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INTEGRAL MEANS AND MAXIMUM AREA INTEGRAL PROBLEMS FOR CERTAIN FAMILY OF p-VALENT FUNCTIONS

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Abstract. The paper considers p-valent functions in the open unit disk. We study the integral means along with the area integral problems for functions belonging to a family of p-valent functions.

1. Introduction

The concept of univalence has a natural extension as described in p-valent function theory. A function f which is analytic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is said to be p-valent in $\mathbb{D}$ if it takes each of its values at most p times ($p \in \mathbb{N}$) in $\mathbb{D}$, that is, if the number of roots of the equation $f(z) = w$ in $\mathbb{D}$, for any w, does not exceed p. For example, $f(z) = z^2$ is a 2-valent in $\mathbb{D}$. Let $\mathcal{A}_p$ denote the family of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \quad p \in \mathbb{N}, \tag{1.1}$$

which are analytic and p-valent in $\mathbb{D}$. The motivation of studying p-valent functions comes from the theory of univalent functions. One of the basic problems in p-valent function theory is to see how results from univalent function theory fit analogously into the theory of p-valent functions for $p \geq 2$. Background on some of the important problems in the theory of p-valent functions, for instance, can be found in [5,9,10,16,17,23,28,29]. However, in this collection, the classical integral means and area problems have not been studied in p-valent setting, which is our objective in the present work.

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The integral means and Dirichlet integral in p-valent theory are defined as follows. For $r \in (0,1]$, consider a function $f \in \mathcal{A}_p$ which has the *integral means*

$$L_1(r,f,p) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r^{2p}}{|f(re^{i\theta})|^2}d\theta, \quad z = re^{i\theta} \in \mathbb{D},$$

and the *Dirichlet integral*

$$\Delta(r,f) := \iint_{\mathbb{D}_r} |f'(z)|^2\, dxdy = \pi p r^{2p} + \pi\sum_{n=1}^{\infty}(n+p)|a_{n+p}|^2 r^{2(n+p)}, \;\; z = x+iy,$$

where $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$. Computing these integrals are known as *the integral means and area problems.*

In this article, we mainly focus on computing the above integrals for a most general family of functions in $\mathcal{A}_p$. For $A \in \mathbb{C}$, $-1 \le B \le 0$ and $A \ne B$, we define the general family by

$$\mathcal{S}_p^*(A,B) := \left\{ f \in \mathcal{A}_p : \frac{zf'(z)}{pf(z)} \prec \frac{1+Az}{1+Bz},\; z \in \mathbb{D} \right\},$$

where the symbol $\prec$ denotes the usual subordination. The function

$$k_{A,B,p}(z) := \begin{cases} z^p(1+Bz)^{((A/B)-1)p} & \text{for } B \ne 0 \\ z^p e^{Apz} & \text{for } B = 0 \end{cases} \tag{1.2}$$

plays the role of an extremal function for the class $\mathcal{S}_p^*(A,B)$. Due to some technical reasons, we are not presenting the precise statements of our main results in this section, however, readers may refer to Sections 3 and 4 for more detail.

One of the motivations to study this form of the integral means comes from the following observations. The integral means are associated with some functionals appearing in planar fluid mechanics concerning isoperimetric problems for moving phase domains; see [31, 32]. For more details on the functionals, reader can refer to Section 3. Another aim to study integral means problem was to solve the Bieberbach conjecture; see [8, 26] and references therein. In 2002, Gromova and Vasil'ev [11] made a conjecture that if $f \in \mathcal{S}^*(\beta) := \mathcal{S}_1^*(1-2\beta,-1)$ for $\beta \in [0,1)$, then the estimate

$$L_1(r,f) := L_1(r,f,1) \le \frac{\Gamma(5-4\beta)}{\Gamma^2(3-2\beta)}$$

holds, where Γ is the classical gamma function. The estimate was proven sharp only for $\beta = 0$ and $\beta = 1/2$. This conjecture has been recently settled by Ponnusamy and Wirths in [21] in a more general setting by considering the family $\mathcal{S}^*(A,B) := \mathcal{S}_1^*(A,B)$ for $-1 \le B < A \le 1$. The class $\mathcal{S}^*(A,B)$ was introduced by Janowski in [13]. In this paper we estimate the quantity $L_1(r,f,p)$ for $f \in \mathcal{S}_p^*(A,B)$, $p \in \mathbb{N}$, $A \in \mathbb{C}$, $-1 \le B \le 0$ and $A \ne B$.

The interest to study area problems comes from computing areas of certain regions in the complex plane. In general it is a difficult problem to find area of an arbitrary region. However, our problem finds exact area formulae of regions that are images of $\mathbb{D}$ under certain functions. In 1990, Yamashita [33] conjectured that

$$\max_{f\in\mathcal{C}} \Delta\left(r, \frac{z}{f}\right) = \pi r^2$$

for each r, $0 < r \leq 1$. The maximum is attained only by the rotations of the function $z/(1-z)$. Here $\mathcal{C}$ denotes the well-known class of convex functions in $\mathcal{A}_1$. In 2013, the Yamashita conjecture was settled in [15] for functions belonging to the family $\mathcal{S}^*(\beta) := \mathcal{S}_1^*(1-2\beta, -1)$. The Yamashita conjecture problem for the class $\mathcal{S}^*(A,B) := \mathcal{S}_1^*(A,B)$, $-1 \leq B < A \leq 1$, was suggested in [21] and it was partially solved by the authors in [27]. In the recent article of Ponnusamy et al. [20], this problem has been completely solved for the full range $A \in \mathbb{C}, -1 \leq B < 0$ and $A \neq B$. In this paper, we discuss the Dirichlet integral $\Delta(r, z^p/f)$ when $f \in \mathcal{S}_p^*(A,B)$ for all $p \in \mathbb{N}$.

Remaining part of the paper is organized as follows. In Section 2, we give preliminary information on the family $\mathcal{S}_p^*(A,B)$ and other basic definitions that are used in the sequel. Sections 3 and 4 deal with the statement of our main results and some of their important consequences. In Section 5, we derive some important results which play a vital role to prove our main results. Section 6 is devoted to the proofs of our main theorems. Finally, in Section 7, we propose some open problems.

2. Basic information

Let f and g be two analytic functions in $\mathbb{D}$. We say that f is *subordinate* to g, written as $f(z) \prec g(z)$, if there exists an analytic function w in $\mathbb{D}$ for which $w(0) = 0, |w(z)| \leq |z| < 1$ such that $f(z) = g(w(z))$. Furthermore, if the function g is univalent in $\mathbb{D}$, then we have the following geometric characterization (see also [8, 14, 18]):

$$f(\mathbb{D}) \subset g(\mathbb{D}) \text{ and } f(0) = g(0) \Leftrightarrow f(z) \prec g(z).$$

A function $f \in \mathcal{A}_p$ is called p-valent starlike of order β if there exists a $\rho > 0$, such that for any z, $\rho < |z| < 1$,

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad 0 \leq \beta < p$$

and

$$\int_0^{2\pi} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) d\theta = 2p\pi, \quad z = re^{i\theta},\ \theta \in [0, 2\pi] \tag{2.1}$$

for each r, $\rho < r < 1$. This integral is just the number of zeros of f in the interior of the circle $|z| = r$ and hence f has p zeros in $\mathbb{D}$, and is in fact p-valent there [5]. We denote by $\mathcal{S}_p^*(\beta)$, the class of all p-valent starlike functions of order β and it was studied by Goluzina in [9]. Obviously, $\mathcal{S}_p^*(0) =: \mathcal{S}_p^*$ is the usual class of p-valent starlike functions which was introduced by Goodman in [10].

For the case $p = 1$, the integral (2.1) is equal to 2π and f has one simple zero in $\mathbb{D}$, and in fact f is univalent in $\mathbb{D}$.

Choosing $A = \lambda e^{-i\alpha}(e^{-i\alpha} - (2\beta/p)\cos\alpha)$ and $B = -\lambda$, the class $\mathcal{S}_p^*(A, B)$ reduces to the class $\mathcal{F}_p(\alpha, \beta, \lambda)$ of functions $f \in \mathcal{A}_p$, satisfying the relation

$$e^{i\alpha}\frac{zf'(z)}{pf(z)} \prec \frac{e^{i\alpha} + \left(e^{-i\alpha} - (2\beta/p)\cos\alpha\right)\lambda z}{1 - \lambda z}, \quad z \in \mathbb{D}$$

or

$$e^{i\alpha}\frac{zf'(z)}{pf(z)} \prec \left(\frac{1 + (1 - (2\beta/p))\lambda z}{1 - \lambda z}\right)\cos\alpha + i\sin\alpha,$$

where $0 < \lambda \leq 1$, $0 \leq \beta < p$, $p \in \mathbb{N}$ and $|\alpha| < \pi/2$. The class $\mathcal{F}_p(\alpha, \beta, \lambda)$ was introduced by Aouf [2]. Obviously, $\mathcal{F}_p(\alpha, \beta, \lambda) \subset \mathcal{F}_p(\alpha, \beta, 1) =: \mathcal{S}_{\alpha,p}(\beta)$. Functions in $\mathcal{S}_{\alpha,p}(\beta)$ are said to be p-valent α-spirallike functions of order β (see [16]). The class $\mathcal{S}_{\alpha,p}(0) =: \mathcal{S}_{\alpha,p}$ is the class of p-valent α-spirallike functions. Recently in [28,29], the authors obtained correct forms of the coefficient bounds for functions to be in the class $\mathcal{F}_p(\alpha, \beta, \lambda)$ and other related classes of p-valent functions. If we let different values of p, α, β and λ in the class $\mathcal{F}_p(\alpha, \beta, \lambda)$, then we get certain subclass of p-valent functions (see for instance [29]). It is easy to see that the function $k_{p,\alpha,\beta,\lambda}$ is defined by

$$k_{p,\alpha,\beta,\lambda}(z) = \frac{z^p}{(1 - \lambda z)^{\xi}}, \quad \xi = 2(p - \beta)e^{-i\alpha}\cos\alpha \tag{2.2}$$

belongs to the class $\mathcal{F}_p(\alpha, \beta, \lambda)$.

