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ON COATOMIC MODULES AND LOCAL COHOMOLOGY MODULES WITH RESPECT TO A PAIR OF IDEALS

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Abstract. In this paper, we show some results on the vanishing and the finiteness of local cohomology modules with respect to a pair of ideals. We also prove that $\mathrm{Supp}(H^{\dim M-1}_{I,J}(M)/JH^{\dim M-1}_{I,J}(M))$ is a finite set.

1. Introduction

Throughout this paper, R is a Noetherian commutative (with non-zero identity) ring and I, J are two ideals of R. It is well-known that the local cohomology theory of Grothendieck is an important tool in commutative algebra and algebraic geometry. In [8], Takahashi, Yoshino and Yoshizawa introduced the module $H^i_{I,J}(M)$ as a generalization of the ordinary local cohomology module $H^i_I(M)$. For an R-module M, the (I, J)-torsion submodule of M is $\Gamma_{I,J}(M) = \{x \in M \mid I^n x \subseteq Jx \text{ for some positive integer } n\}$. They denoted by $H^i_{I,J}$ the i-th right derived functor of the functor $\Gamma_{I,J}$. It is clear that when $J = 0$, the functor $H^i_{I,0}$ coincides with the usual local cohomology functor H^i_I.

When M is a finitely generated R-module, many properties of $H^i_{I,J}(M)$ have been studied in [2], [3], [4], [5], [6] and [8]. We now improve some results of those papers in the case M is a coatomic module. An R-module M is called coatomic if every proper submodule of M is contained in a maximal submodule of M. The coatomic modules were introduced and studied by H. Zöschinger in [9]. In [1] the authors studied some properties of the local cohomology modules $H^i_I(M)$ concerning to coatomic modules. An important result on coatomic modules was shown in [9, Satz 2.4] which says that: Let $(R, \mathfrak{m})$ be a local ring and M an R-module. The following statements are equivalent:

(i) M is a coatomic module;
(ii) There is an integer $t \geq 1$ such that $\mathfrak{m}^t M$ is finitely generated;
(iii) There is an integer $t \geq 1$ such that $M/(0 :_M \mathfrak{m}^t)$ is finitely generated.

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The purpose of this paper is to show some results on the vanishing and the finiteness of local cohomology modules with respect to a pair of ideals $H^i_{I,J}(M)$. Some equivalent conditions of (I,J)-torsion modules when M is a coatomic R-module are shown in Theorem 2.1. Theorem 2.3 shows that if M is a coatomic module of $\dim M > 0$ or a minimax module of $d = \dim M > 1$ over a local ring $(R, \mathfrak{m})$, then the module $H^d_{I,J}(M)$ is artinian and $\operatorname{Att}(H^d_{I,J}(M)) = \{\mathfrak{p} \in \operatorname{Supp}(M) \cap V(J) \mid \operatorname{cd}(I,J,R/\mathfrak{p}) = d\}$. We will see in Proposition 2.4 that if M is a coatomic R-module with $\dim M > 0$ and t is a non-negative integer such that $\operatorname{Supp}(H^i_{I,J}(M)) \subseteq \{\mathfrak{m}\}$ for all $i < t$, then $H^i_{I,J}(M)$ is artinian for all $0 < i < t$. An important result of this paper is Theorem 2.6 which shows that $H^i_{I,J}(M)$ is finitely generated or coatomic for all $i \geq t$ if and only if $H^i_{I,J}(M) = 0$ for all $i \geq t$. When studying the finiteness of support of local cohomology modules with respect to an ideal, M. Aghapournahr and L. Melkersson in [1], Saremi in [7] showed that $\operatorname{Supp}(H^{\dim M-1}_I(M))$ is a finite set. Now, we prove in Theorem 2.13 that in a semi-local ring, the set $\operatorname{Supp}(H^{\dim M-1}_{I,J}(M)/JH^{\dim M-1}_{I,J}(M))$ is finite.

