix} j+1 \\ k+1 \end{Bmatrix} \frac{(-\tau)^j}{j!} M_\alpha^{n-j} f(\tau) \Bigg) \\ &= -\sum_{\ell=m+1}^{n-1} \left(\sum_{k=m}^{\ell-1} \begin{bmatrix} k \\ m \end{bmatrix} (k+1) \begin{Bmatrix} \ell \\ k+1 \end{Bmatrix} \right) \frac{(-\tau)^\ell}{\ell!} w_\ell \\ &\qquad + \sum_{j=m}^{n-1} \sum_{k=m}^{j} \begin{bmatrix} k \\ m \end{bmatrix} \begin{Bmatrix} j+1 \\ k+1 \end{Bmatrix} \frac{(-\tau)^j}{j!} M_\alpha^{n-j} f(\tau), \end{aligned}$$

so that, by Lemma 3.2,

$$A w_m = -\sum_{\ell=m+1}^{n-1} \left(\sum_{k=m}^{\ell-1} \begin{bmatrix} k \\ m \end{bmatrix} (k+1) \begin{Bmatrix} \ell \\ k+1 \end{Bmatrix} \right) \frac{(-\tau)^{\ell-m} m!}{\ell!} w_\ell$$

$$+\sum_{j=m}^{n-1}\sum_{k=m}^{j}\begin{bmatrix}k\\m\end{bmatrix}\begin{Bmatrix}j+1\\k+1\end{Bmatrix}\frac{(-\tau)^{j-m}m!}{j!}M_\alpha^{n-j}f(\tau)$$

$$=-\sum_{\ell=m+1}^{n-1}\binom{\ell}{m}\frac{(-\tau)^{\ell-m}m!}{\ell!}w_\ell+\sum_{j=m}^{n-1}\binom{j}{m}\frac{(-\tau)^{j-m}m!}{j!}M_\alpha^{n-j}f(\tau).$$

Thus we obtain that

$$Aw_m=-\sum_{\ell=m+1}^{n-1}\frac{(-\tau)^{\ell-m}}{(\ell-m)!}w_\ell+\sum_{j=m}^{n-1}\frac{(-\tau)^{j-m}}{(j-m)!}M_\alpha^{n-j}f(\tau)\quad(m=1,\ldots,n-1).$$

Thus the equation (3.3) holds: the proof is complete. □

Note that the equality

$$(t)_{\overline{k}}=\sum_{j=0}^{k}(-1)^{k+j}\begin{bmatrix}k\\j\end{bmatrix}t^j,\quad t\geq 0 \tag{3.17}$$

is derived from (3.4), where $(t)_{\overline{k}}=t(t+1)(t+2)\cdots(t+k-1),\ k\geq 1,\ (t)_{\overline{0}}=1$. Summing up Theorem 2.6, Theorem 3.4 and (3.17) and using the same argument as in the proof of [9, Theorem 2], we arrive at the following result.

Theorem 3.5. *Assume that a solution w_0 of the equation* (3.11) *exists. Then the solution u of the equation* (1.1) *with $u(0)=w_0$ is represented as*

$$\begin{aligned}u(t)=U_\alpha(t,0)\sum_{k=0}^{n-1}\left(\frac{t}{\tau}\right)_{\overline{k}}\frac{1}{k!}\Bigg[&(I-V_\alpha(0))^k w_0\\&-\sum_{i=0}^{k-1}(-1)^i i!(I-V_\alpha(0))^{k-1-i}\sum_{j=i}^{n-1}\begin{Bmatrix}j+1\\i+1\end{Bmatrix}\frac{(-\tau)^j}{j!}M_\alpha^{n-j}f(\tau)\Bigg]\\&+M_\alpha^n f(t),\quad t\geq 0.\end{aligned}\tag{3.18}$$

Corollary 3.6. *The equation* (1.1) *has a solution if and only if the equation* (3.11) *has a solution, that is, $H_{\alpha,n}(n)f\in\mathcal{R}((I-V_\alpha(0))^n)$.*

Finally, we give sufficient conditions on the existence of the solution w_0 to the equation (3.11). Note that for a bounded linear operator $A:X\to X$, one has $\mathcal{R}(A^{n+1})\subset\mathcal{R}(A^n), n=0,1,\ldots$.

Corollary 3.7. *Suppose that*

$$M_\alpha^{n-j}f(\tau)\in\mathcal{R}((I-V_\alpha(0))^{j+1}),\quad j=0,1,\ldots,n-1. \tag{3.19}$$

Then the equation (1.1) *has a solution.*

Proof. Suppose that (3.19) holds. Then for $j=0,1,\ldots,n-1$ there exists $x_j\in X$ such that $M_\alpha^{n-j}f(\tau)=A^{j+1}x_j$, so we have

$$H_{\alpha,n}(n)f=\sum_{j=0}^{n-1}\frac{(-\tau)^j}{j!}\left(\sum_{i=0}^{j}(-1)^i i!A^{j-i}\begin{Bmatrix}j+1\\i+1\end{Bmatrix}\right)A^{n-1-j}A^{j+1}x_j$$

$$= A^n \sum_{j=0}^{n-1} \frac{(-\tau)^j}{j!} \left(\sum_{i=0}^{j} (-1)^i i! A^{j-i} \left\{ \begin{matrix} j+1 \\ i+1 \end{matrix} \right\} \right) x_j.$$

Thus we have the corollary in the above from Corollary 3.6. □

Corollary 3.8. *Suppose that*

$$M_\alpha^{n-j} f(\tau) \in \mathcal{R}((I - V_\alpha(0))^n), \quad j = 0, 1, \ldots, n-1. \tag{3.20}$$

Then the equation (1.1) *has a solution.*

4. Applications to differential equations

We shall illustrate the above-mentioned results by the simple examples for ODEs and PDEs. For a closed linear operator $B : \mathcal{D}(B) \subset X \to X$ we denote by $\sigma(B), \sigma_p(B)$ and $\sigma_n(B)$ the spectrum of B, the point spectrum of B and the set of all normal eigenvalues (see [2] for the definition) of B, respectively.