We note that, by taking special choices of parameters A, B and p in the definition of the class $\mathcal{S}_p^*(A, B)$, we get the following classes which were investigated and studied by several authors. We list down some of them as follows:

(1) $\mathcal{S}_p^*(1 - (2\beta/p), -1) =: \mathcal{S}_p^*(\beta)$.
(2) $\mathcal{S}_p^*(1, -1) =: \mathcal{S}_p^*$.
(3) The class $\mathcal{S}_1^*((1-2\beta)\lambda, -\lambda) =: \mathcal{T}(\lambda, \beta)$ $(0 \leq \beta < 1)$ is studied in [2,27].
(4) $\mathcal{S}_p^*((1 - (2\beta/p))\lambda, -\lambda) =: \mathcal{T}_p(\lambda, \beta)$ (i.e., $\mathcal{F}_p(0, \beta, \lambda) =: \mathcal{T}_p(\lambda, \beta)$), the class of p-valent functions of $\mathcal{T}(\lambda, \beta)$ which is studied in [3].
(5) The class $\mathcal{S}_1^*(\beta) =: \mathcal{S}^*(\beta)$ $(0 \leq \beta < 1)$ is the class of starlike functions of order β which was studied by Robertson in [24].
(6) The class $\mathcal{S}_1^*(1, -1) = \mathcal{S}^*(0) =: \mathcal{S}^*$ denotes the well-known class of starlike functions.

In this paper, we consider functions f in $\mathcal{A}_p$ $(p \in \mathbb{N})$ such that z^p/f is non-vanishing in $\mathbb{D}$, hence it can be represented as Taylor's series of the form

$$\frac{z^p}{f(z)} = 1 + \sum_{n=1}^{\infty} b_{n+p-1}z^n,\ z \in \mathbb{D}. \tag{2.3}$$

We will also make use of the Gaussian Hypergeometric functions defined by

$${}_2F_1(a, b; c; z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n, \quad |z| < 1,$$

where $a, b, c \in \mathbb{C}$ and $c \neq \mathbb{Z}^- = \{0, -1, -2, \ldots\}$. The function ${}_2F_1(a, b; c; z)$ is analytic in $\mathbb{D}$. The Pochhammer symbol $(a)_n$ is defined in terms of the Gamma functions Γ, by

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

In 1882, for $\mathrm{Re(c-a-b)} > 0$ and $z = 1$, Gauss established the following useful relation connected with the Euler gamma function

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} < \infty.$$

Similarly, the function ${}_0F_1(a; z)$ is defined as

$${}_0F_1(a; z) = \sum_{n=0}^{\infty} \frac{1}{(a)_n} \frac{z^n}{n!}, \quad |z| < 1.$$

For basic information about Gaussian Hypergeometric functions, we refer to the well-known text books [1, 22].

3. Integral means problem

For a normalized analytic function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $z \in \mathbb{D}$, we introduce the functional

$$M(r, g, \lambda_1, \lambda_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(re^{i\theta})|^{\lambda_1} |g'(re^{i\theta})|^{\lambda_2} d\theta,$$

where $0 < r \leq 1$ and λ_1, $\lambda_2 \in \mathbb{R}$, which is called the integral means. This functional was introduced and investigated by Gromova and Vasil'ev in [11] and attracts much attention (see [4, 19]).

For $f \in A_p$, let us consider the integral mean

$$M(r, f, p, \lambda_1, \lambda_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda_1} |f'(re^{i\theta})|^{\lambda_2} d\theta$$

for λ_1, $\lambda_2 \in \mathbb{R}$ and $r \in (0, 1)$. For the special cases $\lambda_1 = -2$ and $\lambda_2 = 0$, we find the following interesting integral means such that

$$I_1(r, f, p) := M(r, f, p, -2, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{|f(re^{i\theta})|^2} d\theta.$$

We now state our first main result.

Theorem 3.1. *Let $A \in \mathbb{C}, -1 \leq B \leq 0, A \neq B$ and $p \in \mathbb{N}$. If $f \in \mathcal{S}_p^*(A, B)$ and has the form (2.3), then, for $0 < r \leq 1$, we have*

$$L_1(r, f, p) := r^{2p} I_1(r, f, p) \leq \begin{cases} {}_2F_1\big(\phi p, \overline{\phi} p; 1; B^2\big), & \text{if } B \neq 0; \\ J_0(2ip|A|), & \text{if } B = 0, \end{cases}$$

where $\phi = (A/B) - 1$ and $J_0(z)$ is the Bessel function of order zero. Both inequalities are sharp for the rotations of the function $k_{A,B,p}$ as defined by (1.2).

We remark that when $p = 1$ and $-1 \le B < A \le 1$ in Theorem 3.1, then we obtain [21, Theorem 1]. For $A = 1 - (2\beta/p)$ and $B = -1$, Theorem 3.1 leads to the following immediate consequence.

Corollary 3.2. *For $0 \le \beta < p$ and $p \in \mathbb{N}$, let $f \in \mathcal{S}_p^*(\beta)$. Then we have*

$$L_1(r, f, p) \le \frac{\Gamma(1 + 4(p - \beta))}{\Gamma^2(1 + 2(p - \beta))}, \quad r \in (0, 1].$$

The inequality is sharp.

The case $p = 1$ is recently obtained in [21, Corollary 1]. Moreover, choosing $B = -A$ in Theorem 3.1, we have:

Corollary 3.3. *Let $f \in \mathcal{S}_p^*(A, -A)$ for $0 < A \le 1$ and $p \in \mathbb{N}$. Then we have*

$$L_1(r, f, p) \le {}_2F_1(-2p, -2p; 1; A^2), \quad 0 < r \le 1.$$

The case $p = 1$ is recently obtained in [21, Corollary 2]. If we choose $A = \lambda e^{-i\alpha}(pe^{-i\alpha} - 2\beta \cos\alpha)/p$ and $B = -\lambda$, then

$$\begin{aligned} p\phi = p\left(\frac{A}{B} - 1\right) &= -e^{-i\alpha}(pe^{-i\alpha} - 2\beta\cos\alpha) - p \\ &= -2e^{-i\alpha}(p - \beta)\cos\alpha =: -\xi. \end{aligned}$$

By Theorem 3.1, for $B \neq 0$, we get the following integral means for $f \in \mathcal{F}_p(\alpha, \beta, \lambda)$.

Theorem 3.4. *For $0 < \lambda \le 1$, $0 \le \beta < p$, $p \in \mathbb{N}$ and $|\alpha| < \pi/2$. Let $f \in \mathcal{F}_p(\alpha, \beta, \lambda)$ be such that z^p/f has the form (2.3). Then we have*

$$L_1(r, f, p) := r^{2p} I_1(r, f, p) \le \sum_{n=0}^{\infty} \left| \binom{\xi}{n} \right|^2 \lambda^{2n},$$

where $\xi = 2(p - \beta)e^{-i\alpha}\cos\alpha$. The equality is attained for the functions $k_{p,\alpha,\beta,\lambda}$ as defined by (2.2).

If we let $\lambda = 1$, then Theorem 3.4 yields:

Corollary 3.5. *Let $f \in \mathcal{F}_p(\alpha, \beta, 1) =: \mathcal{S}_{\alpha,p}(\beta)$, for $0 \le \beta < p$, $p \in \mathbb{N}$ and $|\alpha| < \pi/2$. Then we have*

$$L_1(r, f, p) \le \sum_{n=0}^{\infty} \left| \binom{\xi}{n} \right|^2,$$

where $\xi = 2(p - \beta)e^{-i\alpha}\cos\alpha$. The estimate is sharp. In particular, we have the following:

- $L_1(r, f, p) \le \sum_{n=0}^{\infty} \left|\binom{\eta}{n}\right|^2$ *for $f \in \mathcal{F}_p(\alpha, 0, 1) =: \mathcal{S}_{\alpha,p}$ where $\eta = 2pe^{-i\alpha}\cos\alpha$.*
- $L_1(r, f, p) \le \Gamma(1 + 4p)/\Gamma^2(1 + 2p)$ *for $f \in \mathcal{F}_p(0, 0, 1) =: \mathcal{S}_p^*$.*

All inequalities are sharp.

The case $p = 1$ in Corollary 3.5 is obtained in [21, Theorem 2]. The proof of Theorem 3.1 is presented in Section 6.

4. Area integral problem

Let $f \in \mathcal{A}_p$. The area of the multi-sheeted image of the disk $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ $(0 < r \leq 1)$ under f is denoted by $\Delta(r, f)$. Thus, in terms of the coefficients of $f \in \mathcal{A}_p$, $f'(z) = pz^{p-1} + \sum_{n=1}^{\infty}(n+p)a_{n+p}z^{n+p-1}$, with the help of the classical Parseval-Gutzmer formula (see [27]) one gets the relation

$$\Delta(r, f) = \iint_{\mathbb{D}_r} |f'(z)|^2 \, dxdy = \pi p r^{2p} + \pi \sum_{n=1}^{\infty}(n+p)|a_{n+p}|^2 r^{2(n+p)}.$$

This is called the Dirichlet integral of f. Computing this area is known as the *area problem for the functions of type f*. We call f Dirichlet-finite if $\Delta(1, f) < \infty$.