2. Main results

Let I, J be two ideals of R. In [8], the authors denoted by $W(I,J) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J \text{ for some } n \gg 1\}$ and $\tilde{W}(I,J) = \{\mathfrak{a} \lhd R \mid I^n \subseteq \mathfrak{a} + J \text{ for some } n \gg 1\}$. An R-module M is called (I,J)-torsion if $M = \Gamma_{I,J}(M)$. When M is a finitely generated R-module, it follows from [8, 1.9] that M is (I,J)-torsion if and only if M/JM is I-torsion. We have the first result on the equivalent conditions of (I,J)-torsion modules when M is a coatomic R-module.

Theorem 2.1. *Let $(R,\mathfrak{m})$ be a local ring and M a coatomic R-module. The following statements are equivalent*:

(i) *M is (I,J)-torsion*;
(ii) *M/JM is I-torsion*;
(iii) *$H^i_{I,J}(M) = 0$ for all $i > 0$.*

Proof. (i) ⇒ (ii). Trivial.

(i) ⇒ (iii). It follows from [8, 1.13(1)].

(ii) ⇒ (i). Assume that M/JM is an I-torsion R-module. Since M is coatomic, by [9, Satz 2.4] there exists a positive integer t such that $M/(0:_M \mathfrak{m}^t)$ is a finitely generated R-module. Let $N = 0 :_M \mathfrak{m}^t$. It is clear that N is $\mathfrak{m}$-torsion and then N is (I,J)-torsion. We see that

$$\frac{M/N}{J(M/N)} \cong \frac{M}{JM+N} \cong \frac{M/JM}{JM+N/JM}.$$

By the assumption, we can conclude that $(M/N)/J(M/N)$ is I-torsion. Since M/N is finitely generated, we have by [8, 1.9] that M/N is (I,J)-torsion. Now, combining the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

with [8, 1.8 (2)] we get that M is (I, J)-torsion.

(iii) $\Rightarrow$ (i). By [9, Satz 2.4], there is an integer t such that $M/(0 :_M \mathfrak{m}^t)$ is finitely generated. If $M = 0 :_M \mathfrak{m}^t$, then M is an $\mathfrak{m}$-torsion R-module. Since $(R, \mathfrak{m})$ is a local ring, we see that M is (I, J)-torsion. Now assume that $M/(0 :_M \mathfrak{m}^t) \neq 0$. This implies that $\mathfrak{m} \in \operatorname{Supp}(M/(0 :_M \mathfrak{m}^t))$. From the short exact sequence

$$0 \to 0 :_M \mathfrak{m}^t \to M \to M/(0 :_M \mathfrak{m}^t) \to 0$$

we have

$$\operatorname{Supp}(M) = \operatorname{Supp}(M/(0 :_M \mathfrak{m}^t)) \cup \operatorname{Supp}(0 :_M \mathfrak{m}^t) = \operatorname{Supp}(M/(0 :_M \mathfrak{m}^t))$$

and

$$H^i_{I,J}(M) \cong H^i_{I,J}(M/(0 :_M \mathfrak{m}^t))$$

for all $i > 0$. By the assumption, $H^i_{I,J}(M/(0 :_M \mathfrak{m}^t)) = 0$ for all $i > 0$. Therefore, we can conclude that

$$\operatorname{Supp}(M/(0 :_M \mathfrak{m}^t)) \subseteq W(I, J)$$

by [8, 4.2] and the proof is complete. □

Now, if M is a minimax R-module, then we have a similar result. We recall that an R-module M is minimax if there is a finitely generated submodule N of M such that M/N is artinian. Minimax modules were first introduced and studies by H. Zöschinger in [10].

Proposition 2.2. *Let $(R, \mathfrak{m})$ be a local ring and M a minimax R-module. The following statements are equivalent*:

(i) *M is (I, J)-torsion.*
(ii) *M/JM is I-torsion.*

Proof. (i) $\Rightarrow$ (ii). Trivial.