4.1. An example of ODE

Let us consider 3-dimensional 2π-periodic linear systems

$$\left(\frac{d}{dt} - A(t) \right) u(t) = f(t), \tag{4.1}$$

and

$$\left(\frac{d}{dt} - A(t) \right) \left(\frac{d}{dt} - A(t) \right) u(t) = f(t), \tag{4.2}$$

where

$$A(t) = \begin{bmatrix} a & 0 & 0 \\ 0 & \sin t & 1 + \cos t \\ 0 & 0 & \sin t \end{bmatrix}, \ u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}, \ f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}.$$

We assume that $a \in \mathbb{R}, a \neq 0$, so that $e^{2\pi a} \neq 1$. Since $A(t)$ and $\int_s^t A(r)dr$ are commutative, the solution operator $U(t,s)$ of the equation $u'(t) = A(t)u(t)$ is given by

$$\begin{aligned} U(t,s) &= \exp\left(\int_s^t A(r)dr \right) \\ &= \begin{bmatrix} e^{a(t-s)} & 0 & 0 \\ 0 & e^{-\cos t + \cos s} & e^{-\cos t + \cos s}(t + \sin t - s - \sin s) \\ 0 & 0 & e^{-\cos t + \cos s} \end{bmatrix}. \end{aligned} \tag{4.3}$$

$\{U(t,s)\}_{t \geq s}$ is a 2π–periodic evolutionary process on $\mathbb{R}^3$. Clearly,

$$U(t,0) = \begin{bmatrix} e^{at} & 0 & 0 \\ 0 & e^{1-\cos t} & e^{1-\cos t}(t + \sin t) \\ 0 & 0 & e^{1-\cos t} \end{bmatrix}$$

and hence, the monodromy operator $V(0)$ is given by

$$V(0) = \begin{bmatrix} e^{2\pi a} & 0 & 0 \\ 0 & 1 & 2\pi \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.4}$$

Since $e^{2\pi a} \neq 0$, $\sigma_p(V(0)) = \{e^{2\pi a}, 1\}$ and
(4.5)

$$I - V(0) = \begin{bmatrix} 1-e^{2\pi a} & 0 & 0 \\ 0 & 0 & -2\pi \\ 0 & 0 & 0 \end{bmatrix}, \quad (I - V(0))^2 = \begin{bmatrix} (1-e^{2\pi a})^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus the ascent and the descent of $I - V(0)$ are 2.

Let L be the generator of the evolution semigroup $\{T^h\}_{h\geq 0}$ on $P_{2\pi}(\mathbb{R}^3)$ associated with this evolutionary process $\{U(t,s)\}_{t\geq s}$. Then the condition $-Lu = f$ is equivalent that

$$u(t) = U(t,s)u(s) + \int_s^t U(t,r)f(r)dr$$

for any t, s with $t \geq s$. In this case, this means that $u \in P_{2\pi}^{(1)}(\mathbb{R}^3)$ and $u'(t) = A(t)u(t) + f(t)$, that is, $u'(t) - A(t)u(t) = f(t)$. Namely, the equation (4.1) is equivalent to the equation $-Lu = f$; and the equation (4.2) is equivalent to the equation $L^2 u = f$.

For $w \in \mathbb{R}^3$, $w_j, j = 1,2,3$, denotes the j-th component of w. For example, $[M_0^k f(2\pi)]_j$ denotes the j-th component of $M_0^k f(2\pi)$.

To state the propositions, we compute $M_0 f(t)$, $M_0 f(2\pi)$, $M_0^2 f(t)$ and $M_0^2 f(2\pi)$ in advance as follows:

$$M_0 f(t) = \begin{bmatrix} \int_0^t e^{a(t-s)} f_1(s)ds \\ \int_0^t e^{-\cos t+\cos s}(f_2(s)+t+\sin t-s-\sin s)f_3(s))ds \\ \int_0^t e^{-\cos t+\cos s} f_3(s)ds \end{bmatrix},$$

$$M_0^2 f(t) = \begin{bmatrix} \int_0^t (t-s)e^{a(t-s)} f_1(s)ds \\ \int_0^t (t-s)e^{-\cos t+\cos s}(f_2(s)+(t+\sin t-s-\sin s)f_3(s))ds \\ \int_0^t (t-s)e^{-\cos t+\cos s} f_3(s)ds \end{bmatrix},$$

$$M_0 f(2\pi) = \begin{bmatrix} \int_0^{2\pi} e^{a(2\pi-s)} f_1(s)ds \\ \int_0^{2\pi} e^{-1+\cos s}(f_2(s)+(2\pi-s-\sin s)f_3(s))ds \\ \int_0^{2\pi} e^{-1+\cos s} f_3(s)ds \end{bmatrix}, \tag{4.6}$$

$$M_0^2 f(2\pi) = \begin{bmatrix} \int_0^{2\pi} (2\pi-s)e^{a(2\pi-s)} f_1(s)ds \\ \int_0^{2\pi} (2\pi-s)e^{-1+\cos s}(f_2(s)+(2\pi-s-\sin s)f_3(s))ds \\ \int_0^{2\pi} (2\pi-s)e^{-1+\cos s} f_3(s)ds \end{bmatrix}.$$

Proposition 4.1. *Let L be the generator of the evolution semigroup $\{T^h\}_{h\geq 0}$ on the space $P_{2\pi}(\mathbb{R}^3)$ associated with $\{U(t,s)\}_{t\geq s}$ given by* (4.3). *Then*

$$f \in \mathcal{R}(-L) \Longleftrightarrow [M_0 f(2\pi)]_3 = 0, \tag{4.7}$$

$$f \in \mathcal{R}(L^2) \Longleftrightarrow [M_0 f(2\pi)]_3 = 0 \quad \textit{and} \quad [M_0^2 f(2\pi)]_3 = [M_0 f(2\pi)]_2 \tag{4.8}$$

$$\Longleftrightarrow f \in \mathcal{R}((-L)^3), \tag{4.9}$$

and the descent of $-L$ is 2, *that is,*

$$\mathcal{R}(-L) \supsetneq \mathcal{R}((-L)^2) = \mathcal{R}((-L)^3).$$

In fact,

$$f \in \mathcal{R}(-L) \Longleftrightarrow \int_0^{2\pi} e^{\cos s} f_3(s)ds = 0, \tag{4.10}$$