We now state our second main results.

Theorem 4.1. *Let $f \in \mathcal{S}_p^*(A, 0)$, for $0 < |A| \leq 1$ and $p \in \mathbb{N}$, be of the form* (2.3). *Then we have*

$$\max_{f \in \mathcal{S}_p^*(A,0)} \Delta\left(r, \frac{z^p}{f}\right) = \pi |A|^2 p^2 r^2 {}_0F_1(2, |A|^2 p^2 r^2) =: E_A(r, p), \tag{4.1}$$

where r, $0 < r \leq 1$, and the maximum is attained by the rotations of $k_{A,0,p}(z) = z^p e^{Apz}$.

The case $A = 1$ simplifies to:

Corollary 4.2. *If $f \in \mathcal{S}_p^*(1, 0)$ for $p \in \mathbb{N}$, then we have*

$$\max_{f \in \mathcal{S}_p^*(1,0)} \Delta\left(r, \frac{z^p}{f}\right) = \pi p^2 r^2 {}_0F_1(2, p^2 r^2), \quad r \in (0, 1].$$

The maximum is attained by the rotations of the function $k_{1,0,p}(z) = z^p e^{pz}$.

Theorem 4.3. *Let $f \in \mathcal{S}_p^*(A, B)$ for $A \in \mathbb{C}, -1 \leq B < 0$, $A \neq B$, $p \in \mathbb{N}$ and f be of the form* (2.3). *Then, for $0 < r \leq 1$, we have*

$$\max_{f \in \mathcal{S}_p^*(A,B)} \Delta\left(r, \frac{z^p}{f}\right) =: E_{A,B}(r, p),$$

where

$$E_{A,B}(r, p) = \pi |\overline{A} - B|^2 p^2 r^2 {}_2F_1\big(\phi p + 1, \overline{\phi} p + 1; 2; B^2 r^2\big),$$

with $\phi = (A/B) - 1$. The maximum is attained for the rotations of $k_{A,B,p}$ as defined by (1.2).

Moreover, Theorem 4.3, for $A = 1 - (2\beta/p)$ and $B = -1$, gives the following result.

Corollary 4.4. *If $f \in \mathcal{S}_p^*(\beta)$ for $0 \le \beta < p$ and $p \in \mathbb{N}$, then we have*

$$\max_{f\in\mathcal{S}_p^*(\beta)} \Delta\left(r, \frac{z^p}{f}\right) = 4\pi(p-\beta)^2 r^2 {}_2F_1\big((2\beta-2p+1),(2\beta-2p+1);2;r^2\big)$$

for all $r \in (0,1]$. The maximum is attained for the rotations of the function $z^p/(1-z)^{2p-2\beta}$. In particular, for $f \in \mathcal{S}_p^*(0) =: \mathcal{S}_p^*$, one has

$$\max_{f\in\mathcal{S}_p^*} \Delta\left(r, \frac{z^p}{f}\right) = 4\pi p^2 r^2 {}_2F_1\big(1-2p,1-2p;2;r^2\big), \quad r\in(0,1],$$

and the maximum is attained for the rotations of the function $z^p/(1-z)^{2p}$.

For the choice $p = 1$, the above results in this section are obtained in [20]. Choosing $A = (1-(2\beta/p))\lambda$ and $B = -\lambda$ in Theorem 4.3, we find that:

Corollary 4.5. *Let $f \in \mathcal{T}_p(\lambda,\beta)$ for $0 < \lambda \le 1$, $0 \le \beta < p$ and $p \in \mathbb{N}$. Then we have*

$$\max_{f\in\mathcal{T}_p(\lambda,\beta)} \Delta\left(r, \frac{z^p}{f}\right) = 4\pi\lambda^2(p-\beta)^2 r^2 {}_2F_1\big((2\beta-2p+1),(2\beta-2p+1);2;\lambda^2r^2\big),$$

where r, $0 < r \le 1$, and the maximum is attained for the rotations of $z^p/(1-\lambda z)^{2p-2\beta}$. In particular, for $f \in \mathcal{T}_p(1,\beta) =: \mathcal{S}_p^*(\beta)$, we get Corollary 4.4.

We end this section with the following special results.

The case $A = \lambda e^{-i\alpha}(e^{-i\alpha} - (2\beta/p)\cos\alpha)$ and $B = -\lambda$, simplifies that

$$\begin{aligned} p\left(\frac{A}{B}-1\right)+1 &= -e^{-i\alpha}(pe^{-i\alpha} - 2\beta\cos\alpha) - p + 1 \\ &= 1 - 2e^{-i\alpha}(p-\beta)\cos\alpha =: 1-\xi. \end{aligned}$$

By Theorem 4.3, we obtain Yamashita's conjecture on area maximum property for the class $\mathcal{F}_p(\alpha,\beta,\lambda)$.

Theorem 4.6. *Let λ,β,α such that $0 < \lambda \le 1$, $0 \le \beta < p$, $-\pi/2 < \alpha < \pi/2$ and $p \in \mathbb{N}$. If the function f, defined by (2.3), belongs to the class $\mathcal{F}_p(\alpha,\beta,\lambda)$, then we have*

$$\max_{f\in\mathcal{F}_p(\alpha,\beta,\lambda)} \Delta\left(r, \frac{z^p}{f}\right) = E_{\alpha,\beta,\lambda}(r,p),$$

where

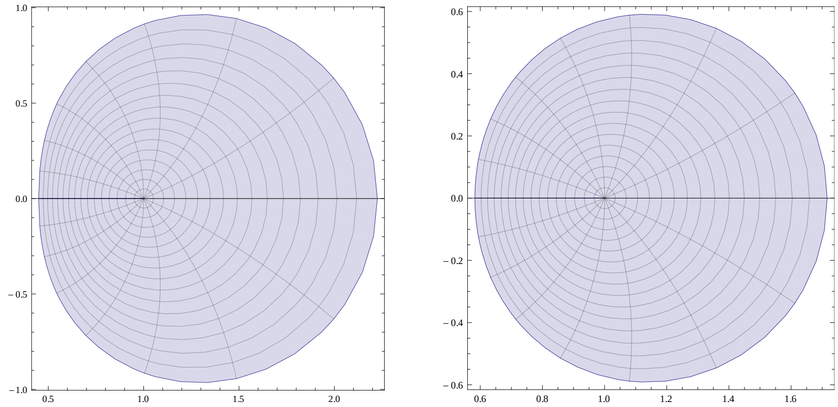
$$E_{\alpha,\beta,\lambda}(r,p) = \pi r^2\lambda^2|\overline{\xi}|^2 {}_2F_1(1-\xi, 1-\overline{\xi}; 2; \lambda^2 r^2),$$

with $\xi = 2(p-\beta)e^{-i\alpha}\cos\alpha$. The maximum is attained for the rotations of $k_{p,\alpha,\beta,\lambda}$ as defined by (2.2).

The case $\lambda = 1$ of Theorem 4.6 gives:

Corollary 4.7. *Let $f \in \mathcal{S}_{\alpha,p}(\beta) := \mathcal{F}_p(\alpha,\beta,1)$ be of the form (2.3). Then we have*

$$\max_{f\in\mathcal{S}_{\alpha,p}(\beta)} \Delta\left(r, \frac{z^p}{f}\right) = \pi r^2|\overline{\xi}|^2 {}_2F_1(1-\xi, 1-\overline{\xi}; 2; r^2), \quad 0 < r \le 1.$$



The image domain $g_{2/5,2}(\mathbb{D})$ The image domain $g_{1/5,3}(\mathbb{D}_{0.9})$

FIGURE 1. Images of the disk $\mathbb{D}_r$ under $g_{2/5,2}$ and $g_{1/5,3}$.

The maximum is attained for the rotations of $k_{p,\alpha,\beta,1}$ *as defined by* (2.2). *In particular, for* $f \in \mathcal{S}_{\alpha,p} := \mathcal{S}_{\alpha,p}(0)$*, one has*

$$\max_{f\in\mathcal{S}_{\alpha,p}} \Delta\left(r, \frac{z^p}{f}\right) = \pi r^2 |\overline{\eta}|^2 {}_2F_1\big(1-\eta, 1-\overline{\eta}; 2; r^2\big), \quad \eta = 2pe^{-i\alpha}\cos\alpha$$

for all $r \in (0, 1]$.

For the choice $p = 1$, Corollaries 4.5 and 4.7 are obtained in [27, Theorem 1.3] and [21, Theorem 3 and Corollary 4], respectively.