(ii) $\Rightarrow$ (i). Since M is a minimax R-module, there exists a finitely generated submodule N of M such that M/N is artinian. From the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

we have the following exact sequence

$$\cdots \to \operatorname{Tor}^R_1(R/J, M/N) \to N/JN \to M/JM \to (M/N)/J(M/N) \to 0.$$

Since M/N is an artinian R-module, we have $\operatorname{Supp}(M/N) \subseteq \{\mathfrak{m}\}$ and then M/N is I-torisional and (I, J)-torsion. Therefore $\operatorname{Tor}^R_1(R/J, M/N)$ is an I-torsion R-module. It follows from the hypothesis that N/JN is I-torsion. Since N is finitely generated, we have by [8, 1.9] that N is an (I, J)-torsion R-module. By [8, 1.8(2)], we imply that M is (I, J)-torsion. □

In [3, 2.1], if M is a finitely generated R-module over a local ring $(R, \mathfrak{m})$ with $\dim M = d$, then $H^d_{I,J}(M)$ is artinian. We now prove that these properties hold for the larger class of coatomic modules of minimax modules instead of the class of finitely generated modules.

Theorem 2.3. *Let* $(R, \mathfrak{m})$ *be a local ring and* M *a coatomic* R*-module with* $d = \dim M > 0$ *or a minimax* R*-module with* $d = \dim M > 1$*. Then* $H^d_{I,J}(M)$ *is artinian and*

$$\operatorname{Att}(H^d_{I,J}(M)) = \{\mathfrak{p} \in \operatorname{Supp}(M) \cap V(J) \mid \operatorname{cd}(I, J, R/\mathfrak{p}) = d\},$$

where $\operatorname{cd}(I, J, M) = \sup\{n \mid H^n_{I,J}(M) \neq 0\}$.

Proof. At first we assume that M is a coatomic R-module. Then there is an integer $k \geq 1$ such that $M/(0 :_M \mathfrak{m}^k)$ is finitely generated by [9, Satz 2.4] and

$$H^i_{I,J}(M) \cong H^i_{I,J}(M/(0 :_M \mathfrak{m}^k))$$

for all $i > 0$. Since $\dim M > 0$, we can conclude that

$$\operatorname{Supp}(M) = \operatorname{Supp}(M/(0 :_M \mathfrak{m}^k))$$

and

$$\dim M = \dim M/(0 :_M \mathfrak{m}^k).$$

From [3, 2.1], $H^d_{I,J}(M/(0 :_M \mathfrak{m}^k))$ is artinian and then $H^d_{I,J}(M)$ is also artinian. Now we have by [2, 2.1],

$$\begin{aligned}\operatorname{Att}(H^d_{I,J}(M)) &= \operatorname{Att}(H^d_{I,J}(M/(0 :_M \mathfrak{m}^k))) \\ &= \{\mathfrak{p} \in \operatorname{Supp}(M/(0 :_M \mathfrak{m}^k)) \cap V(J) \mid \operatorname{cd}(I, J, R/\mathfrak{p}) = d\} \\ &= \{\mathfrak{p} \in \operatorname{Supp}(M) \cap V(J) \mid \operatorname{cd}(I, J, R/\mathfrak{p}) = d\}.\end{aligned}$$

In the case M is a minimax R-module. There exists a short exact sequence

$$0 \to N \to M \to A \to 0,$$

where N is finitely generated and A is artinian. Since $\dim M > 0$ and A is artinian, we have $\operatorname{Supp}(M) = \operatorname{Supp}(N)$ and $\dim M = \dim N$. By applying the functor $\Gamma_{I,J}(-)$ to the above exact sequence, we obtain an exact sequence

$$0 \to H^0_{I,J}(N) \to H^0_{I,J}(M) \to H^0_{I,J}(A) \to H^1_{I,J}(N) \to H^1_{I,J}(M) \to 0$$

and

$$H^i_{I,J}(N) \cong H^i_{I,J}(M)$$

for all $i \geq 2$. Since N is a finitely generated R-module, we have by [3, 2.1] that $H^d_{I,J}(N)$ is artinian and then so is $H^d_{I,J}(M)$. By using [2, 2.1] again, we have

$$\begin{aligned}\operatorname{Att}(H^d_{I,J}(M)) &= \operatorname{Att}(H^d_{I,J}(N)) \\ &= \{\mathfrak{p} \in \operatorname{Supp}(N) \cap V(J) \mid \operatorname{cd}(I, J, R/\mathfrak{p}) = d\} \\ &= \{\mathfrak{p} \in \operatorname{Supp}(M) \cap V(J) \mid \operatorname{cd}(I, J, R/\mathfrak{p}) = d\}\end{aligned}$$

and the proof is complete. ∎

Note that, if M is a minimax R-module with $\dim M = 1$, then we see that

$$\mathrm{Att}(H^1_{I,J}(M)) \subseteq \{\mathfrak{p} \in \mathrm{Supp}(M) \cap V(J) \mid \mathrm{cd}(I, J, R/\mathfrak{p}) = 1\}.$$