(4.11)

$$f \in \mathcal{R}((-L)^2) = \mathcal{R}((-L)^3) \Longleftrightarrow \begin{cases} \int_0^{2\pi} e^{\cos s} f_3(s)ds = 0 \\ \int_0^{2\pi} e^{\cos s}(f_2(s) - \sin s f_3(s))ds = 0. \end{cases}$$

Proof. From Corollary 3.6, $f \in \mathcal{R}((-L)^n)$ if and only if $H_{0,n}(n)f \in \mathcal{R}((I - V(0))^n)$ for $n = 1, 2, \ldots$. Since $I - V(0)$ and $(I - V(0))^2$ are given as (4.5) and

$$(I - V(0))^3 = \begin{bmatrix} (1 - e^{2\pi a})^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we have

$$H_{0,1}(1)f \in \mathcal{R}(I - V(0)) \Longleftrightarrow [H_{0,1}(1)f]_3 = 0, \tag{4.12}$$

$$H_{0,2}(2)f \in \mathcal{R}((I - V(0))^2) \Longleftrightarrow [H_{0,2}(2)f]_j = 0 \quad (j = 2, 3), \tag{4.13}$$

$$H_{0,3}(3)f \in \mathcal{R}((I - V(0))^3) \Longleftrightarrow [H_{0,3}(3)f]_j = 0 \quad (j = 2, 3). \tag{4.14}$$

By the formula of $H_{\alpha,n}(k)$ given by (3.10), we have

$$H_{0,1}(1)f = M_0 f(2\pi), \tag{4.15}$$

$$H_{0,2}(2)f = AM_0^2 f(2\pi) + (-2\pi)(A - I)M_0 f(2\pi), \tag{4.16}$$

$$\begin{aligned} H_{0,3}(3)f &= \sum_{j=0}^{2} \frac{(-2\pi)^j}{j!} \left(\sum_{i=0}^{j} (-1)^i i! A^{j-i} \begin{Bmatrix} j+1 \\ i+1 \end{Bmatrix} \right) A^{2-j} M_0^{3-j} f(2\pi) \\ &= A^2 M_0^3 f(2\pi) + (-2\pi)(A - I)AM_0^2 f(2\pi) \\ &\quad + \frac{(2\pi)^2}{2!} \left(A^2 - 3A + 2I\right) M_0 f(2\pi), \end{aligned} \tag{4.17}$$

where $A = I - V(0)$.

At first, the equation (4.15) implies that

$$[H_{0,1}(1)f]_3 = [M_0 f(2\pi)]_3.$$

Thus the assertion (4.12) is rewritten as

$$H_{0,1}(1)f \in \mathcal{R}(I - V(0)) \Longleftrightarrow [M_0 f(2\pi)]_3 = 0.$$

The j-th component $[H_{0,2}(2)f]_j$ is given by

$$[H_{0,2}(2)f]_j = [AM_0^2 f(2\pi)]_j + 2\pi[V(0)M_0 f(2\pi)]_j.$$

Since the third row of $A = I - V(0)$ is $[0, 0, 0]$, we obtain that

$$[H_{0,2}(2)f]_3 = 2\pi[V(0)M_0 f(2\pi)]_3 = 2\pi[M_0 f(2\pi)]_3.$$

In view of $I - V(0)$ and $V(0)$, we have

$$[H_{0,2}(2)f]_2 = -2\pi[M_0^2 f(2\pi)]_3 + 2\pi\left([M_0 f(2\pi)]_2 + 2\pi[M_0 f(2\pi)]_3\right).$$

Thus the assertion (4.13) is rewritten as

$$\begin{aligned} &H_{0,2}(2)f \in \mathcal{R}((I - V(0))^2) \\ \Longleftrightarrow\ &[M_0 f(2\pi)]_3 = 0 \ \text{ and } \ [M_0^2 f(2\pi)]_3 = [M_0 f(2\pi)]_2. \end{aligned}$$

To obtain $[H_{0,3}(3)f]_j, j = 2, 3$, we observe that the second row and the third row of A^2, A^3 are $[0, 0, 0]$. Thus, for $j = 2, 3$,

$$\begin{aligned} [H_{0,3}(3)f]_j &= 2\pi[AM_0^2 f(2\pi)]_j + 2\pi^2[(-3A + 2I)M_0 f(2\pi)]_j \\ &= 2\pi[AM_0^2 f(2\pi)]_j + 2\pi^2[(3V(0) - I)M_0 f(2\pi)]_j. \end{aligned}$$

In case $j = 3$, we obtain

$$\begin{aligned} [H_{0,3}(3)f]_3 &= 2\pi^2[(3V(0) - I)M_0 f(2\pi)]_3 \\ &= 2\pi^2 \times 2 \times [M_0 f(2\pi)]_3 = 4\pi^2[M_0 f(2\pi)]_3, \end{aligned}$$

and in case $j = 2$,

$$\begin{aligned} [H_{0,3}(3)f]_2 &= (2\pi)(-2\pi)[M_0^2 f(2\pi)]_3 + 2\pi^2[(3V(0) - I)M_0 f(2\pi)]_2 \\ &= (2\pi)(-2\pi)[M_0^2 f(2\pi)]_3 + 2\pi^2\left(2[M_0 f(2\pi)]_2 + 6\pi[M_0 f(2\pi)]_3\right) \\ &= 4\pi^2\left(-[M_0^2 f(2\pi)]_3 + [M_0 f(2\pi)]_2\right) + 12\pi^3[M_0 f(2\pi)]_3. \end{aligned}$$

Thus the assertion (4.13) is rewritten as

$$\begin{aligned} &H_{0,3}(3)f \in \mathcal{R}((I - V(0))^3) \\ \Longleftrightarrow\ &[M_0 f(2\pi)]_3 = 0 \ \text{ and } \ [M_0^2 f(2\pi)]_3 = [M_0 f(2\pi)]_2. \end{aligned}$$

These results up to this point are summarized as in (4.7), (4.8) and (4.9). In particular, $-L$ has the descent ≤ 2.