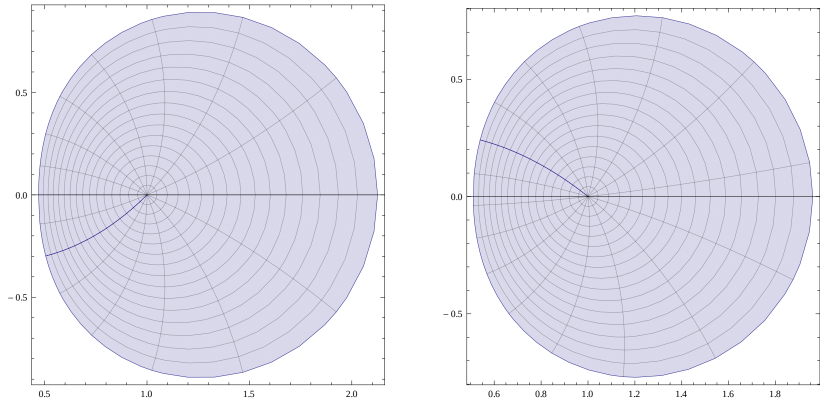
Proofs of Theorems 4.1 and 4.3 are presented in Section 6. To see the bounds for the Dirichlet finite function, we denote

$$E_A(1, p) = \pi p^2 |A|^2 \sum_{n=0}^{\infty} \frac{1}{(1)_n (2)_n} p^{2n} |A|^{2n},$$

$$E_{A,B}(1, p) = \pi p^2 |\overline{A} - B|^2 \sum_{n=0}^{\infty} \frac{(p\phi + 1)_n (p\overline{\phi} + 1)_n}{(2)_n (1)_n} B^{2n} \quad \text{and}$$

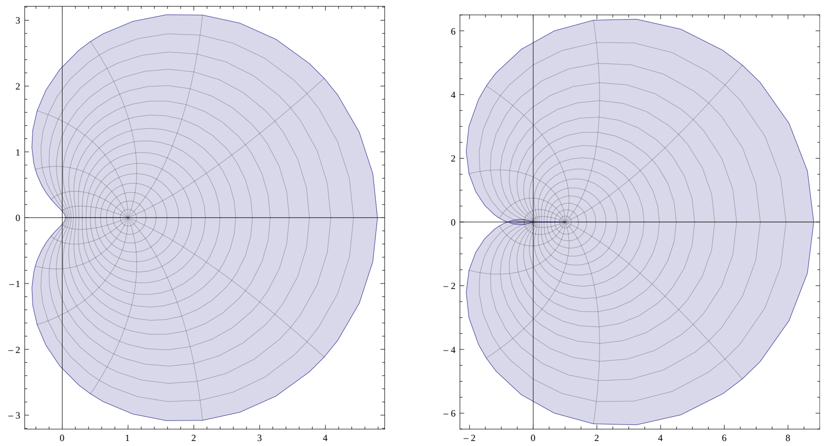
$$E_{\alpha,\beta,\lambda}(1, p) = \pi \lambda^2 |\overline{\xi}|^2 \sum_{n=0}^{\infty} \frac{(1-\xi)_n (1-\overline{\xi})_n}{(2)_n (1)_n} \lambda^{2n}.$$

The images of the disk $\mathbb{D}_r$ ($r \in (0, 1]$) under the extremal functions $g_{A,p}(z) := z^p/k_{A,p}(z) = e^{-Apz}$, $g_{A,B,p}(z) := z^p/k_{A,B,p}(z) = (1 + Bz)^{(1-A/B)p}$ and $z^p/k_{p,\alpha,\beta,\lambda}(z) =: l_{p,\alpha,\beta,\lambda}(z) = (1 - \lambda z)^{\xi}$ and numerical values of $E_A(r, p)$, $E_{A,B}(r, p)$ and $E_{\alpha,\beta,\lambda}(r, p)$ are described in Figures 1–5 and Tables 1 & 2, respectively, for several values of $A, B, \alpha, \beta, \lambda, r$ and p. We remind the reader that for $B = -1$, $E_{A,B}(1, p)$ is finite only if $2 > \mathrm{Re}(2 + p(\phi + \overline{\phi})/B)$, i.e., if $\mathrm{Re}\, A > -1$.



The image domain $g_{(1+i)/3,2}(\mathbb{D}_{0.8})$ The image domain $g_{2/8-i/5,3}(\mathbb{D}_{0.7})$

FIGURE 2. Images of the disk $\mathbb{D}_r$ under $g_{(1+i)/3,2}$ and $g_{2/8-i/5,3}$.

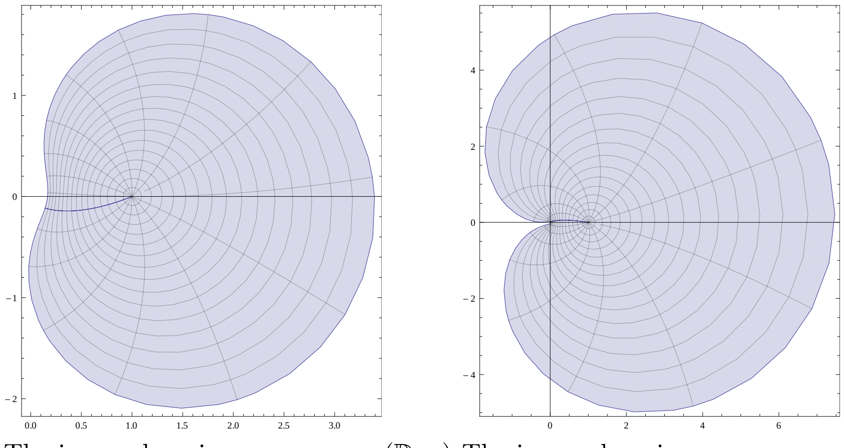


The image domain $g_{2/5,-3/5,52}(\mathbb{D})$ The image domain $g_{1/5,-9/10,3}(\mathbb{D}_{0.9})$

FIGURE 3. Images of the disk $\mathbb{D}_r$ under $g_{2/5,-3/5,2}$ and $g_{1/5,-9/10,3}$.

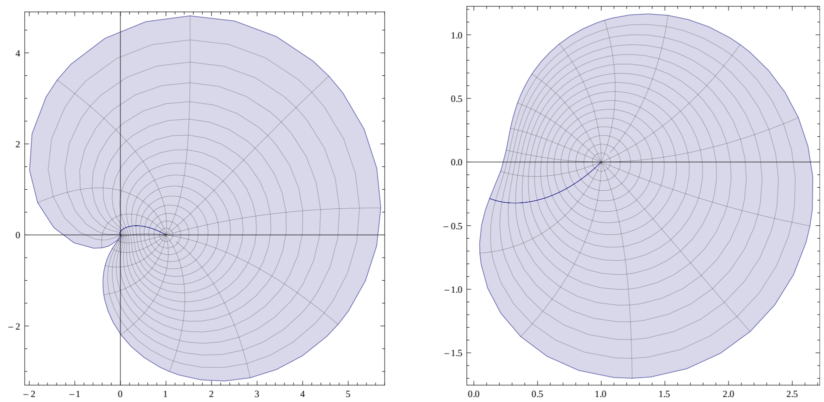
TABLE 1. Approximate values of $E_A(r,p)$ and $E_{A,B}(r,p)$

p	A	r	**Approximate Values of** $E_A(r,p)$	B	**Approximate Values of** $E_{A,B}(r,p)$
2	$2/5$	1	2.7264	$-3/5$	26.19994
3	$1/5$	0.9	1.05631	$-9/10$	112.473
2	$(1+i)/3$	0.8	2.34613	$-1/2$	10.5859
3	$2/8-i/5$	0.7	1.76615	$-99/100$	26.98



The image domain $g_{(1+i)/3,-0.5,2}(\mathbb{D}_{0.8})$ The image domain $g_{2/8-i/5,0.99,3}(\mathbb{D}_{0.7})$

FIGURE 4. Images of the disk $\mathbb{D}_r$ under $g_{(1+i)/3,-0.5,2}$ and $g_{2/8-i/5,0.99,3}$.



The image domain $l_{2,-\pi/6,1,0.9}(\mathbb{D})$ The image domain $l_{3,\pi/4,1.5,0.6}(\mathbb{D}_{0.9})$

FIGURE 5. Images of the disk $\mathbb{D}_r$ under $l_{2,-\pi/6,1,0.9}$ and $l_{3,\pi/4,1.5,0.6}$.

TABLE 2. Approximate values of $E_{\alpha,\beta,\lambda}(r,p)$

p	α	β	λ	r	**Approximate Values of** $E_{\alpha,\beta,\lambda}(r,p)$
2	$-\pi/6$	1	9/10	1	11.2667
3	$\pi/4$	1.5	3/5	0.9	7.1980

In the next section, we present the following crucial lemmas which play important roles for the proofs of our main results.

5. Preparatory results

We first present a necessary coefficient condition for a function $f \in \mathcal{S}_p^*(A,B)$.

Lemma 5.1. *Let $f \in \mathcal{S}_p^*(A,B)$ for $A \in \mathbb{C}, -1 \le B \le 0$, $A \neq B$, $p \in \mathbb{N}$ and f be of the form* (2.3). *Then*

$$\sum_{k=1}^{\infty} \left(k^2 - |kB + (\overline{A} - B)p|^2\right) |b_{k+p-1}|^2 \le |\overline{A} - B|^2 p^2$$

holds. Equality is attained for the function $k_{A,B,p}$ as defined by (1.2).

