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2-TYPE SURFACES AND QUADRIC HYPERSURFACES SATISFYING $\langle \Delta x, x \rangle = $ const.

Changrim Jang and Haerae Jo

Abstract. Let M be a connected n-dimensional submanifold of a Euclidean space E^{n+k} equipped with the induced metric and Δ its Laplacian. If the position vector x of M is decomposed as a sum of three vectors $x = x_1 + x_2 + x_0$ where two vectors x_1 and x_2 are non-constant eigen vectors of the Laplacian , i.e., $\Delta x_i = \lambda_i x_i, i = 1, 2$ ($\lambda_i \in R$) and x_0 is a constant vector, then, M is called a 2-type submanifold. In this paper we showed that a 2-type surface M in E^3 satisfies $\langle \Delta x, x - x_0 \rangle = c$ for a constant c, where $\langle \ , \ \rangle$ is the usual inner product in E^3, then M is an open part of a circular cylinder. Also we showed that if a quadric hypersurface M in a Euclidean space satisfies $\langle \Delta x, x \rangle = c$ for a constant c, then it is one of a minimal quadric hypersurface, a genaralized cone , a hypersphere, and a spherical cylinder .

1. Introduction

Let M be an n-dimensional submanifold of the $(n+k)$-dimensionl Euclidean space E^{n+k}, equipped with the induced metric. Denote by Δ the Laplacian of M. If the position vector x of M in E^{n+k} can be decomposed as a finite sum of non-constant eigenvectors of Δ, we shall say that M is of finite-type. More precisely, M is said to be of q-type if the position vector x of M can be expressed as in the following form:

$$x = x_0 + x_{i_1} + \cdots + x_{i_q},$$

where x_0 is a constant vector, and $x_{i_j} (j = 1, \cdots, q)$ are non-constant vectors in E^{n+k} such that $\Delta x_{i_j} = \lambda_{i_j} x_{i_j}$, $\lambda_{i_j} \in R$, $\lambda_{i_1} < \cdots < \lambda_{i_q}$. The notion of finite-type submanifolds has been introduced by B.-Y. Chen [1]. Many results concerning this subject are obtained during last three decades. One of the interesting research areas on this subject is a classification of 2-type sumanifolds. Th.Hasanis and Th.Vlachos proved that the only 2-type surface in the three dimenional sphere S^3 is an open part of a product of two circles of different radii [4]. Also they proved that a spherical hypersurface M is of 2-type if and only if it

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has constant scalar curvature and mean curvature [5]. In [2] B.-Y.Chen studied a special 2-type surface M in E^3 whose postion vector x can be decomposed as a sum of two non-constant eingenvectors $x = x_1 + x_2$, $\Delta x_1 = 0$, $\Delta x_2 = \lambda x_2$, $0 \neq \lambda \in R$. Such a 2-type surface is said to be of null 2-type. Especillay he proved that the only null 2-type surface in E^3 is a circular cylinder. Many studies on null 2-type submanifolds are followed. But untill now generally 2-type surfaces are not classified. We can notice that every known finite-type hypersurface M satisfies the condition $\langle \Delta x, x \rangle = c$ for a constant c, where x is the position vector of M and $\langle\ ,\ \rangle$ denotes the usual inner product in Euclidean space. Note that the condition $\langle \Delta x, x \rangle = c$ for a constant c is not coordinate invariant. Sometimes a parallel translation is necessary to see that this condition can be satisfied. So we would like to study finite-type submanifold satisfying the codndition $\langle \Delta x, x \rangle = c$ for a constant c. In Section 3 we will show that if a 2-type surface M in E^3 satisfies the condition $\langle \Delta x, x - x_0 \rangle = c$ for a constant c, then it is an open part of a circular cylinder. In [3] B.-Y. Chen, F. Dillen and H. Z. Song proved that if M is a quadric hypersurface of finite-type in a Euclidean space, then M is one of a minimal quadric hypersurface, a spherical cylinder, and a hypersphere. In Section 4, we will show that if a quadric hypersurface M in a Euclidean space satisfies the condition $\langle \Delta x, x \rangle = c$ for a constant c, then it is one of a minimal quadric hypersurface, a generalized cone , a hypersphere, and a spherical cylinder.

2. Preliminaries

Consider an n-dimensional submanifold M of E^{n+1} and denote $\bar{\nabla}$ and ∇ the usaual Riemannian connection of E^{n+1} and the induced connection on M, respectively. The formulas of Gauss and Weingarten are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1}$$
$$\bar{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2}$$

for vector fields X, Y tangent to M and ξ normal to M, where h is the second fundamental form , D the normal connnection, and A the shape operator of M. For each normal vector ξ at a point $p \in M$, the shape operator A_ξ is a self adjoint operator of the tangent space T_pM at p. The second fundamental form h and the shape operator A are related by

$$\langle A_\xi X, Y \rangle = \langle h(X, Y), \xi \rangle, \tag{3}$$

where $\langle\ ,\ \rangle$ is the usaual inner product in E^{n+1}. Let v be an E^{n+1}-valued smooth function on M, and let $\{e_1, e_2, \cdots, e_n\}$ be a local orthornomal frame field of M. We define

$$\Delta v = \sum_{i=1}^{n} (\bar{\nabla}_{e_i} \bar{\nabla}_{e_i} v - \bar{\nabla}_{\nabla_{e_i} e_i} v).$$

It is well known that the position vector x and the mean curvature vector H of M in E^{n+1} satisfy

$$\Delta x = H. \tag{4}$$

Let e_{n+1} be a local unit normal vector to M. Since the mean curvature vector H is normal to M, we have $H = \langle H, e_{n+1}\rangle e_{n+1}$. The function $\langle H, e_{n+1}\rangle$ is called mean curvature function and it will be denoted by α.