In view of (4.6), we obtain

$$[M_0 f(2\pi)]_3 = 0 \Longleftrightarrow \int_0^{2\pi} e^{-1+\cos s} f_3(s)ds = 0. \tag{4.18}$$

This assertion together with (4.7) yields the assertion (4.10).

The condition $[M_0^2 f(2\pi)]_3 = [M_0 f(2\pi)]_2$ becomes

$$\begin{aligned}&\int_0^{2\pi} (2\pi - s)e^{-1+\cos s} f_3(s)ds \\ = &\int_0^{2\pi} e^{-1+\cos s}(f_2(s) + (2\pi - s - \sin s)f_3(s))ds,\end{aligned}$$

or

$$[M_0^2 f(2\pi)]_3 = [M_0 f(2\pi)]_2 \Longleftrightarrow \int_0^{2\pi} e^{-1+\cos s}(f_2(s) - \sin s f_3(s))ds = 0.$$

This assertion together with (4.8) and (4.9) yields the assertion (4.11).

Furthermore, if $f_3(s) \equiv 0$, then $[M_0 f(2\pi)]_3 = 0$. If $f_3(s) \equiv 0$ and $f_2(s) \equiv 1$, then $[M_0^2 f(2\pi)]_3 \neq [M_0 f(2\pi)]_2$. This implies that $\mathcal{R}(-L) \supsetneq \mathcal{R}(L^2)$, so that the descent of $-L$ is equal to 2. □

We check the sufficient conditions given in Corollary 3.7. If

$$M_\alpha^{n-j} f(\tau) \in \mathcal{R}((I - V_\alpha(0))^{j+1}), \quad j = 0, 1, \ldots, n-1,$$

then the equation (1.1) has a solution. If $n = 2$, $\alpha = 0$, then the condition (3.19) becomes

$$M_0^2 f(2\pi) \in \mathcal{R}((I - V(0))), \quad M_0 f(2\pi) \in \mathcal{R}((I - V(0))^2).$$

These conditions are equivalent to the conditions

$$[M_0^2 f(2\pi)]_3 = 0, \quad [M_0 f(2\pi)]_2 = 0, \quad [M_0 f(2\pi)]_3 = 0. \tag{4.19}$$

If $n = 3$, $\alpha = 0$, then the condition (3.19) becomes

$$M_0^3 f(2\pi) \in \mathcal{R}((I - V(0))), M_0^2 f(2\pi) \in \mathcal{R}((I - V(0))^2), M_0 f(2\pi) \in \mathcal{R}((I - V(0))^3).$$

Next we solve the equations $-Lu = f$, $(-L)^2 u = f$. At first the following result holds.

Proposition 4.2. *The solution u of the equation $-Lu = f$ for $f \in \mathcal{R}(-L)$ is given by $u(t) = U(t, 0)w_0 + M_0 f(t)$, where*

$$w_0 = \begin{bmatrix} \frac{1}{1-e^{2\pi a}} \int_0^{2\pi} e^{a(2\pi - s)} f_1(s)ds \\ c \\ -\frac{1}{2\pi} \int_0^{2\pi} e^{-1+\cos s}(f_2(s) - (s + \sin s)f_3(s))ds \end{bmatrix}$$

for any constant c. If $f \in \mathcal{R}(L^2)$, then

$$w_0 = \begin{bmatrix} \frac{1}{1-e^{2\pi a}} \int_0^{2\pi} e^{a(2\pi - s)} f_1(s)ds \\ c \\ \frac{1}{2\pi} \int_0^{2\pi} e^{-1+\cos s}(s f_3(s))ds \end{bmatrix}$$

for any constant c.

Proposition 4.3. *The solution u of $L^2u = f$ for $f \in \mathcal{R}(L^2)$ is given by $u(t) = U(t,0)(w_0 + tw_1) + M_0^2 f(t)$ with*

$$w_0 = \begin{bmatrix} \frac{1}{(1-e^{2\pi a})^2}\int_0^{2\pi}(2\pi - s + se^{2\pi a})e^{a(2\pi-s)}f_1(s)ds \\ c_1 \\ c_2 \end{bmatrix}, \tag{4.20}$$

$$w_1 = \begin{bmatrix} \frac{1}{1-e^{2\pi a}}\int_0^{2\pi} e^{a(2\pi-s)}f_1(s)ds \\ -c_2 + \frac{1}{2\pi}\int_0^{2\pi} se^{-1+\cos s}(f_2(s) + (2\pi - s - \sin s)f_3(s))ds \\ \frac{1}{2\pi}\int_0^{2\pi} se^{-1+\cos s}f_3(s)ds \end{bmatrix} \tag{4.21}$$

for any constants c_1 and c_2.

Proof. The solution u of $L^2u = f$ for $f \in \mathcal{R}(L^2)$ is given by Theorem 3.4 as

$$u(t) = U(t,0)(w_0 + tw_1) + M_0^2 f(t)$$

with w_0 and w_1 such that

$$(I - V(0))^2 w_0 = H_{0,2}(2)f, \quad w_1 = \frac{1}{2\pi}[(I - V(0))w_0 - H_{0,2}(1)f].$$

From the first equation $w_0 = {}^t[w_{10}, w_{20}, w_{30}]$ is given by

$$w_{10} = \frac{1}{(1-e^{2\pi a})^2}[H_{0,2}(2)f]_1, \quad w_{20} = c_1, w_{30} = c_2$$

for any constants c_1 and c_2. Thus w_0 is given by (4.20), since

$$\begin{aligned}[H_{0,2}(2)f]_1 &= [(I - V(0))M_0^2 f(2\pi)]_1 + 2\pi[V(0)M_0 f(2\pi)]_1 \\ &= (1 - e^{2\pi a})[M_0^2 f(2\pi)]_1 + 2\pi e^{2\pi a}[M_0 f(2\pi)]_1 \\ &= (1 - e^{2\pi a})\int_0^{2\pi}(2\pi - s)e^{a(2\pi-s)}f_1(s)ds \\ &\quad + 2\pi e^{2\pi a}\int_0^{2\pi} e^{a(2\pi-s)}f_1(s)ds \\ &= \int_0^{2\pi}(2\pi - s + se^{2\pi a})e^{a(2\pi-s)}f_1(s)ds.\end{aligned}$$