Proof. Let $f \in \mathcal{S}_p^*(A,B)$ and $g(z) := z^p/f(z)$. Then by subordination principle, we obtain

$$\frac{zg'(z)}{pg(z)} = \frac{(B-A)zw(z)}{1+Bzw(z)}, \quad z \in \mathbb{D},$$

where $w(0) = 1$ in $\mathbb{D}$. Substituting this in the series expansion (2.3) of g, we get

$$\sum_{k=1}^{\infty} kb_{k+p-1}z^{k-1} = -\left((A-B)p + \sum_{k=1}^{\infty} \big(kB + (A-B)p\big)b_{k+p-1}z^k\right) w(z).$$

It is equivalent to

$$\sum_{k=1}^{n} kb_{k+p-1}z^{k-1} + \sum_{k=n+1}^{\infty} c_k z^{k-1}$$
$$= -\left((A-B)p + \sum_{k=1}^{n-1} \big(kB + (A-B)p\big)b_{k+p-1}z^k\right) w(z)$$

for certain coefficients c_k. By Clunie's method [6] (see also [7,25,26]) for $n \in \mathbb{N}$, since $|w(z)| < 1$ in $\mathbb{D}$, we find

$$\sum_{k=1}^{n} k^2|b_{k+p-1}|^2 r^{2k-2} \le |A-B|^2p^2 + \sum_{k=1}^{n-1} \big|kB + (A-B)p\big|^2 |b_{k+p-1}|^2 r^{2k},$$

it holds for all $r \in (0,1)$ and for all large n. It is equivalent to

$$\sum_{k=1}^{n} k^2|b_{k+p-1}|^2 r^{2k-2} - \sum_{k=1}^{n-1} \big|kB + (A-B)p\big|^2 |b_{k+p-1}|^2 r^{2k} \le |A-B|^2p^2. \tag{5.1}$$

If we take $r \to 1^-$ and allow $n \to \infty$, then we get the desired inequality

$$\sum_{k=1}^{\infty} \left(k^2 - \big|kB + (\overline{A} - B)p\big|^2\right) |b_{k+p-1}|^2 \le |\overline{A} - B|^2 p^2.$$

Equality occurs in the above inequality for the function $k_{A,B,p}$ as defined by (1.2). The proof of Lemma 5.1 is complete. □

Lemma 5.2. *Let* $0 < |A| \leq 1$ *and* $f \in \mathcal{S}_p^*(A, 0)$. *For* $|z| < r$, *suppose that*

$$\frac{z^p}{f(z)} = 1 + \sum_{k=1}^{\infty} b_{k+p-1} z^k \text{ and } e^{-Apz} = 1 + \sum_{k=1}^{\infty} c_{k+p-1} z^k, \quad r \in (0, 1].$$

Then for all $N \in \mathbb{N}$,

$$\sum_{k=1}^{N} k|b_{k+p-1}|^2 r^{2k} \leq \sum_{k=1}^{N} k|c_{k+p-1}|^2 r^{2k} \tag{5.2}$$

holds.

Proof. It is enough to prove the lemma for $0 < A \leq 1$. From the relation (5.1) for $B = 0$, we get

$$\sum_{k=1}^{n-1} (k^2 - A^2 p^2 r^2)|b_{k+p-1}|^2 r^{2k-2} + n^2 |b_{n+p-1}|^2 r^{2n-2} \leq A^2 p^2.$$

Multiplying by r^2 on both sides, we obtain

$$\sum_{k=1}^{n-1} (k^2 - A^2 p^2 r^2)|b_{k+p-1}|^2 r^{2k} + n^2 |b_{n+p-1}|^2 r^{2n} \leq A^2 p^2 r^2. \tag{5.3}$$

Obviously, the series expansion of e^{-Apz} shows that the equality, when $n \to \infty$, in (5.3) attains with $b_{k+p-1} = c_{k+p-1}$.

We split remaining part of the proof into three following steps.

Step-I: Cramer's rule.

We consider the inequalities corresponding to (5.3) for $n = 1, 2, \ldots, N$ and multiply the n^{th} coefficient by a factor $\lambda_{n,N}$. These factors are chosen in such a way that the addition of the left sides of the modified inequalities results the left side of (5.2) and hence from the modified inequalities, we get

$$\sum_{k=1}^{N} k|b_{k+p-1}|^2 r^{2k} \leq A^2 p^2 r^2 \lambda_{n,N}. \tag{5.4}$$

First, we shall evaluate the suitable multipliers $\lambda_{n,N}$ by Cramer's rule. Secondly, in Step-II, we will prove that these multipliers are all positive. Finally, from (5.2) and (5.4), we will prove the inequality

$$A^2 p^2 r^2 \lambda_{n,N} \leq \sum_{k=1}^{N} k|c_{k+p-1}|^2 r^{2k} \tag{5.5}$$

in Step-III. Here $c_{k+p-1} = (Ap)^k/(k!)$.

For the calculation of the factors $\lambda_{n,N}$, we get the following system of linear equations

$$(5.6) \qquad k = k^2\lambda_{k,N} + \sum_{n=k+1}^{N} \lambda_{n,N}(k^2 - A^2p^2r^2), \quad k = 1, 2, \dots, N.$$

Since the matrix of this system is an upper triangular matrix with positive integers as diagonal elements, the solution of this system is uniquely determined. Cramer's rule allows us to write the solution of the system (5.6) in the form

$$\lambda_{n,N} = \frac{((n-1)!)^2}{(N!)^2} \operatorname{Det} A_{n,N},$$

where $A_{n,N}$ is the $(N-n+1) \times (N-n+1)$ matrix constructed as follows:

$$A_{n,N} = \begin{bmatrix} n & n^2 - A^2p^2r^2 & \cdots & n^2 - A^2p^2r^2 \\ n+1 & (n+1)^2 & \cdots & (n+1)^2 - A^2p^2r^2 \\ \vdots & \vdots & \vdots & \vdots \\ N & 0 & \cdots & N^2 \end{bmatrix}.$$

Determinants of these matrices can be obtained by expanding, according to Laplace's rule with respect to the last row, wherein the first coefficient is N and the last one is N^2. The rest of the entries are zeros. This expansion and a mathematical induction lead to the following formula: if $k \le N-1$, then

$$\lambda_{k,N} = \lambda_{k,N-1} - \frac{1}{N}\left(1 - \frac{A^2p^2r^2}{k^2}\right) \prod_{m=k+1}^{N-1} \left(\frac{A^2p^2r^2}{m^2}\right).$$

We see that the sequence $\{\lambda_{k,N}\}$ is strictly decreasing in N when $k \in \mathbb{N}$ is fixed and $N \ge k$, i.e., $\lambda_{k,N} < \lambda_{k,N-1}$ with

$$(5.7) \qquad \lambda_k := \lim_{N\to\infty} \lambda_{k,N} = \frac{1}{k} - \left(1 - \frac{A^2p^2r^2}{k^2}\right) \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{m=k+1}^{n-1} \left(\frac{A^2p^2r^2}{m^2}\right).$$

To prove that $\lambda_{k,N} > 0$ for all $N \in \mathbb{N}, 1 \le k \le N$, it is adequate to show that $\lambda_k \ge 0$ for $k \in \mathbb{N}$. This will be completed in Step II. But before that we want to remark that the proof of the said inequality is sufficient for the proof of the theorem, since, as we remarked for (5.3), equality holds for $b_{k+p-1} = c_{k+p-1}$.

Step-II: Positivity of the multipliers.

In this step, we show that

$$\sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{m=k+1}^{n-1} \left(\frac{A^2p^2r^2}{m^2}\right) \le \frac{1}{k\left(1 - \frac{A^2p^2r^2}{k^2}\right)} = \frac{1}{k} \sum_{n=k+1}^{\infty} \left(\frac{A^2p^2r^2}{k^2}\right)^n,$$

which is indeed easy to prove, i.e., from (5.7), $\lambda_k \ge 0$.

Step-III:
Since the sequence $\{\lambda_{n,N}\}$ is strictly decreasing in N for each fixed n, $n \leq N$, i.e., $\lambda_{n,N} < \lambda_{n,n}$, so that

$$A^2p^2r^2\lambda_{n,N} < A^2p^2r^2\lambda_{n,n} = \frac{A^2p^2r^2}{n}$$
$$< A^2p^2r^2 \leq \sum_{k=1}^{N} \frac{k(Ap)^{2k}}{(k!)^2} r^{2k}.$$

This means that inequality (5.5) holds. The proof of our lemma is complete. □

Lemma 5.3. *Let $f \in \mathcal{S}_p^*(A,B)$ for $A \in \mathbb{C}, -1 \leq B < 0$, $A \neq B$ and $p \in \mathbb{N}$. Suppose that*

$$(1-Bz)^{(1-(A/B))p} = 1 + \sum_{k=1}^{\infty} d_{k+p-1}z^k \quad \textit{and} \quad \frac{z^p}{f(z)} = 1 + \sum_{k=1}^{\infty} b_{k+p-1}z^k$$

for all r, $0 < r \leq 1$. Then the inequality

$$\sum_{k=1}^{N} k|b_{k+p-1}|^2 r^{2k} \leq \sum_{k=1}^{N} k|d_{k+p-1}|^2 r^{2k} \tag{5.8}$$

is valid for all $N \in \mathbb{N}$.

Proof. Multiplying by r^2 on both sides of the inequality (5.1), we obtain

$$\sum_{k=1}^{n-1} \left(k^2 - \left|kB + (A-B)p\right|^2 r^2\right) |b_{k+p-1}|^2 r^{2k} + n^2|b_{n+p-1}|^2 r^{2n} \leq |A-B|^2p^2r^2.$$

Set for an abbreviation $\phi := (A/B) - 1$ and rewrite the last inequality in the form

$$\sum_{k=1}^{n-1} \left(k^2 - |k+p\phi|^2 B^2 r^2\right) |b_{k+p-1}|^2 r^{2k} + n^2|b_{n+p-1}|^2 r^{2n} \leq B^2p^2r^2|\phi|^2. \tag{5.9}$$

It is apparent that in the inequality (5.9), the equality is attained for the function $(1-Bz)^{(1-(A/B))p}$ with $b_{k+p-1} = d_{k+p-1}$, when $n \to \infty$.