3. 2-type surface in E^3 satisfying $\langle \Delta x, x - x_0\rangle = const.$

Let M be a 2-type surface in E^3. Then its position vector x is expressed in the form

$$x = x_0 + x_1 + x_2,$$

where x_0 is a constant vector, and $x_i (i = 1, 2)$ are nonconstant vectors in E^3 such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in R$, $\lambda_1 \neq \lambda_2$. By (4) we have $\Delta x = H = \lambda_1 x_1 + \lambda_2 x_2$ and $\Delta^2 x = \Delta H = \lambda_1^2 x_1 + \lambda_2^2 x_2$. Thus

$$\Delta^2 x = (\lambda_1 + \lambda_2)\Delta x - \lambda_1 \lambda_2 (x - x_0). \tag{5}$$

The general basic formula of ΔH derived in [1] plays important role in the study of low type. In particular, if M is a surface in E^3, it reduces to

$$\Delta H = (\Delta\alpha - \alpha ||A_{e_3}||^2)e_3 - 2\alpha A_{e_3}(\text{grad}\alpha) - \alpha \text{grad}\alpha, \tag{6}$$

where α is the mean curvature function and e_3 a unit normal vector of M in E^3. By comparing the tangential part of both (5) and (6), we find

$$\lambda_1 \lambda_2 (x - x_0)^T = 2A_{e_3}(\text{grad}\alpha) + \alpha \text{grad}\alpha, \tag{7}$$

where $(x - x_0)^T$ means the tangential part of the vector $x - x_0$. Now suppose that

$$\langle \Delta x, x - x_0\rangle = c \tag{8}$$

holds for a constant c. Let $\{e_1, e_2\}$ be a local orthonormal frame of M. Since

$$\begin{aligned}
\Delta\langle \Delta x, x - x_0\rangle &= \sum_{i=1}^{2} e_i e_i \langle \Delta x, x - x_0\rangle - \sum_{i=1}^{2} \nabla_{e_i} e_i \langle \Delta x, x - x_0\rangle \\
&= \sum_{i=1}^{2} e_i (\langle \bar{\nabla}_{e_i}(\Delta x), x - x_0\rangle + \langle \Delta x, e_i\rangle) \\
&\quad - \sum_{i=1}^{2} (\langle \bar{\nabla}_{\nabla_{e_i} e_i}(\Delta x), x - x_0\rangle + \langle \Delta x, \nabla_{e_i} e_i\rangle) \\
&= \sum_{i=1}^{2} e_i \langle \bar{\nabla}_{e_i}(\Delta x), x - x_0\rangle - \sum_{i=1}^{2} \langle \bar{\nabla}_{\nabla_{e_i} e_i}(\Delta x), x - x_0\rangle \\
&= \sum_{i=1}^{2} (\langle \bar{\nabla}_{e_i} \bar{\nabla}_{e_i}(\Delta x), x - x_0\rangle + \langle \bar{\nabla}_{e_i}(\Delta x), e_i\rangle) \\
&\quad - \sum_{i=1}^{2} \langle \bar{\nabla}_{\nabla_{e_i} e_i}(\Delta x), x - x_0\rangle \\
&= \langle \Delta(\Delta x), x - x_0\rangle + \sum_{i=1}^{2} \langle \bar{\nabla}_{e_i}(\Delta x), e_i\rangle \\
&= \langle \Delta^2 x, x - x_0\rangle + \sum_{i=1}^{2} \langle D_{e_i}(\Delta x) - A_{\Delta x} e_i, e_i\rangle \text{ (by (2))} \\
&= \langle \Delta^2 x, x - x_0\rangle - \sum_{i=1}^{2} \langle A_{\Delta x} e_i, e_i\rangle \\
&= \langle \Delta^2 x, x - x_0\rangle - \sum_{i=1}^{2} \langle \Delta x, h(e_i, e_i)\rangle \text{ (by (3))} \\
&= \langle \Delta^2 x, x - x_0\rangle - \langle \Delta x, \Delta x\rangle,
\end{aligned}$$

(8) and $\Delta x = H = \alpha e_3$ imply

$$\langle \Delta^2 x, x - x_0\rangle - \alpha^2 = 0. \tag{9}$$

From (5), (8) and (9), we get

$$(\lambda_1 + \lambda_2)c - \lambda_1\lambda_2\langle x - x_0, x - x_0\rangle - \alpha^2 = 0.$$