Since $H_{0,2}(1)f = M_0^2 f(2\pi) - 2\pi M_0 f(2\pi)$ from the definition (3.10),

$$w_1 = \frac{1}{2\pi}\left((I - V(0))w_0 - [M_0^2 f(2\pi) - 2\pi M_0 f(2\pi)]\right).$$

We set $w_1 = {}^t[w_{11}, w_{21}, w_{31}]$. Then

$$\begin{aligned} w_{11} &= \frac{1}{2\pi}\left((1 - e^{2\pi a})w_{10} - [[M_0^2 f(2\pi)]_1 - 2\pi[M_0 f(2\pi)]_1\right) \\ &= \frac{1}{2\pi}\left(\frac{1 - e^{2\pi a}}{(1-e^{2\pi a})^2}\int_0^{2\pi}(2\pi - s + se^{2\pi a})e^{a(2\pi-s)}f_1(s)ds\right. \\ &\qquad \left. - \int_0^{2\pi}(2\pi - s)e^{a(2\pi-s)}f_1(s)ds + 2\pi\int_0^{2\pi} e^{a(2\pi-s)}f_1(s)ds\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\frac{1}{(1-e^{2\pi a})}\int_0^{2\pi}(2\pi-(1-e^{2\pi a})s)e^{a(2\pi-s)}f_1(s)ds \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} se^{a(2\pi-s)}f_1(s)ds \\
&= \frac{1}{1-e^{2\pi a}}\int_0^{2\pi} e^{a(2\pi-s)}f_1(s)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
w_{21} &= \frac{1}{2\pi}\left((-2\pi)w_{30} - [[M_0^2 f(2\pi)]_2 - 2\pi[M_0 f(2\pi)]_2\right) \\
&= -c_2 - \frac{1}{2\pi}\left(\int_0^{2\pi}(2\pi-s)e^{-1+\cos s}(f_2(s)+(2\pi-s-\sin s)f_3(s))ds\right. \\
&\quad \left. -2\pi\int_0^{2\pi} e^{-1+\cos s}(f_2(s)+(2\pi-s-\sin s)f_3(s))ds\right) \\
&= -c_2 - \frac{1}{2\pi}\int_0^{2\pi}(-s)e^{-1+\cos s}(f_2(s)+(2\pi-s-\sin s)f_3(s))ds.
\end{aligned}
$$

Since the third row of $I-V(0)$ is $[0,0,0]$ and $[M_0 f(2\pi)]_3 = 0$, we have

$$
\begin{aligned}
w_{31} &= -\frac{1}{2\pi}[M_0^2 f(2\pi)]_3 = -\frac{1}{2\pi}\int_0^{2\pi}(2\pi-s)e^{-1+\cos s}f_3(s)ds \\
&= -[M_0 f(2\pi)]_3 - \frac{1}{2\pi}\int_0^{2\pi} e^{-1+\cos s}(-sf_3(s))ds \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{-1+\cos s}sf_3(s)ds.
\end{aligned}
$$

Thus w_1 is given by (4.21). $\square$

Corollary 4.4. *If*

$$f(t) = {}^t[\cos t, e^{-\cos t}\cos t, 0], \tag{4.22}$$

then the solution u of $L^2 u = f$ is given by

$$u(t) = \begin{bmatrix} \frac{(a^2-1)\cos t - 2a\sin t}{(a^2+1)^2} \\ (c_1 + c_2\sin t)e^{1-\cos t} + (1-\cos t)e^{-\cos t} \\ c_2 e^{1-\cos t} \end{bmatrix} \tag{4.23}$$

for any constants c_1 and c_2.

4.2. An example of PDE

Denote by $X = L^2([0,\pi],\mathbb{C})$ the space of all square integrable functions from $[0,\pi]$ to $\mathbb{C}$. Then, X is a Hilbert space with the usual inner product $\langle\cdot,\cdot\rangle$ such that

$$\langle w, z\rangle = \int_0^\pi w(x)\overline{z(x)}dx, \ \ w, z \in X.$$

Let us consider a partial differential equation of the form

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + (\alpha(t)-\gamma)u(t,x) + f(t,x), \quad 0 \le x \le \pi,\ t \ge 0 \tag{4.24}$$

$$u(t,0) = u(t,\pi) = 0,\ t \ge 0, \tag{4.25}$$

where $\gamma \in \mathbb{R}$, $\alpha(t)$ is a π-periodic, continuous scalar-valued functions and $f \in P_\pi(X)$.

First, we define a linear operator A_0 by $A_0 u = u''$ for $u \in \mathcal{D}(A_0)$, where

$$\mathcal{D}(A_0) = \big\{u \in X : u \in C^1([0,\pi],\mathbb{C}), u' \text{ is absolutely continuous}, \\ u'' \in X, u(0) = u(\pi) = 0\big\}.$$

Then A_0 is a closed linear operator defined on the dense domain $\mathcal{D}(A_0)$ and a self-adjoint operator on X. Moreover, A_0 generates a C_0-compact semigroup $T_0(t)$ on X such that $\|T_0(t)\| = e^{-t}$ for $t \ge 0$. Since A_0 is a self-adjoint operator on X, so is $T_0(t)$ for each $t \ge 0$, cf. [5, Corollary 4.5, p. 31], or [11, Corollary 10.6, p. 41].