We split remaining part of the proof into three following steps.

Step-I: Cramer's rule.
We consider the inequalities corresponding to (5.9) for $n = 1, 2, \ldots, N$ and multiply the nth coefficient by a factor $\lambda_{n,N}$. These factors are chosen in such a way that the addition of the left sides of the modified inequalities results the left side of (5.8) leading to the inequality

$$\sum_{k=1}^{N} k|b_{k+p-1}|^2 r^{2k} \leq B^2p^2r^2|\phi|^2\lambda_{n,N}. \tag{5.10}$$

First, we solve for the suitable multipliers $\lambda_{n,N}$ by Cramer's rule. Secondly, in Step II, we prove that these multipliers are all positive. Finally, from (5.8) and (5.10), we prove the inequality

$$B^2p^2r^2|\phi|^2\lambda_{n,N} \leq \sum_{k=1}^{N} k|d_{k+p-1}|^2r^{2k}, \quad n=1,2,\ldots,N \tag{5.11}$$

in Step III. Here $d_{k+p-1} = B^k(p\phi)_k/(k!)$.

For computing the factors $\lambda_{n,N}$, we solve the following system of linear equations

$$k = k^2\lambda_{k,N} + \left(k^2 - |k+p\phi|^2B^2r^2\right)\sum_{n=k+1}^{N}\lambda_{n,N}, \quad k=1,2,\ldots,N. \tag{5.12}$$

Since the matrix of this system is an upper triangular matrix with positive integers as diagonal elements, the solution of this system is uniquely determined. Cramer's rule allows us to write the solution of the system (5.12) in the form

$$\lambda_{n,N} = \frac{((n-1)!)^2}{(N!)^2}\,\mathrm{Det}\,A_{n,N},$$

where $A_{n,N}$ is the $(N-n+1)\times(N-n+1)$ matrix constructed as follows:

$$A_{n,N} = \begin{bmatrix} n & n^2-|n+p\phi|^2B^2r^2 & \cdots & n^2-|n+p\phi|^2B^2r^2 \\ n+1 & (n+1)^2 & \cdots & (n+1)^2-|n+1+p\phi|^2B^2r^2 \\ \vdots & \vdots & \vdots & \vdots \\ N & 0 & \cdots & N^2 \end{bmatrix}.$$

Determinants of these matrices can be found by expanding according to Laplace's rule with respect to the last row, wherein the first coefficient is N and the last one is N^2. The rest of the entries are zeros. This expansion and a mathematical induction result in the following formula. If $k \leq N-1$, then

$$\lambda_{k,N} = \lambda_{k,N-1} - \frac{1}{N}\left(1-\left|1+\frac{p\phi}{k}\right|^2B^2r^2\right)\prod_{m=k+1}^{N-1}\left(\left|1+\frac{p\phi}{m}\right|^2B^2r^2\right). \tag{5.13}$$

Note that $U_{k,p} := \left(1-|1+(p\phi/k)|^2\,B^2r^2\right)$ in (5.13) may be positive as well as negative for some $k\in\mathbb{N}$. For instance, see Table 3.

Case (i): Suppose that $U_{k,p}$ is non-positive.

From (5.13), we see that, the sequence $\{\lambda_{k,N}\}$ is strictly increasing in N for every fixed $k\in\mathbb{N}, k\leq N-1$, i.e.,

$$\lambda_{k,N} - \lambda_{k,N-1} > 0$$

so that

$$\lambda_{k,N} > \lambda_{k,N-1} > \cdots > \lambda_{k,k} = 1/k > 0,$$

and thus $\lambda_k \geq 0$ when $N\to\infty$ as required.

Case (ii): Suppose that $U_{k,p}$ is non-negative.

TABLE 3. Signs of the constant $U_{k,p}$

k	A	B	p	r	$U_{k,p}$
1	0.9	-0.6	2	0.4	0.0784
2	3	-0.4	2	0.8	-4.76
3	$3-i$	-0.9	2	0.2	0.8666
2	0.8	-0.7	5	0.9	-6.5350
3	0.5	-1	5	0.6	0.19
2	$2+3i$	-0.8	5	0.3	-7.5221

From (5.13), for each fixed $k \in \mathbb{N}$, $N \geq k$, the sequence $\{\lambda_{k,N}\}$ is strictly decreasing in N, i.e., $\lambda_{k,N} - \lambda_{k,N-1} < 0$ with

$$\begin{aligned} \lambda_k &:= \lim_{N\to\infty} \lambda_{k,N} \\ &= \frac{1}{k} - \left(1 - \left|1 + \frac{p\phi}{k}\right|^2 B^2 r^2\right) \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{m=k+1}^{n-1} \left(\left|1 + \frac{p\phi}{m}\right|^2 B^2 r^2\right). \end{aligned} \tag{5.14}$$

To show that $\lambda_{k,N} > 0$ for all $N \in \mathbb{N}, k \in [1, N]$, it is enough to show that $\lambda_k \geq 0$ for $k \in \mathbb{N}$. Proof of this will be completed in Step II. But before that, we want to note that the proof of the said inequality is adequate for the proof of the theorem, since, we observed in the beginning of the proof, equality is obtained for $b_{k+p-1} = d_{k+p-1}$.

Step-II: Positivity of the multipliers.
Let for an abbreviation

$$S_k = \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{m=k+1}^{n-1} \left(\left|1 + \frac{p\phi}{m}\right|^2 B^2 r^2\right), \quad k \in \mathbb{N}.$$

We now show that

$$S_k \leq \frac{1}{k\left(1 - \left|1 + \frac{p\phi}{k}\right|^2 B^2 r^2\right)}.$$

From the equation (5.14), we get

$$\lambda_k = \frac{1}{k} - S_k + \left(\left|1 + \frac{p\phi}{k}\right|^2 B^2 r^2\right) S_k.$$

Again set for an abbreviation

$$T_k = \frac{1}{k} + \left(\left|1 + \frac{p\phi}{k}\right|^2 B^2 r^2\right) S_k.$$

It is enough to prove that

$$T_k \leq \frac{1}{k\left(1-\left|1+\frac{p\phi}{k}\right|^2 B^2r^2\right)}. \tag{5.15}$$

To prove (5.15) we use the inequality

$$\frac{1}{n\left(1-\left|1+\frac{p\phi}{n}\right|^2 B^2r^2\right)} > \frac{1}{(n+1)\left(1-\left|1+\frac{p\phi}{n+1}\right|^2 B^2r^2\right)} \tag{5.16}$$

(this inequality follows from the fact that $n\left(1-\left|1+(p\phi/n)\right|^2 B^2r^2\right)$ is increasing in n which can be easily verified by the derivative test) and the identity

$$\frac{1}{n\left(1-\left|1+\frac{p\phi}{n}\right|^2 B^2r^2\right)} = \frac{1}{n} + \frac{\left|1+\frac{p\phi}{n}\right|^2 B^2r^2}{n\left(1-\left|1+\frac{p\phi}{n}\right|^2 B^2r^2\right)}, \tag{5.17}$$

which are admissible for each $n \in \mathbb{N}$. Repeated application of (5.16) and (5.17) for $n = k, k+1, \ldots, Q$ results in the inequality

$$\begin{aligned}\frac{1}{k\left(1-\left|1+\frac{p\phi}{k}\right|^2 B^2r^2\right)} &> \sum_{n=k}^{Q} \frac{1}{n} \prod_{m=k}^{n-1} \left(\left|1+\frac{p\phi}{m}\right|^2 B^2r^2\right) \\ &\quad + \frac{\prod_{m=k}^{Q}\left(\left|1+\frac{p\phi}{m}\right|^2 B^2r^2\right)}{Q\left(1-\left|1+\frac{p\phi}{Q}\right|^2 B^2r^2\right)} \\ &=: S_{k,Q} + R_{k,Q} \text{ for } k \leq Q.\end{aligned}$$

Since $R_{k,Q} > 0$, allow the limit as $Q \to \infty$, we get

$$\frac{1}{k\left(1-\left|1+\frac{p\phi}{k}\right|^2 B^2r^2\right)} \geq \lim_{T\to\infty} S_{k,Q} = \sum_{n=k}^{\infty} \frac{1}{n} \prod_{m=k}^{n-1} \left(\left|1+\frac{p\phi}{m}\right|^2 B^2r^2\right) = Q_k,$$

and we complete the inequality (5.15).

Step-III:

In this step, we prove (5.11). Taking the left side of (5.11) for $N = 2, n = 1$ and using the inequality (5.13), we obtain

$$\begin{aligned}B^2p^2r^2|\phi|^2\lambda_{1,2} &= B^2p^2r^2|\phi|^2\left(\lambda_{1,1} - \frac{1}{2}\left(1-|1+p\phi|^2B^2r^2\right)\right) \\ &= \frac{B^2p^2r^2|\phi|^2}{2} + \frac{B^4p^2r^4|\phi|^2|1+p\phi|^2}{2} \quad (\text{since } \lambda_{1,1} = 1)\end{aligned}$$

$$\leq B^2p^2r^2|\phi|^2 + \frac{B^4p^2r^4|\phi|^2|1+p\phi|^2}{2} = \sum_{k=1}^{2} \frac{k|(p\phi)_k|^2}{(k!)^2}(Br)^{2k}.$$

Since, $d_{k+p-1} = B^k(p\phi)_k/(k!)$, then the inequality (5.11) holds for $N = 2$, $n = 1$.