Differentiatiating both sides of the above equation in the direction of a tangent vector X on M, we find

$$-2\lambda_1\lambda_2\langle x - x_0, X\rangle - 2\alpha X(\alpha) = 0$$

or

$$X(\alpha) = -\frac{\lambda_1\lambda_2}{\alpha}\langle X, (x - x_0)^T\rangle.$$

This implies that

$$\text{grad}\alpha = -\frac{\lambda_1\lambda_2}{\alpha}(x - x_0)^T. \tag{10}$$

Lemma 3.1. *Let M be a 2-type surface in E^3 whose position vector x is expressed as $x = x_0 + x_1 + x_2$, where x_0 is a constant vector, and $x_i (i = 1, 2)$ are nonconstant vectors in E^3 such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in R$, $\lambda_1 \neq \lambda_2$. Assume that $\langle \Delta x, x - x_0\rangle = c$ holds for a constant c. Then the mean curvature function α of M is constant.*

Proof. Suppose that α is nonconstant. If M is of null 2-type, then M is a circular cylinder [2], which implies that the mean curvature function α is constant. So the assumptiom implies that M is not of null 2-type. Substituting (10) into (7) we get

$$A_{e_3}(x - x_0)^T = -\alpha(x - x_0)^T,$$

which implies that gradα is a principal vector of the shape operator A_{e_3} and the corresponding principal curvature is $-\alpha$. Since α is the sum of two principal curvatures, the other principal curvature is 2α. Let $\{e_1, e_2\}$ be a local orthonormal frame of M such that e_1 is parallel to gradα. Note that $e_2(\alpha) = 0$. By the Coddazzi equations, we have

$$e_1(2\alpha) = (-\alpha - 2\alpha)\omega_{12}(e_2) = -3\alpha\omega_{12}(e_2), \tag{11}$$

$$e_2(-\alpha) = (-\alpha - 2\alpha)\omega_{12}(e_1) = -3\alpha\omega_{12}(e_1), \tag{12}$$

where ω_{12} is the connection form of $\{e_1, e_2\}$. Since α is nonzero and $e_2(\alpha) = 0$, from (12) it follows that $\omega_{12}(e_1) = 0$. From (11) we have $\omega_{12}(e_2) = -\frac{2e_1(\alpha)}{3\alpha}$. This and $\omega_{12}(e_1) = 0$ implies that

$$\omega_{12} = -\frac{2e_1(\alpha)}{3\alpha}\theta_2, \tag{13}$$

where $\{\theta_1, \theta_2\}$ denotes the dual 1-forms of $\{e_1, e_2\}$. Since grad$\alpha = e_1(\alpha)e_1$, by (10) we find

$$\langle x - x_0, e_2\rangle = 0.$$

Differentiating both sides of the above in the direction of e_2, we find

$$1 + \langle x - x_0, \bar{\nabla}_{e_2}e_2\rangle = 0. \tag{14}$$

By (1) and $h(e_2, e_2) = 2\alpha e_3$ we have

$$\bar{\nabla}_{e_2}e_2 = h(e_2, e_2) + \nabla_{e_2}e_2 = 2\alpha e_3 + \omega_{21}(e_2)e_1.$$

Substituting this into (14) and we find

$$1 + 2\langle x - x_0, \alpha e_3\rangle + \omega_{21}(e_2)\langle x - x_0, e_1\rangle = 0.$$

By using (8), (10), (13) and considering grad$\alpha = e_1(\alpha)e_1$ it follows that

$$1 + 2c - \frac{2(e_1(\alpha))^2}{3\lambda_1\lambda_2} = 0$$

from the above equation. This implies that $e_1(\alpha)$ is a constant. Since $d\omega_{12} = -K\theta_1 \wedge \theta_2$, where K is the Gauss curvature of M, from (13) and the structural equation $d\theta_2 = \omega_{21} \wedge \theta_1$, we get

$$\begin{aligned} -K\theta_1 \wedge \theta_2 &= -\frac{2e_1(\alpha)}{3}(-\frac{e_1(\alpha)}{\alpha^2}\theta_1 \wedge \theta_2) - \frac{2e_1(\alpha)}{3\alpha}(-\frac{2e_1(\alpha)}{3\alpha}\theta_1 \wedge \theta_2) \\ &= \frac{10e_1(\alpha)^2}{9\alpha^2}\theta_1 \wedge \theta_2. \end{aligned}$$

Since $K = -2\alpha^2$, from this we have $18\alpha^4 = 10(e_1(\alpha))^2$, which implies that α is constant. This is a contradiction. □

Proposition 3.2. *Let M be a 2-type surface in E^3 whose position vector x is expressed as $x = x_0 + x_1 + x_2$, where x_0 is a constant vector, and $x_i (i = 1, 2)$ are nonconstant vectors in E^3 such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in R$, $\lambda_1 \neq \lambda_2$. Assume that $\langle \Delta x, x - x_0 \rangle = c$ holds for a constant c. Then M is of null 2-type , i.e., M is an open part of a circular cylinder.*