The point spectrum of A_0 is given by

$$\sigma_p(A_0) = \{-n^2 : n = 1, 2, \dots\}.$$

For $n = 1, 2, \dots$, the function $\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$ is a basis of the null space $\mathcal{N}(-n^2 I - A_0)$, and $\{\phi_n\}_{n=1}^\infty$ is a complete orthonormal basis of X. Thus every $w \in X$ is represented as $w = \sum_{n=1}^\infty \langle w, \phi_n\rangle \phi_n$. Since $T_0(t)\phi_n = e^{-n^2 t}\phi_n$, we obtain that

$$T_0(t)w = \sum_{k=1}^\infty e^{-k^2 t}\langle w, \phi_k\rangle \phi_k, \quad w \in X, \tag{4.26}$$

and that

$$\mathcal{N}(e^{-n^2 t} I - T_0(t)) = \{c\phi_n(x) : c \in \mathbb{C}\}. \tag{4.27}$$

The equation (4.26) implies that $0 \notin \sigma_p(T(t))$, $t \ge 0$. Then

$$\sigma_p(T_0(t)) = \{e^{-n^2 t} : n = 1, 2, \dots\} = e^{t\sigma_p(A_0)},$$

since $e^{t\sigma_p(A_0)} \subset \sigma(T_p(t)) \subset e^{t\sigma_p(A_0)} \cup \{0\}$ in general (see [11, Theorem 2.4, Chap. 2]). Note that $\sigma(A_0) \subset \mathbb{R}$ since A_0 is self-adjoint; $e^{t\sigma(A_0)} \subset \sigma(T_0(t)) \setminus \{0\}, t \ge 0$ by the spectral mapping theorem of semigroups; $\sigma(T_0(t)) \setminus \{0\} = \sigma_p(T_0(t))$ since $T_0(t)$ is a compact operator. Therefore, it follows that $\sigma(A_0) = \sigma_p(A_0)$. Moreover,$\sigma_p(A_0) = \sigma_n(A_0)$ and $e^{t\sigma_n(A_0)} = \sigma_n(T_0(t))$, since these equations hold for compact semigroups (see [7, Corollary 2.14]). Therefore,

$$\sigma(A_0) = \sigma_p(A_0) = \sigma_n(A_0), \\ \sigma(T_0(t)) \setminus \{0\} = \sigma_p(T_0(t)) = \sigma_n(T_0(t)) = e^{t\sigma_n(A_0)},\ t > 0.$$

The operator $T_0(t), t \ge 0$ has the following properties. To show this we refer to the following lemma.

Lemma 4.5 ([12, Theorem 7.9, V.7]). *If T is a compact operator on a Banach space X and if $\lambda \in \mathbb{C}$, then the ascent and descent of $\lambda I - T$ are both finite* (*and hence equal*). *If the ascent is p, then*

$$X = \mathcal{R}((\lambda I - T)^p) \oplus \mathcal{N}((\lambda I - T)^p),$$

where both subspaces are closed.

We note that in general, if S is a self-adjoint operator on a Hilbert space, then $\mathcal{N}(S^2) = \mathcal{N}(S)$. Indeed, if $S^2 x = 0$, then $0 = \langle S^2 x, x\rangle = \langle Sx, Sx\rangle$, whence $Sx = 0$. Thus $\mathcal{N}(S^2) \subset \mathcal{N}(S)$, whence $\mathcal{N}(S^2) = \mathcal{N}(S)$.

Lemma 4.6. *The following results hold*:

(1) *The ascent and the descent of $e^{-n^2 t} I - T_0(t), n = 1, 2, \ldots,$ are both* 1, *and $\mathcal{N}(e^{-n^2 t} I - T_0(t))$ is given by* (4.27).

(2) $X = \mathcal{N}(e^{-n^2 t} I - T_0(t)) \oplus \mathcal{R}(e^{-n^2 t} I - T_0(t))$.

(3)

$$\mathcal{N}(e^{-n^2 t} I - T_0(t)) = (\mathcal{R}(e^{-n^2 t} I - T_0(t)))^{\perp},$$
$$(\mathcal{N}(e^{-n^2 t} I - T_0(t)))^{\perp} = \mathcal{R}(e^{-n^2 t} I - T_0(t)).$$

Proof. Since $T_0(t)$ is a self-adjoint operator, the ascent of $e^{-n^2 t} I - T_0(t)$ is 1. The remainder of Lemma follows from Lemma 4.5 together with the well-known results about the orthonormal complements, ranges and null spaces (cf. [12, p. 244]). □

Next, we define a closed linear operator A by

$$Au = A_0 u - \gamma u \quad \text{for} \quad u \in \mathcal{D}(A) = \mathcal{D}(A_0),$$

which is the generator of the C_0-semigroup $T(t) = e^{-\gamma t} T_0(t)$ on X.

If we set

$$A(t) = A + \alpha(t) I_X = A_0 + (\alpha(t) - \gamma) I_X \quad \text{for} \quad \in \mathbb{R},$$

then the equation (4.24) is a perturbation of the homogeneous equation

$$\frac{d}{dt} u(t) = A(t) u(t). \tag{4.28}$$

Set

$$a(t,s) = \int_s^t \alpha(r) dr, \quad a(t) = \int_0^t \alpha(r) dr.$$

Clearly, $a(t,s) = a(t) - a(s)$, $a(t+\pi, s+\pi) = a(t,s)$ and $a(t+\pi) = a(t) + a(\pi)$. Then the solution operator $U(t,s)$ of the equation (4.28) is represented as

$$U(t,s) = e^{a(t,s)} T(t-s) = e^{a(t) - a(s) - \gamma(t-s)} T_0(t-s), \ t \geq s.$$

Then,

$$V(0) := U(\pi, 0) = e^{a(\pi) - \gamma\pi} T_0(\pi).$$

Since $T_0(\pi)$ is a compact operator, so is $V(0)$. The operator $V(0)$ has the following properties.

Lemma 4.7. *Set*

$$\alpha_n = \frac{a(\pi)}{\pi} - \gamma - n^2, \; n = 1, 2, \ldots.$$

Then the following results hold:

(1) $\sigma_p(V(0)) = \sigma_n(V(0))) = \{e^{\pi\alpha_n} : n = 1, 2, \ldots\}$.

(2) *The ascent and the decent of the operator* $e^{\pi\alpha_n}I - V(0)$ *are both* 1.