Now, we can complete the proof by the method of induction. Assume that the inequality (5.11) is true for $N = m$, i.e.,

$$B^2p^2r^2|\phi|^2\lambda_{n,m} \leq \sum_{k=1}^{m} k|d_{k+p-1}|^2 r^{2k}, \quad n = 1, 2, \ldots, m. \tag{5.18}$$

Then for $N = m+1$, using the inequality (5.13), we deduce that

$$\begin{aligned}
&B^2p^2r^2|\phi|^2\lambda_{n,m+1}\\
&= B^2p^2r^2|\phi|^2 \left[\lambda_{n,m} - \frac{1}{m+1}\left(1 - \left|1+\frac{p\phi}{n}\right|^2 B^2r^2\right)\right.\\
&\quad \left.\times \prod_{t=n+1}^{m}\left(\left|1+\frac{p\phi}{t}\right|^2 B^2r^2\right)\right]\\
&\leq \sum_{k=1}^{m} k|d_{k+p-1}|^2 r^{2k} - \frac{1}{m+1}\left(1 - \left|1+\frac{p\phi}{n}\right|^2 B^2r^2\right)\\
&\quad \times \prod_{t=n+1}^{m}\left(\left|1+\frac{p\phi}{t}\right|^2 B^2r^2\right) B^2p^2r^2|\phi|^2 \quad \text{(by (5.18))}\\
&= \sum_{k=1}^{m} k|d_{k+p-1}|^2 r^{2k} - \frac{1}{m+1}\prod_{t=n+1}^{m}\left(\left|1+\frac{p\phi}{t}\right|^2 B^2r^2\right) B^2p^2r^2|\phi|^2\\
&\quad + \frac{1}{m+1}\prod_{t=n}^{m}\left(\left|1+\frac{p\phi}{t}\right|^2 B^2r^2\right) B^2p^2r^2|\phi|^2\\
&\leq \sum_{k=1}^{m} \frac{k|(p\phi)_k|^2}{(k!)^2}(Br)^{2k} + \frac{1}{m+1}\prod_{t=n}^{m}\left(\left|1+\frac{p\phi}{t}\right|^2 B^2r^2\right) B^2p^2r^2|\phi|^2,
\end{aligned}$$

since $d_{k+p-1} = B^k(p\phi)_k/(k!)$. The last inequality implies that

$$\begin{aligned}
B^2p^2r^2|\phi|^2\lambda_{n,m+1} \leq &\sum_{k=1}^{m} \frac{k|(p\phi)_k|^2}{(k!)^2}(Br)^{2k}\\
&+ \frac{1}{m+1}\prod_{t=1}^{m}\left(\left|1+\frac{p\phi}{t}\right|^2 B^2r^2\right) B^2p^2r^2|\phi|^2
\end{aligned}$$

or equivalently,

$$B^2p^2r^2|\phi|^2\lambda_{n,m+1}$$

$$\leq \sum_{k=1}^{m} \frac{k|(p\phi)_k|^2}{(k!)^2}(Br)^{2k} + \frac{(m+1)(B^2r^2)^{m+1}}{(1)_{m+1}^2} \prod_{t=1}^{m} \left(\left|1+\frac{p\phi}{t}\right|^2 \right) (1)_m^2 p^2|\phi|^2$$
$$= \sum_{k=1}^{m} \frac{k|(p\phi)_k|^2}{(k!)^2}(Br)^{2k} + \frac{(m+1)(B^2r^2)^{m+1}}{(1)_{m+1}^2} \prod_{t=1}^{m} \left(|t+p\phi|^2 \right) p^2|\phi|^2$$
$$= \sum_{k=1}^{m} \frac{k|(p\phi)_k|^2}{(k!)^2}(Br)^{2k} + \frac{(m+1)(B^2r^2)^{m+1}}{(1)_{m+1}^2} |(p\phi)_{m+1}|^2$$
$$= \sum_{k=1}^{m+1} \frac{k|(p\phi)_k|^2}{(k!)^2}(Br)^{2k}.$$

Hence, we obtain the desired inequality (5.11).

The proof of Lemma 5.3 is complete. □

6. Proofs of the main results

Proof of Theorem 3.1

Let $f \in \mathcal{S}_p^*(A,B)$. We apply the theorem of Hallenbeck and Ruschewey [12, Theorem 2] and get

$$\frac{f(z)}{z^p} \prec \frac{1}{(1+Bz)^{(1-(A/B))p}}, \quad z \in \mathbb{D},$$

so that

$$\frac{z^p}{f(z)} \prec (1+Bz)^{(1-(A/B))p} =: \chi_{A,B,p}(z), \quad z \in \mathbb{D}, \tag{6.1}$$

where

$$\chi_{A,B,p}(z) = \frac{z^p}{k_{A,B,p}(z)} = \begin{cases} (1+Bz)^{(1-(A/B))p} & \text{if } B \neq 0 \\ e^{-Apz} & \text{if } B = 0 \end{cases}$$

and the function $k_{A,B,p}$ defined in (1.2). For $B \neq 0$, we rewrite the quantity $\chi_{A,B,p}(z)$ in hypergeometric function notation and get

$$\chi_{A,B,p}(z) = \begin{cases} {}_2F_1(p\phi,1;1;-Bz) & \text{if } B \neq 0 \\ e^{-Apz} & \text{if } B = 0 \end{cases}$$
$$=: \sum_{n=0}^{\infty} d_{n+p-1} z^n \tag{6.2}$$

with $\phi = (A/B) - 1$ and

$$d_{n+p-1} = \begin{cases} \frac{(-1)^n (p\phi)_n B^n}{n!} & \text{if } B \neq 0 \\ \frac{(-1)^n (Ap)^n}{n!} & \text{if } B = 0. \end{cases}$$

From (6.1), z^p/f and $\chi_{A,B,p}$ are two analytic functions and have the series representation (2.3) and (6.2) (with $b_{p-1} = 1 = d_{p-1}$), respectively. Then by

Rogosinski's result (see [8, 26]), we get

$$\sum_{n=0}^{k} |b_{n+p-1}|^2 r^{2n} \le \sum_{n=0}^{k} |d_{n+p-1}|^2 r^{2n}$$

for $0 < r < 1$ and $k \in \mathbb{N}$. Thus, from (6.1) and (6.2), we obtain

$$\sum_{n=0}^{k} |b_{n+p-1}|^2 r^{2n} \le \begin{cases} \displaystyle\sum_{n=0}^{k} \frac{(p\phi)_n (p\overline{\phi})_n}{(n!)^2} B^{2n} r^{2n} & \text{if } B \neq 0 \\ \displaystyle\sum_{n=0}^{k} \frac{1}{(n!)^2} \big(p|A|\big)^{2n} r^{2n} & \text{if } B = 0. \end{cases}$$

If we take $r \to 1$ and allow $k \to \infty$, then we find the inequality

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} |b_{n+p-1}|^2 &\le \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{(p\phi)_n (p\overline{\phi})_n}{(n!)^2} B^{2n} & \text{if } B \neq 0 \\ \displaystyle\sum_{n=0}^{\infty} \frac{1}{(n!)^2} \big(p|A|\big)^{2n} & \text{if } B = 0 \end{cases} \\ &= \begin{cases} {}_2F_1\big(p\phi, p\overline{\phi}; 1; B^2\big) & \text{if } B \neq 0 \\ J_0(2ip|A|) & \text{if } B = 0, \end{cases} \end{aligned}$$

where $J_0(z)$ is the Bessel function of zero order (see [30] for its definition).

Now, we evaluate the integral means for the function z^p/f and get

$$\begin{aligned} L_1(r, f, p) := r^{2p} I_1(r, f, p) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{r^{2p}}{|f(re^{i\theta})|^2} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{z^p}{f(z)} \right|^2 d\theta \\ &= 1 + \sum_{n=1}^{\infty} |b_{n+p-1}|^2 r^{2n} \\ &\le 1 + \sum_{n=1}^{\infty} |b_{n+p-1}|^2, \end{aligned}$$

which establishes the desired inequality. The result is sharp and it can be easily verified by considering the function $z^p/k_{A,B,p}$, defined in (1.2).

Proof of Theorem 4.1

Suppose $f \in \mathcal{S}_p^*(A)$, $0 < |A| \le 1$ and $p \in \mathbb{N}$. It is enough to prove the theorem for $0 < A \le 1$. By the definition of $\mathcal{S}_p^*(A)$, we get

$$\frac{zf'(z)}{pf(z)} \prec 1 + Az = \frac{zk'_{A,p}(z)}{pk_{A,p}(z)}, \ z \in \mathbb{D}.$$

Let $g(z) = z^p/f(z)$ be of the form (2.3). Then using the theorem of Hallenbeck and Ruschewey [12, Theorem 2] and subordinate property, we get

$$g(z) \prec e^{-Apz} = \frac{z^p}{k_{A,p}(z)}.$$

By rewriting the last subordination relation in power series form, we have

$$1+\sum_{k=1}^{\infty} b_{k+p-1}z^k \prec e^{-Apz} = 1+\sum_{k=1}^{\infty} c_{k+p-1}z^k,$$

where $c_{k+p-1} = (-1)^k (Ap)^k/(k!)$. Now, by Lemma 5.2, for $r \in (0,1]$, we have

$$\sum_{k=1}^{N} k|b_{k+p-1}|^2 r^{2k} \leq \sum_{k=1}^{N} k|c_{k+p-1}|^2 r^{2k},\ N \in \mathbb{N}.$$

If we assume $N \to \infty$, then it follows

$$\pi \sum_{k=1}^{\infty} k|b_{k+p-1}|^2 r^{2k} \leq \pi \sum_{k=1}^{\infty} k|c_{k+p-1}|^2 r^{2k},$$

i.e.,

$$\Delta\left(r, \frac{z^p}{f}\right) \leq \Delta\left(r, \frac{z^p}{k_{A,p}}\right).$$

It is easy to simplify that

$$\Delta\left(r, z^p/k_{A,p}\right) = \pi|A|^2 p^2 r^2 {}_0F_1(2, |A|^2 p^2 r^2) = E_A(r,p),$$

then we get the desired identity (4.1). The maximum is attained by rotations of $k_{A,p}(z) = z^p e^{Apz}$.