It should be mentioned that the above result is not true when $\dim M = 0$. The example is similar to [1, 3.5]. On the other hand, if R is not a local ring and $\dim M = 0$, then $H^0_{I,J}(M)$ is not artinian. Let $R = \mathbb{Z}, M = (\mathbb{Z}_2)^{\mathbb{N}}$ and $I = 2\mathbb{Z}, J = 4\mathbb{Z}$. We see that $\dim M = 0$ and $H^0_{I,J}(M) = M$ is not artinian.

We see in [6, 2.8] that $H^i_{I,J}(M)$ is artinian for all $i < t$ if M is a minimax module such that $\mathrm{Supp}(H^i_{I,J}(M)) \subseteq \{\mathfrak{m}\}$ for all $i < t$. Now, we consider in the case M is a coatomic module.

Proposition 2.4. *Let $(R, \mathfrak{m})$ be a local ring, M a coatomic R-module with $\dim M > 0$. Assume that t is a non-negative integer such that $\mathrm{Supp}(H^i_{I,J}(M)) \subseteq \{\mathfrak{m}\}$ for all $i < t$. Then $H^i_{I,J}(M)$ is artinian for all $0 < i < t$.*

Proof. It follows from the proof of Theorem 2.3, there is an integer $k \geq 1$ such that $M/(0 :_M \mathfrak{m}^k)$ is finitely generated and $\mathrm{Supp}(M) = \mathrm{Supp}(M/(0 :_M \mathfrak{m}^k))$. By the hypothesis we see that $\mathrm{Supp}(H^i_{I,J}(M/(0 :_M \mathfrak{m}^k)) \subseteq \{\mathfrak{m}\}$ for all $i < t$. Since finitely generated modules are minimax modules, we have by [6, 2.8] that $H^i_{I,J}(M/(0 :_M \mathfrak{m}^k))$ is artinian for all $i < t$. Note that $H^i_{I,J}(M/(0 :_M \mathfrak{m}^k)) \cong H^i_{I,J}(M)$ for all $i > 0$ and which completes the proof. □

When M is a finitely generated module, in [8] we see that $H^i_{I,J}(M) = 0$ for all $i > \dim M/JM$. Now, we give an extension of this property in the case M is a coatomic R-module.

Proposition 2.5. *Let $(R, \mathfrak{m})$ be a local ring and M a coatomic R-module. The following statements hold*:

(i) *If $J \neq R$, then $H^i_{I,J}(M) = 0$ for all $i > \dim M/JM$.*
(ii) *Suppose that $\sqrt{I + J} = \mathfrak{m}$. Then $\sup\{n \mid H^n_{I,J}(M) \neq 0\} = \dim M/JM$.*

Proof. (i) If $\dim M/JM = -1$, then $M = JM$. Since M is coatomic, there is an integer $t \geq 1$ such that $\mathfrak{m}^t M$ is finitely generated by [9, Satz 2.4]. This implies that M is finitely generated since $M = J^t M \subseteq \mathfrak{m}^t M$. Therefore $M = 0$ by Nakayama's Lemma.

Now suppose that $\dim M/JM \geq 0$. By the assumption on M, there exists an integer $t \geq 1$ such that $M/(0 :_M \mathfrak{m}^t)$ is finitely generated. Let $N = 0 :_M \mathfrak{m}^t$, now the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

gives rise a long exact sequence

$$\cdots \to N/JN \stackrel{\alpha}{\to} M/JM \to (M/N)/J(M/N) \to 0.$$

Note that

$$\mathrm{Supp}(\mathrm{Im}\,\alpha) \subseteq \mathrm{Supp}(N/JN) \subseteq \mathrm{Supp}(N) \subseteq \{\mathfrak{m}\}.$$

This implies that $\dim(\operatorname{Im}\alpha) \leq 0$. If $M = N$, then we can easily check the claim. So in the remainder of the proof, we may and do assume that $N \subsetneq M$. Now from the short exact sequence

$$0 \to \operatorname{Im}\alpha \to M/JM \to (M/N)/J(M/N) \to 0$$

we get

$$\dim M/JM = \dim(M/N)/J(M/N).$$

Since M/N is a finitely generated R-module, we have $H^i_{I,J}(M/N) = 0$ for all $i > \dim M/JM$ by [8, 4.3]. Now the conclusion follows from the isomorphism $H^i_{I,J}(M) \cong H^i_{I,J}(M/N)$ for all $i > 0$.