Proof. By Lemma 3.1, the mean curvature function α of M is constant. By (6) it implies that $\Delta^2 x = \Delta H$ is normal to M. From (5) it follows that $\lambda_1\lambda_2(x - x_0)$ is normal to M. If M is not of null 2-type, then the vector $x - x_0$ is normal to M. This is impossible. Thus M is of null 2-type. Consequently M is an open part of a circular cylinder [2]. □

4. Quadric hypersurfaces satisfying $\langle \Delta x, x \rangle = const.$

Consider the set M of points $(x_1, \cdots, x_{n+1})$ in the $(n+1)$-dimensional Euclidean space E^{n+1} satisfying the following equation of the second degree:

$$\sum_{i,j=1}^{n+1} a_{ij} x_i x_j + \sum_{i=1}^{n+1} b_i x_i + d = 0, \tag{15}$$

where a_{ij}, b_j, d are real numbers. The equation can be experssed as in the following form

$$\langle Ax + b, x \rangle + d = 0,$$

where $\langle\ ,\ \rangle$ is the usual inner product of E^{n+1}, for the matrix $A = (a_{ij})$ and vectors $x = \begin{bmatrix} x_1 \\ \vdots \\ x_{n+1} \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ \vdots \\ b_{n+1} \end{bmatrix}$. We can assume without loss of generality that the matrix $A = (a_{ij})$ is symmetric and A is not a zero matrix. If the left side of the equation (15) is reducible polynomial, then M is a hyperplane or a union of two hyperplanes. In this paper we assume that the polynomial given by the left side of (15) is irreducible over real numbers. In general the whole set M does not form a submanifold of E^{n+1}. Instead it can be shown that the subset $M' = \{x = \begin{bmatrix} x_1 \\ \vdots \\ x_{n+1} \end{bmatrix} \in M | 2Ax + b \neq 0\}$ is an n-dimensional submanifold of E^{n+1} by using the implicit function theorem. In this paper, we mean the hypersurface M' by a quadric hypersurface M described by (15). We will study a quadric hypersurface M satisfying the condition $\langle \Delta x, x\rangle = c$ for a constant c, where x is the position vector of M and Δ its Laplacian. Note that the condition $\langle \Delta x, x\rangle = c$ for a constant c is invariant under an orthogonal transformation. So without loss of generalty we may assume that the matrix A is diagonal with digonal entries $\lambda_1, \cdots, \lambda_{n+1}$. So the equation (15) can be written as

$$\sum_{i=1}^{n+1} \lambda_i x_i^2 + \sum_{i=1}^{n+1} b_i x_i + d = 0, \tag{16}$$

or

$$\langle Ax + b, x\rangle + d = 0, \tag{17}$$

where A is the diagonal matrix diag$[\lambda_1, \cdots, \lambda_{n+1}]$. Note again that we only consider the case that the left side of (16) is irreducible. First of all we will investigate some basic properties of quadric hypersurface M and classify the minimal quadric hypersurfaces in an elementary way.

Lemma 4.1. *The vector $2Ax + b$ is a nozero normal vector to M.*

Proof. Differentiating both sides of (17) in the direction of a tangent vector field X of M , we find

$$\langle AX, x\rangle + \langle Ax + b, X\rangle = 0$$

or

$$\langle 2Ax + b, X\rangle = 0.$$

This implies that $2Ax+b$ is normal to M. By assumptiom $2Ax+b$ is nonzero. □

Lemma 4.2. *Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame of M . Then the following holds.*

$$\sum_{i=1}^{n} \langle 2Ae_i, e_i\rangle + \langle 2Ax + b, \Delta x\rangle = 0. \tag{18}$$

Proof. Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame of M . By Lemma 4.1 we have

$$\langle 2Ax + b, e_i\rangle = 0$$

for $i = 1, 2, \cdots, n$. Differentiating the above equation in the direction of e_i we find

$$\langle 2Ae_i, e_i\rangle + \langle 2Ax + b, h(e_i, e_i)\rangle = 0,$$

where h is the second fundamental form of M . Since $\Delta x = \sum_{i=1}^{n} h(e_i, e_i)$, by summing up over i we get (18). □

It is already well-known that the only minimal quadric hypersurfaces are cones described in the following lemma. But we will prove it by using Lemma 4.2.

Lemma 4.3. *If M is a minimal quadric hypersurface, then by a parallel translation and an orthogonal coordinate change, it can be described by*

$$(l-1)\sum_{i=1}^{k} x_i^2 + (1-k)\sum_{i=k+1}^{k+l} x_i^2 = 0$$

for integers $k, l(k, l > 1, k+l \leq n+1)$.