The proof of our theorem is complete. □

Proof of Theorem 4.3

Let $g(z) = z^p/f(z)$ be of the form (2.3). Now, by the definition of $\mathcal{S}_p^*(A,B)$, we obtain

$$\frac{zf'(z)}{pf(z)} \prec \frac{1+Az}{1+Bz} = \frac{zk'_{A,B,p}(z)}{pk_{A,B,p}(z)}.$$

By Hallenbeck and Ruschewey's result [12] and subordinate principle, we find that

$$g(z) \prec (1+Bz)^{(1-(A/B))p} = \frac{z^p}{k_{A,B,p}(z)}.$$

Suppose, $z^p/k_{A,B,p}$ has the power series representation $1+\sum_{k=1}^{\infty} d_{k+p-1}z^n$ with $d_{k+p-1} = (-1)^k B^k (p\phi)_k/(k!)$. Then it follows from Lemma 5.3, for $N \in \mathbb{N}$,

$$\sum_{k=1}^{N} k|b_{k+p-1}|^2 r^{2k} \leq \sum_{k=1}^{N} k|d_{k+p-1}|^2 r^{2k}, \quad 0 < r \leq 1,$$

which implies that

$$\pi \sum_{k=1}^{\infty} k|b_{k+p-1}|^2 r^{2k} \leq \pi \sum_{k=1}^{\infty} k|d_{k+p-1}|^2 r^{2p},$$

i.e.,

$$\Delta\left(r, \frac{z^p}{f}\right) \leq \Delta\left(r, \frac{z^p}{k_{A,B,p}}\right).$$

By the area formula for $z^p/k_{A,B,p}$, we easily have

$$\begin{aligned}\pi^{-1}\Delta\left(r,\frac{z}{k_{A,B,p}}\right) &= \sum_{k=1}^{\infty} k|d_{k+p-1}|^2 r^{2k}\\ &= \sum_{k=1}^{\infty} k\frac{(p\phi)_k(p)\overline{\phi})_k}{(1)_k^2}B^{2k}r^{2k}\\ &= B^2p^2r^2|\phi|^2\sum_{k=0}^{\infty}\frac{(p\phi+1)_k(p\overline{\phi}+1)_k}{(2)_k(1)_k}B^{2k}r^{2k}.\end{aligned}$$

Hence,

$$\Delta\left(r,\frac{z}{k_{A,B,p}}\right) = \pi|\overline{A}-B|^2p^2r^2{}_2F_1\big(p\phi+1,p\overline{\phi}+1;2;B^2r^2\big) = E_{A,B}(r,p),$$

and the proof of Theorem 3.4 is complete. □

7. Concluding remarks and open problem

For $-1\leq B\leq 0$ and $A\in\mathbb{C}$, $A\neq B$, define

$$\mathcal{C}_p(A,B) := \left\{f\in\mathcal{A}_p:\ \frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\frac{1+Az}{1+Bz},\quad z\in\mathbb{D}\right\}.$$

The choices $A=1-(2\beta/p)$ and $B=-1$ turn the class $\mathcal{C}_p(A,B)$ into the class $\mathcal{C}_p(\beta)$, the class of p-valent convex of order β. The class $\mathcal{C}_p(0) =: \mathcal{C}_p$ is the usual class of p-valent convex functions. The results of this paper (e.g. Theorems 4.1 and 4.3) motivate the following problems for further research in this direction:

Open problem 7.1. It would be interesting to know the solution of the maximal area integral problem for functions of type z^p/f when f ranges over $\mathcal{C}_p(A,B)$. In particular, the problem is still open for $f\in\mathcal{C}_p(\beta)$, $0\leq\beta<p$. The case $p=1$ is also stated in [21] (see also [20, 27]).

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References

[1] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University press, U.K., 1999.
[2] M. K. Aouf, *Coefficient estimates for some classes of p-valent functions*, Internat. J. Math. Math. Sci. **11** (1988), no. 1, 47–54.
[3] ———, *A generalization of multivalent functions with negative coefficients*, J. Korean Math. Soc. **25** (1988), no. 1, 53–66.
[4] A. Baernstein II, *Integral means, univalent functions and circular symmetrization*, Acta Math. **133** (1974), 139–169.

[5] G. R. Blakley, *Classes of p-valent starlike functions*, Proc. Amer. Math. Soc. **13** (1962), 152–157.
[6] J. Clunie, *On meromorphic schlicht functions*, J. London Math. Soc. **34** (1959), 215–216.
[7] J. Clunie and F. R. Keogh, *On starlike and convex schlicht functions*, J. London Math. Soc. **35** (1960), 229–233.
[8] P. L. Duren, *Univalent Functions*, Springer-Verlag, New York, 1983.
[9] E. G. Goluzina, *On the coefficients of a class of functions, regular in a disk and having an integral representation in it*, J. Soviet Math. **6** (1974), 606–617.
[10] A. W. Goodman, *On the Schwarz-Christoffel transformation and p-valent functions*, Trans. Amer. Math. Soc. **68** (1950), 204–223.
[11] L. Gromova and A. Vasil'ev, *On integral means of star-like functions*, Proc. Indian Acad. Sci. Math. Sci. **112** (2002), no. 4, 563–570.
[12] D. J. Hallenbeck and St. Ruscheweyh, *Subordination by convex functions*, Proc. Amer. Math. Soc. **52** (1975), 191–195.
[13] W. Janowski, *Some extremal problems for certain families of analytic functions*, Ann. Polon. Math. **28** (1973), 297–326.
[14] S. S. Miller and P. T. Mocanu, *Differential Subordinations*: *Theory and Applications*, Vol. 225, Marcel Dekker, New York and Basel, 2000.
[15] M. Obradović, S. Ponnusamy, and K.-J. Wirths, *A proof of Yamashita's conjecture on area integral*, Comput. Methods Funct. Theory **13** (2013), no. 3, 479–492.
[16] D. A. Patil and N. K. Thakare, *On coefficient bound of p-valent λ-spiral functions of order α*, Indian J. Pure Appl. Math. **10** (1979), no. 7, 842–853.
[17] ———, *On convex hulls and extreme points of p-valent starlike and convex classes with applications*, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.) **27(75)** (1983), no. 2, 145–160.
[18] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht in Göttingen, 1975.
[19] ———, *Boundary Behaviour of Conformal Maps*, Berlin: Springer, 1992.
[20] S. Ponnusamy, S. K. Sahoo, and N. L. Sharma, *Maximal area integral problem for certain class of univalent analytic functions*, Mediterr. J. Math. **13** (2016), no. 2, 607–623.
[21] S. Ponnusamy and K.-J. Wirths, *On the problem of Gromova and Vasil'ev on integral means, and Yamashita's conjecture for spirallike functions*, Ann. Acad. Sci. Fenn. Math. **39** (2014), no. 2, 721–731.
[22] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960.
[23] G. L. Reddy and K. S. Padmanabhan, *On analytic functions with reference to the Bernardi integral operator*, Bull. Austral. Math. Soc. **25** (1982), no. 3, 387–396.
[24] M. S. Robertson, *On the theory of univalent functions*, Ann. of Math. **37** (1936), no. 2, 374–408.
[25] ———, *Quasi-subordination and coefficient conjectures*, Bull. Amer. Math. Soc. **76** (1970), 1–9.
[26] W. Rogosinski, *On the coefficients of subordinate functions*, Proc. London Math. Soc. **48** (1943), no. 2, 48–82.
[27] S. K. Sahoo and N. L. Sharma, *On area integral problem for analytic functions in the starlike family*, J. Class. Anal. **6** (2015), no. 1, 72–83.
[28] ———, *A note on a class of p-valent starlike functions of order beta*, Sib. Math. J. **57** (2016), no. 2, 364–368.
[29] N. L. Sharma, *A note on Coefficient estimates for some classes of p-valent functions*, Ukrainian Math. J., to appear.
[30] N. M. Temme, *Special Functions*: *An Introduction to the Classical Functions of Mathematical Physics*, Wiley-Interscience, New York, 1996.
[31] A. Vasil'ev, *Univalent functions in two dimensional free boundary problems*, Acta Appl. Math. **79** (2003), no. 3, 249–280.

[32] A. Vasil'ev and I. Markina, *On the geometry of Hele-Shaw flows with small surface tension*, Interface Free Bound. **5** (2003), no. 2, 183–192.
[33] S. Yamashita, *Area and length maxima for univalent functions*, Bull. Austral. Math. Soc. **41** (1990), no. 3, 435–439.

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