(ii) Combining [8, 4.5] with the isomorphism $H^i_{I,J}(M) \cong H^i_{I,J}(M/N)$ for all $i > 0$, we get the assertion. □

We are going to state and prove one of main results of this paper. The following theorem is a generalization of [1, 3.9] which shows a relationship on the vanishing, the finiteness and the coatomicness of $H^i_{I,J}(M)$.

Theorem 2.6. *Let $(R, \mathfrak{m})$ be a local ring, M a finitely generated R-module and t a positive integer. The following statements are equivalent*:

(i) $H^i_{I,J}(M) = 0$ *for all* $i \geq t$;

(ii) $H^i_{I,J}(M)$ *is finitely generated for all* $i \geq t$;

(iii) $H^i_{I,J}(M)$ *is coatomic for all* $i \geq t$.

Proof. (i) ⇒ (ii) ⇒ (iii). Trivial.

(iii) ⇒ (i). The proof is by induction on $\dim M$. Let $n = \dim M$. If $n = 0$, then $H^i_{I,J}(M) = 0$ for all $i > 0$.

Let $n > 0$, it follows from [8, 1.13] that

$$H^i_{I,J}(M) \cong H^i_{I,J}(M/\Gamma_{I,J}(M))$$

for all $i > 0$. Denote by $\overline{M} = M/\Gamma_{I,J}(M)$, it is clear that $\overline{M}$ is (I, J)-torsion-free. This implies that $\overline{M}$ is $\mathfrak{a}$-torsion-free for all $\mathfrak{a} \in \tilde{W}(I, J)$. In particular, $\overline{M}$ is $\mathfrak{m}$-torsion-free and there is an element $x \in \mathfrak{m}$ which is regular on $\overline{M}$. Now, the short exact sequence

$$0 \to \overline{M} \xrightarrow{.x} \overline{M} \to \overline{M}/x\overline{M} \to 0$$

induces a long exact sequence

$$\cdots \to H^i_{I,J}(\overline{M}) \xrightarrow{.x} H^i_{I,J}(\overline{M}) \to H^i_{I,J}(\overline{M}/x\overline{M}) \to \cdots .$$

By the assumption, $H^i_{I,J}(\overline{M}/x\overline{M})$ is coatomic for all $i \geq t$. Since $\dim(\overline{M}/x\overline{M}) < \dim(\overline{M}) \leq n$ and $\overline{M}$ is a finitely generated R-module, it follows from the inductive hypothesis that $H^i_{I,J}(\overline{M}/x\overline{M}) = 0$ for all $i \geq t$. Now the long exact sequence yields

$$H^i_{I,J}(\overline{M}) = xH^i_{I,J}(\overline{M})$$

for all $i \geq t$. Note that coatomic modules satisfy Nakayama's Lemma. Thus $H^i_{I,J}(\overline{M}) = 0$ for all $i \geq t$, and the proof is complete. □

We may improve these results as follows.

Corollary 2.7. *Let $(R, \mathfrak{m})$ be a local ring, M a coatomic R-module and t a positive integer. The following statements are equivalent*:

(i) $H^i_{I,J}(M) = 0$ *for all* $i \geq t$;

(ii) $H^i_{I,J}(M)$ *is finitely generated for all* $i \geq t$;

(iii) $H^i_{I,J}(M)$ *is coatomic for all* $i \geq t$.