Proof. Let M be a minmal quadric hypersurfaces described by (16). Since the condition minimality is invariant under any parallel translation and orthogonal coordinate change, we may write the equation (16) as

$$\sum_{i=1}^{s} \lambda_i x_i^2 + d = 0 \ (\ \lambda_i \neq 0,\ i = 1, \cdots, s \leq n+1)$$

or

$$\sum_{i=1}^{s} \lambda_i x_i^2 + \sum_{i=s+1}^{s+t} b_i x_i + d = 0.$$
$$(\ \lambda_i \neq 0,\ i = 1, \cdots, s,\ b_j \neq 0, j = s+1, \cdots, s+t \leq n+1)$$

We will show that the second description is impossible. Suppose that M is described by the second equation. Let $e_1, \cdots, e_n$ be a local orthonormal frame of M . Since $2Ax + b$ is a normal vector field of M. Thus $e_1, \cdots, e_n$ and $\frac{2Ax+b}{|2Ax+b|}$ form a Euclidean orthonormal frame, where $|2Ax + b|$ means the magnitude of the vector $2Ax + b$. So we have

$$\sum_{i=1}^{n} \langle 2Ae_i, e_i \rangle + \langle 2A\frac{2Ax+b}{|2Ax+b|}, \frac{2Ax+b}{|2Ax+b|} \rangle = \mathrm{tr}(2A),$$

where $\mathrm{tr}(2A)$ is the trace of the matrix $2A$. Since M is minimal, it follows from (18) and the above equation that

$$\langle 2A(2Ax+b), 2Ax+b \rangle = \mathrm{tr}(2A)\langle 2Ax+b, 2Ax+b \rangle. \tag{19}$$

Since $b_{s+1} \neq 0$, M can be locally considered as a graph of the function $x_{s+1} = \frac{1}{b_{s+1}}(-d - \sum_{i=1}^{s} \lambda_i x_i^2 - \sum_{i=s+2}^{s+t} b_i x_i)$. The equation (19) can be written as

$$\sum_{i=1}^{s} 4\lambda_i^2(\mathrm{tr}(2A) - 2\lambda_i)x_i^2 - \mathrm{tr}(2A)\sum_{i=s+1}^{s+t} b_i^2 = 0.$$

As $x_1, \cdots, x_s$ are independent variables, from the above equation, we have $\lambda_i = \mathrm{tr}(A), i = 1, \cdots, s$ and $\mathrm{tr}(2A)\sum_{i=s+1}^{s+t} b_i^2 = 0$. From this we find $\lambda_i = 0, i = 1, \cdots, s$, which is a contradiction. Thus we know that $b = 0$, which implies $\langle Ax, x \rangle + d = 0$, or $\sum_{i=1}^{s} \lambda_i x_i^2 + d = 0$. The equation (19) can be simplified as

$$\langle A^2x, Ax \rangle = \mathrm{tr}(A)\langle Ax, Ax \rangle. \tag{20}$$

Without loss of generality we may consider M as a graph of the function $x_1 = \pm\frac{1}{\sqrt{\lambda_1}}\sqrt{-d - \sum_{i=2}^{s}\lambda_i x_i^2}$. Substituting this into (20) we get

$$\sum_{i=2}^{s}\lambda_i(\lambda_i^2 - \mathrm{tr}(A)\lambda_i - \lambda_1^2 + \mathrm{tr}(A)\lambda_1)x_i^2 - \lambda_1 d(\lambda_1 - \mathrm{tr}(A)) = 0.$$

From this we have

$$\lambda_i^2 - \mathrm{tr}(A)\lambda_i - \lambda_1^2 + \mathrm{tr}(A)\lambda_1 = 0,\ i = 2, \cdots, s,\ \lambda_1 d(\lambda_1 - \mathrm{tr}(A)) = 0.$$

From the second equation, we have $d = 0$ or $\lambda_1 = \mathrm{tr}(A)$. If $\lambda_1 = \mathrm{tr}(A)$, then the first equation and the condition $\lambda_i \neq 0, i = 2, \cdots, s$ we find $\lambda_i = \mathrm{tr}(A), i = 1, \cdots, s$, which implies that $s = 1$ or M and thus $\lambda_1 x_1^2 + d$ is reducible. So we have $d = 0$. The first equation is factorized into

$$(\lambda_i - \lambda_1)(\lambda_i - (\mathrm{tr}(A) - \lambda_1)) = 0,$$

which implies that $\lambda_i = \lambda_1$ or $\lambda_i = \mathrm{tr}(A) - \lambda_1$, $i = 1, \cdots, s$. If all $\lambda_i = \lambda_1$, then $\lambda\sum_{i=1}^{s} x_i^2 = 0$ or $x_1 = \cdots = x_s = 0$, which is impossible. So without loss of generality, we may assume that $\lambda_1 = \cdots = \lambda_k$ and $\lambda_{k+1} = \cdots = \lambda_s$ for some positive integer $k, 1 \leq k < s$. Suppose that $k = 1$. Then, since $\mathrm{tr}(A) = \lambda_1 + (s-1)(\mathrm{tr}(A) - \lambda_1)$, $(s-2)(\mathrm{tr}(A) - \lambda_1) = 0$. This implies that $s = 2$ or $\mathrm{tr}(A) - \lambda_1 = 0$. In any cases, the polynomial $\sum_{i=1}^{s}\lambda_i x_i^2$ is reducible. So we may assume that $1 < k < s-1$. Let $\lambda_1 = \lambda$, $\mathrm{tr}(A) - \lambda_1 = \mu$ and $s - k = l$. From $\mathrm{tr}(A) = k\lambda + l\mu$ and $\mu = \mathrm{tr}(A) - \lambda$, we have $\mu = \frac{1-k}{l-1}\lambda$. So given quadric hypersurface can be discribed as