(3)

$$\mathcal{N}(e^{\pi\alpha_n}I - V(0)) = \{c\phi_n : c \in \mathbb{C}\}, \tag{4.29}$$

$$\mathcal{R}(e^{\pi\alpha_n}I - V(0)) = \{c\phi_n : c \in \mathbb{C}\}^\perp. \tag{4.30}$$

Proof. (1) Since

$$\sigma_p(T_0(\pi)) = \sigma_n(T_0(\pi)) = \{e^{-n^2\pi} : n = 1, 2, \ldots\},$$

and

$$e^{a(\pi)-\gamma\pi-n^2\pi}I - V(0) = e^{a(\pi)-\gamma\pi}(e^{-n^2\pi}I - T_0(\pi)),$$

we have

$$\sigma_p(V(0)) = \sigma_n(V(0))) = \{e^{\pi\alpha_n} : n = 1, 2, \ldots\}.$$

The assertions (2) and (3) follow from Lemma 4.6. □

It is easy to show that $\{U(t,s)\}_{t\geq s}$ is a π-periodic evolutionary process on X. Let L be the generator of the evolution semigroup $\{T^h\}_{h\geq 0}$ on $P_\pi(X)$ associated with $\{U(t,s)\}_{t\geq s}$. Then the operator L has the following properties.

Lemma 4.8. *Set*

$$\alpha_{n,k} = \alpha_n + 2k\sqrt{-1} : n = 1, 2, \ldots, k = 0, \pm 1, \pm 2, \ldots.$$

Then the following results hold:

(1) $\sigma_p(L) = \sigma_n(L) = \{\alpha_{n,k} : n = 1, 2, \ldots, k = 0, \pm 1, \pm 2, \ldots\}$.

(2) *The ascent and the decent of the operator* $\alpha_{n,k}I - L$ *are both* 1 *and* $\dim \mathcal{N}(\alpha_{n,k}I - L) = 1$.

Proof. It follows from [7, Proposition 4.2] that

$$\sigma_p(L) = \sigma_n(L) = \{\alpha_{n,k} : n = 1, 2, \ldots, k = 0, \pm 1, \pm 2, \ldots\}.$$

Since

$$\mathcal{N}((\alpha_{n,k}I - L)^m) \cong \mathcal{N}((e^{\pi\alpha_n}I - V(0))^m), \quad m = 1, 2, \ldots, \tag{4.31}$$

by [9, Corollary 4.4], where $\cong$ means the relation of being isomorphic, the ascent and the decent of the operator $\alpha_{n,k}I - L$ are both 1, cf. [7, Theorem 3.4]. The assertions (4.29) and (4.31) imply that

$$\dim \mathcal{N}(\alpha_{n,k}I - L) = 1$$

for all k. □

Let us find the solution u of

$$(\alpha_n I - L)^2 u = f, \quad n \neq 1, \tag{4.32}$$

where

$$f(t,x) = e^{a(t)-\frac{a(\pi)}{\pi}t}\phi_1(x) = \sqrt{\frac{2}{\pi}}e^{a(t)-\frac{a(\pi)}{\pi}t}\sin x. \tag{4.33}$$

Then we obtain the following result.

Proposition 4.9. *The solution of the equation* (4.32) *with* f *given by* (4.33) *is represented as*

$$u(t,x) = e^{a(t)-\frac{a(\pi)}{\pi}t}\sqrt{\frac{2}{\pi}}\left(\frac{1}{(n^2-1)^2}\sin x + c\sin nx\right), \tag{4.34}$$

where c *is any constant.*

Proof. We define a π-periodic function $b(t)$ by

$$b(t) = a(t) - \frac{a(\pi)}{\pi}t.$$

With the notation of (2.1), we have

$$\begin{aligned} U_{\alpha_n}(t,s) &= e^{-\alpha_n(t-s)}U(t,s) \\ &= e^{-\left(\frac{a(\pi)}{\pi}-\gamma-n^2\right)(t-s)}e^{a(t,s)-\gamma(t-s)}T_0(t-s) \\ &= e^{-(\frac{a(\pi)}{\pi}-n^2)(t-s)}e^{a(t)-a(s)}T_0(t-s) \quad t \geq s, \\ &= e^{n^2(t-s)+b(t)-b(s)}T_0(t-s) \end{aligned}$$

and

$$V_{\alpha_n}(0) = e^{n^2\pi}T_0(\pi).$$

Note that

$$I - V_{\alpha_n}(0) = e^{n^2\pi}(e^{-n^2\pi}I - T_0(\pi)).$$

By Lemma 4.7 the operator $V_{\alpha_n}(0)$ has the following properties.

(i) The ascent and the decent of the operator $I - V_{\alpha_n}(0)$ are both 1.

(ii) $\mathcal{N}(I - V_{\alpha_n}(0)) = \{c\phi_n : c \in \mathbb{C}\}$ and $\mathcal{R}(I - V_{\alpha_n}(0)) = \{c\phi_n : c \in \mathbb{C}\}^{\perp}$.

Now we will apply Theorem 2.6 to the equation (4.32). To do so, we calculate w_0, w_1 and $M_{\alpha_n}^2 f(t)$. Since

$$\begin{aligned} U_{\alpha_n}(t,s)f(s) &= e^{n^2(t-s)+b(t)-b(s)}T_0(t-s)b(s)\phi_1 \\ &= e^{b(t)+(n^2-1)(t-s)}\phi_1 \quad (t \geq s), \end{aligned}$$

we have

$$M_{\alpha_n}f(t) = \int_0^t e^{b(t)+(n^2-1)(t-s)}\phi_1 ds = e^{b(t)}\frac{e^{(n^2-1)t}-1}{n^2-1}\phi_1,$$

and

$$M_{\alpha_n}^2 f(t) = \int_0^t (t-s)e^{b(t)+(n^2-1)(t-s)}\phi_1 dr$$
$$= e^{b(t)} \left(\frac{te^{(n^2-1)t}}{n^2-1} - \frac{e^{(n^2-1)t}-1}{(n^2-1)^2} \right) \phi_1,$$

so that

$$M_{\alpha_n} f(\pi) = \frac{e^{(n^2-1)\pi}-1}{n^2-1}\phi_1$$

and

$$M_{\alpha_n}^2 f(\pi) = \left(\frac{\pi e^{(n^2-1)\pi}}{n^2-1} - \frac{e^{(n^2-1)\pi}-1}{(n^2-1)^2} \right) \phi_1.$$