Proof. Since M is a coatomic R-module, there is an integer $k \geq 1$ such that $M/(0 :_M \mathfrak{m}^k)$ is finitely generated by [9, Satz 2.4]. Therefore, we have the isomorphisms

$$H^i_{I,J}(M) \cong H^i_{I,J}(M/(0 :_M \mathfrak{m}^k))$$

for all $i > 0$. The assertion follows immediate from 2.6. □

Corollary 2.8. *Let $(R, \mathfrak{m})$ be a local ring, M a minimax R-module and $t > 1$ a positive integer. The following statements are equivalent*:

(i) $H^i_{I,J}(M) = 0$ *for all* $i \geq t$;

(ii) $H^i_{I,J}(M)$ *is finitely generated for all* $i \geq t$;

(iii) $H^i_{I,J}(M)$ *is coatomic for all* $i \geq t$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Trivial. We now prove (iii) $\Rightarrow$ (i). Since M is a minimax R-module, there is a short exact sequence

$$0 \to N \to M \to A \to 0,$$

where N is finitely generated and A is artinian. By applying the functor $\Gamma_{I,J}(-)$ to the above exact sequence, we get a long exact sequence

$$0 \to H^0_{I,J}(N) \to H^0_{I,J}(M) \to H^0_{I,J}(A) \to H^1_{I,J}(N) \to H^1_{I,J}(M) \to 0$$

and

$$H^i_{I,J}(N) \cong H^i_{I,J}(M)$$

for all $i \geq 2$. By the hypothesis, $H^i_{I,J}(N)$ is coatomic for all $i \geq t$. It follows from 2.6 that $H^i_{I,J}(N) = 0$ for all $i \geq t$ and which completes the proof. □

Corollary 2.9. *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module with $\mathrm{cd}(I, J, M) > 0$. Then $H^{cd(I,J,M)}_{I,J}(M)$ is not finitely generated.*

Combining [8, 4.5] with 2.9, we have an immediate consequence.

Corollary 2.10. *Let $(R, \mathfrak{m})$ be a local ring, M a finitely generated R-module with $\dim(M/JM) > 0$ and $\sqrt{I + J} = \mathfrak{m}$. Then $H^{\dim M/JM}_{I,J}(M)$ is not finitely generated.*

In [5, Theorem 2], if M is a finitely generated with finite dimension and t is a positive integer such that $H^i_{I,J}(M) = 0$ for all $i > t$, then $H^t_{I,J}(M)/\mathfrak{a}H^t_{I,J}(M) = 0$ for all $\mathfrak{a} \in \tilde{W}(I, J)$. This property will be extended in the case M is a coatomic module.

Proposition 2.11. *Let $(R,\mathfrak{m})$ be a local ring and M a coatomic R-module. Suppose that t is a positive integer such that $H^i_{I,J}(M)=0$ for all $i>t$. Then $H^t_{I,J}(M)/\mathfrak{a}H^t_{I,J}(M)=0$ for all $\mathfrak{a}\in\tilde{W}(I,J)$.*

Proof. Since M is a coatomic R-module, there is an integer $k\geq 1$ such that $M/(0:_M \mathfrak{m}^k)$ is finitely generated. The proof above gives

$$H^t_{I,J}(M)\cong H^t_{I,J}(M/(0:_M \mathfrak{m}^k)).$$

Hence, the assertion follows from [5, Theorem 2]. □

Corollary 2.12. *Let $(R,\mathfrak{m})$ be a local ring and M a coatomic R-module. Assume that $\operatorname{cd}(I,J,M)>0$ and K is a proper submodule of $H^{\operatorname{cd}(I,J,M)}_{I,J}(M)$. Then $H^{\operatorname{cd}(I,J,M)}_{I,J}(M)/K$ is not a coatomic R-module.*

Proof. Suppose that the conclusion is false. It follows from the definition of coatomic modules, there exists a submodule L of $H^{\operatorname{cd}(I,J,M)}_{I,J}(M)$ such that we have a short exact sequence

$$0\to L/K\to H^{\operatorname{cd}(I,J,M)}_{I,J}(M)/K\to R/\mathfrak{m}\to 0.$$

Let $\mathfrak{a}\in\tilde{W}(I,J)$, by applying the functor $R/\mathfrak{a}\otimes_R -$ to the above exact sequence, there is a following exact sequence

$$\cdots\to L/\mathfrak{a}L+K\to H^{\operatorname{cd}(I,J,M)}_{I,J}(M)/\mathfrak{a}H^{\operatorname{cd}(I,J,M)}_{I,J}(M)+K\to R/\mathfrak{m}\to 0.$$