$$\lambda\sum_{i=1}^{k} x_i^2 + \frac{1-k}{l-1}\lambda\sum_{i=k+1}^{k+l} x_i^2 = 0$$

or

$$(l-1)\sum_{i=1}^{k} x_i^2 + (1-k)\sum_{i=k+1}^{k+l} x_i^2 = 0 \tag{21}$$

for some two positive integers $k, l > 1, k + l \leq n + 1$. Conversely, we can show that a quadric hypersurface described by (21) is a minimal hypersurface. Let M be a quadric hypersurface in E^{n+1} discribed by (21). The equation (21) can be written as $\langle Ax, x\rangle = 0$, where A is an $(n+1)\times(n+1)$ diagonal matrix with diagonals $l-1, \cdots, l-1, 1-k, \cdots, 1-k, 0, \cdots, 0$. Let $e_1, \cdots, e_n$ be a local orthonormal frame of M. Since $\frac{Ax}{|Ax|}$ is a unit normal vector to M, we have

$$\langle Ae_1, e_1\rangle + \cdots + \langle Ae_n, e_n\rangle + \langle A\frac{Ax}{|Ax|}, \frac{Ax}{|Ax|}\rangle = \mathrm{tr}(A) = k(l-1) + l(1-k) = l - k. \tag{22}$$

By using (21) we have

$$\begin{aligned}\langle A\frac{Ax}{|Ax|},\frac{Ax}{|Ax|}\rangle &= \frac{(l-1)^3\sum_{i=1}^k x_i^2+(1-k)^3\sum_{i=k+1}^{k+l} x_i^2}{(l-1)^2\sum_{i=1}^k x_i^2+(1-k)^2\sum_{i=k+1}^{k+l} x_i^2}\\ &= \frac{(l-1)^3\sum_{i=1}^k x_i^2+(1-l)(1-k)^2\sum_{i=1}^k x_i^2}{(l-1)^2\sum_{i=1}^k x_i^2+(1-l)(1-k)\sum_{i=1}^k x_i^2}\\ &= l-k.\end{aligned}$$

So from (22) and the above equation we get

$$\langle Ae_1,e_1\rangle+\cdots+\langle Ae_n,e_n\rangle=0. \tag{23}$$

By similar computation in Lemma 4.2, we have

$$\langle Ae_1,e_1\rangle+\cdots+\langle Ae_n,e_n\rangle+\langle Ax,\Delta x\rangle=0.$$

This and (23) imply that $\langle Ax,\Delta x\rangle=0$. Subsequently we have $\Delta x=0$. So we can conclude that M is minimal. □

From now on we assume that M is a quadric hypersurface described by $\langle Ax+b,x\rangle+d=0$ for an $(n+1)\times(n+1)$ daigoanl matrix A with diagonal entries $\lambda_1,\cdots,\lambda_{n+1}$ and a constant vector $b=\begin{bmatrix} b_1\\ \vdots\\ b_{n+1}\end{bmatrix}$ in E^{n+1} and satisfies $\langle\Delta x,x\rangle=c$ for a constant c .

Lemma 4.4. *Assume that $c\neq 0$. Then the following holds.*

$$tr(2A)\langle 2Ax+b,2Ax+b\rangle\langle 2Ax+b,x\rangle-\langle 2A(2Ax+b),2Ax+b\rangle\langle 2Ax+b,x\rangle$$
$$+c\langle 2Ax+b,2Ax+b\rangle^2=0.$$

Proof. Let $\{e_1,\cdots,e_n\}$ be a local orthonormal frame of M . Then by Lemma 4.2 the following holds.

$$\sum_{i=1}^n\langle 2Ae_i,e_i\rangle+\langle 2Ax+b,\Delta x\rangle=0. \tag{24}$$

Also we have

$$\sum_{i=1}^n\langle 2Ae_i,e_i\rangle+\langle 2A\frac{2Ax+b}{|2Ax+b|},\frac{2Ax+b}{|2Ax+b|}\rangle=\mathrm{tr}(2A). \tag{25}$$

Since both $2Ax+b$ and Δx are normal to M, there exists a scalar function $f(x)$ defined on M such that $\Delta x=f(x)(2Ax+b)$. This and (24) imply that $\sum_{i=1}^n\langle 2Ae_i,e_i\rangle=-f(x)\langle 2Ax+b,2Ax+b\rangle$. Substituting this into (25), we have

$$\mathrm{tr}(2A)-\frac{\langle 2A(2Ax+b),2Ax+b\rangle}{\langle 2Ax+b,2Ax+b\rangle}+f(x)\langle 2Ax+b,2Ax+b\rangle=0.$$

From this and $\langle \Delta x, x\rangle = f(x)\langle 2Ax+b, x\rangle = c$, it follows that

$$\mathrm{tr}(2A) - \frac{\langle 2A(2Ax+b), 2Ax+b\rangle}{\langle 2Ax+b, 2Ax+b\rangle} + \frac{c}{\langle 2Ax+b, x\rangle}\langle 2Ax+b, 2Ax+b\rangle = 0$$

or

$$\mathrm{tr}(2A)\langle 2Ax+b, 2Ax+b\rangle\langle 2Ax+b, x\rangle - \langle 2A(2Ax+b), 2Ax+b\rangle\langle 2Ax+b, x\rangle$$
$$+c\langle 2Ax+b, 2Ax+b\rangle^2 = 0.$$

□

We proceed two cases seperately.