Thus

$$H_{\alpha_n,2}(1)f = M_{\alpha_n}^2 f(\pi) - \pi M_{\alpha_n} f(\pi) = \left(\frac{1-e^{(n^2-1)\pi}}{(n^2-1)^2} + \frac{\pi}{n^2-1} \right) \phi_1.$$

Since

$$(I - V_{\alpha_n}(0))\phi_1 = (I - e^{n^2\pi}T_0(\pi))\phi_1 = (1 - e^{(n^2-1)\pi})\phi_1, \tag{4.35}$$

we have

$$H_{\alpha_n,2}(2)f = (I - V_{\alpha_n}(0))(M_{\alpha_n}^2 f(\pi) - \pi M_{\alpha_n} f(\pi)) + \pi M_{\alpha_n} f(\pi)$$
$$= (1 - e^{(n^2-1)\pi}) \left(\frac{1-e^{(n^2-1)\pi}}{(n^2-1)^2} + \frac{\pi}{n^2-1} \right) \phi_1 - \pi \frac{1-e^{(n^2-1)\pi}}{n^2-1}\phi_1$$
$$= \frac{(1-e^{(n^2-1)\pi})^2}{(n^2-1)^2}\phi_1.$$

Therefore, the equation (3.11) for w_0 in Theorem 3.4 becomes

$$(I - V_{\alpha_n}(0))^2 w_0 = \frac{(1-e^{(n^2-1)\pi})^2}{(n^2-1)^2}\phi_1. \tag{4.36}$$

The properties (i) and (ii) of $V_{\alpha_n}(0)$ imply that the function in the right side of (4.36) belongs to $\mathcal{R}((I - V_{\alpha_n}(0))^2)(= \mathcal{R}(I - V_{\alpha_n}(0)))$, that is, the equation (4.36) has a solution w_0. In fact, it follows from (4.35) that

$$(I - V_{\alpha_n}(0))^2\phi_1 = (1 - e^{(n^2-1)\pi})^2\phi_1,$$

and hence

$$(I - V_{\alpha_n}(0))^2 \left(\frac{1}{(n^2-1)^2}\phi_1 \right) = \frac{(1-e^{(n^2-1)\pi})^2}{(n^2-1)^2}\phi_1.$$

Then the function $w_0 = \frac{1}{(n^2-1)^2}\phi_1$ is a special solution of the inhomogeneous linear equation (4.36). Since $\mathcal{N}((I - V_{\alpha_n}(0))^2) = \mathcal{N}(I - V_{\alpha_n}(0))$, the general solution of (4.36) is given by

$$w_0 = \frac{1}{(n^2-1)^2}\phi_1 + c\phi_n,$$

where c is any constant. Since

$$(I - V_{\alpha_n}(0))w_0 = \frac{1 - e^{(n^2-1)\pi}}{(n^2-1)^2}\phi_1,$$

it follows from Theorem 3.4 that

$$\pi w_1 = (I - V_\alpha(0))w_0 - H_{\alpha_n,2}(1)f = -\frac{\pi}{n^2-1}\phi_1,$$

that is,

$$w_1 = -\frac{1}{n^2-1}\phi_1.$$

Therefore, by Theorem 2.6 the solution $u(t)$ of the equation (4.32) is represented as

$$\begin{aligned}
u(t) &= U_{\alpha_n}(t,0)(w_0 + tw_1) + M^2_{\alpha_n} f(t) \\
&= e^{n^2 t + b(t)} T_0(t) \left(\frac{1}{(n^2-1)^2}\phi_1 + c\phi_n - \frac{t}{n^2-1}\phi_1 \right) \\
&\quad + e^{b(t)} \left(\frac{te^{(n^2-1)t}}{n^2-1} - \frac{e^{(n^2-1)t} - 1}{(n^2-1)^2} \right) \phi_1 \\
&= e^{(n^2-1)t + b(t)} \left(\frac{1}{(n^2-1)^2}\phi_1 - \frac{t}{n^2-1}\phi_1 \right) + e^{b(t)} c\phi_n \\
&\quad + e^{(n^2-1)t + b(t)} \left(\frac{t}{n^2-1} - \frac{1}{(n^2-1)^2} \right) \phi_1 + e^{b(t)} \frac{1}{(n^2-1)^2}\phi_1 \\
&= e^{b(t)} \left(\frac{1}{(n^2-1)^2}\phi_1 + c\phi_n \right),
\end{aligned}$$

which shows (4.34). □

References

[1] T. Arakawa, T. Ibukiyama, and M. Kaneko, *Bernoulli Numbers and Zeta Functions*, With an appendix by Don Zagier, Springer, Japan, 2014.

[2] F. E. Browder, *On the spectral theory of elliptic differential operators I*, Math. Ann. **142** (1960), 22–130.

[3] C. Chicone and Y. Latushkin, *Evolution semigroups in dynamical systems and differential equations*, Mathematical Surveys and Monographs **70**, American Mathematical Society, 1999.

[4] D. Daners and P. K. Medina, *Abstract evolution equations, periodic problems and applications*, Pitman Research Notes in Math. Ser. 279, Longman Scientific & Technical, 1992.

[5] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Univ. Press, 1985.

[6] Y. Hino, T. Naito, V. M. Nguyen, and J. S. Shin, *Almost Periodic Solutions of Differential Equations in Banach Spaces*, Taylor & Francis, 2002.

[7] D. Kim, R. Miyazaki, T. Naito, and J. S. Shin, *Normal eigenvalues in evolutionary process*, J. Korean Math. Soc. **53** (2016), no. 4, 895–908.

[8] R. Miyazaki, D. Kim, T. Naito, and J. S. Shin, *Fredholm operators, evolution semigroups, and periodic solutions of nonlinear periodic systems*, J. Differential Equations **257** (2014), no. 11, 4214–4247.

[9] ———, *Generalized eigenspaces of generators of evolution semigroups*, J. Math. Anal. Appl. **433** (2016), no. 2, 991–1005.

[10] T. Naito and V. M. Nguyen, *Evolution semigroups and spectral criteria for almost periodic solutions of periodic evolution equations*, J. Differential Equations **152** (1999), no. 2, 358–376.

[11] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[12] A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, Second Edition, John Wiley & Sons, Inc. 1980.

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