Note that $H^{\operatorname{cd}(I,J,M)}_{I,J}(M)/\mathfrak{a}H^{\operatorname{cd}(I,J,M)}_{I,J}(M)+K$ is a homomorphic image of

$$H^{\operatorname{cd}(I,J,M)}_{I,J}(M)/\mathfrak{a}H^{\operatorname{cd}(I,J,M)}_{I,J}(M).$$

Consequently, we can conclude that $H^{\operatorname{cd}(I,J,M)}_{I,J}(M)/\mathfrak{a}H^{\operatorname{cd}(I,J,M)}_{I,J}(M)+K=0$ by 2.11. This implies that $R/\mathfrak{m}=0$ which is a contradiction. □

Next, we will consider the dimension of $H^i_{I,J}(M)$ and the support of $H^{d-1}_{I,J}(M)$ where $d=\dim M$. In [1, 3.3] or [7, 2.3], when studying the local cohomology modules with respect to an ideal, the authors showed that $\dim H^i_I(M)\leq d-i$ and $\operatorname{Supp}(H^{d-1}_I(M))$ is a finite set. The proof of next theorem is based on these results.

Theorem 2.13. *Let M be a finitely generated R-module with $d=\dim M<\infty$. Then*

(i) $\dim H^i_{I,J}(M)\leq d-i$.
(ii) *If R is a semi-local ring, then* $\operatorname{Supp}(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M))$ *is finite.*

Proof. (i) Our proof starts with the observation that

$$H^i_{I,J}(M)=\varinjlim_{\mathfrak{a}\in\tilde{W}(I,J)} H^i_{\mathfrak{a}}(M).$$

This implies that

$$\operatorname{Supp}(H^i_{I,J}(M)) \subseteq \bigcup_{\mathfrak{a}\in \tilde{W}(I,J)} \operatorname{Supp}(H^i_{\mathfrak{a}}(M)).$$

From [7, 2.3], $\dim(H^i_{\mathfrak{a}}(M)) \leq \dim M - i$ for all $\mathfrak{a} \in \tilde{W}(I,J)$. We conclude that $\dim(H^i_{I,J}(M)) \leq \dim M - i$.

(ii) We prove by induction on $d = \dim M$. It is nothing to prove when $d = 0$. If $d = 1$, we see that $H^0_{I,J}(M)$ is finitely generated. Since $\dim(H^0_{I,J}(M)) \leq 1$ by (i), it follows that

$$\operatorname{Supp}(H^0_{I,J}(M)) \subseteq \operatorname{Min}(H^0_{I,J}(M)) \cup \operatorname{Max}(R).$$

Since $H^0_{I,J}(M)$ is finitely generated, we can conclude that $\operatorname{Supp}(H^0_{I,J}(M))$ is finite. Let $d > 1$, we now assume that the statement is true for all non-zero finitely generated modules with dimension less than $\dim M$. Now the short exact sequence

$$0 \to \Gamma_J(M) \to M \to M/\Gamma_J(M) \to 0$$

induces a long exact sequence

$$\cdots H^{d-1}_{I,J}(\Gamma_J(M)) \xrightarrow{f} H^{d-1}_{I,J}(M) \xrightarrow{g} H^{d-1}_{I,J}(M/\Gamma_J(M)) \xrightarrow{h} H^d_{I,J}(\Gamma_J(M)) \cdots .$$

It follows from [8, 2.5] that $H^i_{I,J}(\Gamma_J(M)) \cong H^i_I(\Gamma_J(M))$ for all $i \geq 0$. On the other hand $\dim \Gamma_J(M) \leq \dim M$, so in the view of [7, 2.5] we see that $\operatorname{Supp}(H^{d-1}_{I,J}(\Gamma_J(M)))$ is finite. This implies that $\operatorname{Supp}(\operatorname{Im} f)$ is finite. Since $H^d_{I,J}(\Gamma_J(M))$ is artinian, the support of $\operatorname{Im} h$ is finite. We now have two short exact sequences

$$0 \to \operatorname{Im} f \to H^{d-1}_{I,J}(M) \to \operatorname{Im} g \to 0$$

and

$$0 \to \operatorname{Im} g \to H^{d-1}_{I,J}(M/\Gamma_J(M)) \to \operatorname{Im} h \to 0.$$