Case 1. $\langle \Delta x, x\rangle = 0$

If $\Delta x = 0$, then M is a minimal hypersurface. Assume that M is nonminimal, that is, $\Delta x \neq 0$. As both of Δx and $2Ax+b$ are normal to M, there exists a nonzero scalar funtion $f(x)$ defined on M such that $\Delta x = f(x)(2Ax+b)$. From $0 = \langle \Delta x, x\rangle = f(x)\langle 2Ax+b, x\rangle$, we get $\langle 2Ax+b, x\rangle = 0$. From this and $\langle Ax+b, x\rangle + d = 0$, we have $\langle Ax, x\rangle = d$. We can deduce that Ax is a normal vector field of . Since $2Ax+b$ is also normal, we can see that if b is nonzero vector, then b is a constant normal vector of M. As M is not a hyperplane, it is impossible. So we can say that $b = 0$ and consequently $\langle Ax, x\rangle = 0$. Therefore we can conclude that a quadric hypersurface satisfies $\langle \Delta x, x\rangle = 0$, then M is a minimal quadric hypersurface described in Lemma 4.3 or a nonminimal quadric hypersurface described by $\langle Ax, x\rangle = 0$ for a diagonal matrix A.

Case 2. $\langle \Delta x, x\rangle = c \neq 0$

First we will show that if $\lambda_i = 0$, then $b_i = 0$ for $i \in \{1, \cdots, n+1\}$. Suppose that $\lambda_1 = 0$ and $b_1 \neq 0$. Then M can be locally considered a graph of function $x_1 = \frac{1}{b_1}(-d - \sum_{i=2}^{n+1}\lambda_i x_i^2 - \sum_{i=2}^{n+1} b_i x_i)$, since $\langle Ax+b, x\rangle = d$. Lemma 4.2 and $\langle Ax+b, x\rangle + d = 0$ imply that

$$\mathrm{tr}(2A)\langle 2Ax+b, 2Ax+b\rangle(\langle Ax, x\rangle - d) - \langle 2A(2Ax+b), 2Ax+b\rangle(\langle Ax, x\rangle - d)$$
$$+c\langle 2Ax+b, 2Ax+b\rangle^2 = 0. \quad (26)$$

We can observe the left side of (26) is a polynomial of $x_2, \cdots, x_{n+1}$, which are independent variables. So it must be identically zero. If we consider the coefficients of the term x_i^4, $i = 2, \cdots, n+1$ of this polynomial, we find

$$4\mathrm{tr}(2A)\lambda_i^3 - 8\lambda_i^4 + 16c\lambda_i^4 = 0, \ i = 2, \cdots, n+1.$$

This implies that

$$\lambda_i = 0 \text{ or } (2-4c)\lambda_i = \mathrm{tr}(2A), \ i = 2, \cdots, n+1. \quad (27)$$

Now consider the coefficients of $x_i^2 x_j^2 (2 \leq i, j \leq n+1, i \neq j)$. Then we find

$$4\mathrm{tr}(2A)(\lambda_i^2\lambda_j + \lambda_j^2\lambda_i) - 8(\lambda_i^3\lambda_j + \lambda_j^3\lambda_i) + 32c\lambda_i^2\lambda_j^2 = 0. \quad (28)$$

If $2-4c=0$, then from (27) we find $\mathrm{tr}(2A)=0$. This and (28) imply that all λ_i are equally zero. It's a contradiction. So we can see that $2-4c\neq 0$. Consequently from (27) we may assume that

$$\lambda_i=\lambda\neq 0,\ i=2,\cdots,k$$

and

$$\lambda_i=0,\ i=k+1,\cdots,n+1.$$

So the equation (26) can be written as

$$\begin{aligned}\mathrm{tr}(2A)(4\lambda^2\sum_{i=2}^{k}x_i^2+4\lambda\sum_{i=2}^{k}b_ix_i+\sum_{i=1}^{n+1}b_i^2)(\lambda\sum_{i=2}^{k}x_i^2-d)\\ -(8\lambda^3\sum_{i=2}^{k}x_i^2+8\lambda^2\sum_{i=2}^{k}b_ix_i+2\lambda\sum_{i=2}^{k}b_i^2)(\lambda\sum_{i=2}^{k}x_i^2-d)\\ +c(4\lambda^2\sum_{i=2}^{k}x_i^2+4\lambda\sum_{i=2}^{k}b_ix_i+\sum_{i=1}^{n+1}b_i^2)^2=0.\end{aligned}\tag{29}$$