By applying the functor $R/J \otimes_R -$ to above short exact sequences, we obtain the following exact sequences

$$\cdots \to \operatorname{Im} f/J \operatorname{Im} f \to H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M) \to \operatorname{Im} g/J \operatorname{Im} g \to 0$$

and

$$\begin{aligned} \cdots \to \operatorname{Tor}^R_1(R/J, \operatorname{Im} h) \to \operatorname{Im} g/J \operatorname{Im} g \to \\ \to H^{d-1}_{I,J}(M/\Gamma_J(M))/JH^{d-1}_{I,J}(M/\Gamma_J(M)) \to \operatorname{Im} h/J \operatorname{Im} h \to 0. \end{aligned}$$

The proof is complete by showing that

$$\operatorname{Supp}(H^{d-1}_{I,J}(M/\Gamma_J(M))/JH^{d-1}_{I,J}(M/\Gamma_J(M)))$$

is finite. Let $\overline{M} = M/\Gamma_J(M)$, we see that $\overline{M}$ is J-torsion free. Then there is an element $x \in J$ which is $\overline{M}$-regular. Now the short exact sequence

$$0 \to \overline{M} \xrightarrow{\cdot x} \overline{M} \to \overline{M}/x\overline{M} \to 0$$

induces the following exact sequence

$$\cdots \to H^{d-2}_{I,J}(\overline{M}/x\overline{M}) \to H^{d-1}_{I,J}(\overline{M}) \xrightarrow{.x} H^{d-1}_{I,J}(\overline{M}) \to \cdots .$$

This gives us an exact sequence

$$H^{d-2}_{I,J}(\overline{M}/x\overline{M})/JH^{d-2}_{I,J}(\overline{M}/x\overline{M}) \to (0 :_{H^{d-1}_{I,J}(\overline{M})} x)/J(0 :_{H^{d-1}_{I,J}(\overline{M})} x) \to 0.$$

Since $\dim(\overline{M}/x\overline{M}) \leq d-1$, we get by the inductive hypothesis that

$$\operatorname{Supp}(H^{d-2}_{I,J}(\overline{M}/x\overline{M})/JH^{d-2}_{I,J}(\overline{M}/x\overline{M}))$$

is finite and then so is $\operatorname{Supp}((0 :_{H^{d-1}_{I,J}(\overline{M})} x)/J(0 :_{H^{d-1}_{I,J}(\overline{M})} x))$. Since $x \in J$, it follows that the homomorphism

$$(0 :_{H^{d-1}_{I,J}(\overline{M})} x)/J(0 :_{H^{d-1}_{I,J}(\overline{M})} x) \to H^{d-1}_{I,J}(\overline{M})/JH^{d-1}_{I,J}(\overline{M})$$

is surjective. Therefore $\operatorname{Supp}(H^{d-1}_{I,J}(\overline{M})/JH^{d-1}_{I,J}(\overline{M}))$ is finite, and the proof is complete. □

Corollary 2.14. *Let M be a finitely generated R-module with finite dimension $d = \dim M$. Then*

$$\operatorname{Supp}(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M)) \subseteq \operatorname{Ass}(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M)) \cup \operatorname{Max}(R).$$

Proof. It follows from 2.13 that $\dim(H^{d-1}_{I,J}(M)) \leq 1$, we see that

$$\dim(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M)) \leq 1.$$

Therefore $\operatorname{Supp}(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M))$ contains minimal prime ideals of

$$\operatorname{Ass}(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M))$$

and maximal ideals, which completes the proof. □

In the case R is not a semi-local ring, we will see that $\operatorname{Supp}(H^{\dim M-1}_{I,J}(M))$ is not finite.

Example 2.15. Let $R = M = \mathbb{Z}$ and $I = 2\mathbb{Z}$, $J = 4\mathbb{Z}$. We see that $\dim M = 1$ and M is (I,J)-torsion. However, $\operatorname{Supp}(H^0_{I,J}(M)) = \operatorname{Spec}(\mathbb{Z})$ is an infinite set.

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