If we consider the coefficient of the term x_2x_i $(i=3,\cdots,k)$ of the left side of (29), it is equal to $32c\lambda^2b_2b_i$, which must be zero. Suppose that $b_2\neq 0$. It follows that $b_i=0,\ i=3,\cdots,k$. So the coefficients of the terms x_3^2 and x_2^2 are equals to

$$-4d\lambda^2\mathrm{tr}(2A)+\mathrm{tr}(2A)\lambda(\sum_{i=1}^{n+1}b_i^2)+8d\lambda^3-2\lambda^2b_2^2+8c\lambda^2(\sum_{i=1}^{n+1}b_i^2)$$

and

$$-4d\lambda^2\mathrm{tr}(2A)+\mathrm{tr}(2A)\lambda(\sum_{i=1}^{n+1}b_i^2)+8d\lambda^3-2\lambda^2b_2^2+8c\lambda^2(\sum_{i=1}^{n+1}b_i^2)+16\lambda^2cb_2^2,$$

respectively. Since both of them are equal to zero, we get $b_2=0$, which is a contradiction. So we can say that $b_i=0, i=2,\cdots,k$. The equation (26) can be rewritten as

$$\begin{aligned}\mathrm{tr}(2A)(4\lambda^2\sum_{i=2}^{k}x_i^2+b_1^2+\sum_{i=k+1}^{n+1}b_i^2)(\lambda\sum_{i=2}^{k}x_i^2-d)\\ -8\lambda^3(\sum_{i=2}^{k}x_i^2)(\lambda\sum_{i=2}^{k}x_i^2-d)+c(4\lambda^2\sum_{i=2}^{k}x_i^2+b_1^2+\sum_{i=k+1}^{n+1}b_i^2)^2=0.\end{aligned}\tag{30}$$

Then the coefficients of $(\sum_{i=2}^{k}x_i^2)^2$, $\sum_{i=2}^{k}x_i^2$ and the constant term of the left side of (30) are equal to

$$4\lambda^3(\mathrm{tr}(2A)-2\lambda+4c\lambda),$$

$$\mathrm{tr}(2A)(-4d\lambda^2+(b_1^2+\sum_{i=k+1}^{n+1}b_i^2)\lambda)+8d\lambda^3+8c(b_1^2+\sum_{i=k+1}^{n+1}b_i^2)\lambda^2$$

and

$$-\mathrm{tr}(2A)d(b_1^2+\sum_{i=k+1}^{n+1}b_i^2)+c(b_1^2+\sum_{i=k+1}^{n+1}b_i^2)^2,$$

respectively. They must be equal to zero. Substituting $\mathrm{tr}(2A)=2(k-1)\lambda$ into the above coefficients, we have

$$k-1=1-2c,$$

$$(k-1)(-4d\lambda+b_1^2+\sum_{i=k+1}^{n+1}b_i^2)+4d\lambda+4c(b_1^2+\sum_{i=k+1}^{n+1}b_i^2)=0$$

and

$$-2(k-1)d\lambda(b_1^2+\sum_{i=k+1}^{n+1}b_i^2)+c(b_1^2+\sum_{i=k+1}^{n+1}b_i^2)^2=0.$$

Substituting the first equation into the second one and the third one, we find

$$(b_1^2+\sum_{i=k+1}^{n+1}b_i^2)+8cd\lambda+2c(b_1^2+\sum_{i=k+1}^{n+1}b_i^2)=0$$

and

$$-2d\lambda+4cd\lambda+c(b_1^2+\sum_{i=k+1}^{n+1}b_i^2)=0.$$

Multiplying the number 2 at both sides of the second equation and subtracting it from the first equation, we get $4d\lambda=-(b_1^2+\sum_{i=k+1}^{n+1}b_i^2)$. Substituting this into the first equation, we find $b_1^2+\sum_{i=k+1}^{n+1}b_i^2=0$, which is a contradiction. So we may assume that $\lambda_i=0$ implies that $b_i=0$, $i=1,\cdots,n+1$. Thus $\langle Ax+b,x\rangle+d=0$ can be written as $\sum_{i=1}^k\lambda_i x_i^2+\sum_{i=1}^k b_i x_i+d=0$ or $\sum_{i=1}^k\lambda_i(x_i+\frac{b_i}{2\lambda_i})^2=e$ for a constant e and the equation (26) can be given as

$$\mathrm{tr}(2A)\{4\sum_{i=1}^k\lambda_i^2(x_i+\frac{b_i}{2\lambda_i})^2(\sum_{i=1}^k\lambda_i x_i^2-d)-(8(\sum_{i=1}^k\lambda_i^3(x_i+\frac{b_i}{2\lambda_i})^2)(\sum_{i=1}^k\lambda_i x_i^2-d)$$
$$+c(4\sum_{i=1}^k\lambda_i^2(x_i+\frac{b_i}{2\lambda_i})^2)^2=0$$

or

$$(\sum_{i=1}^k\lambda_i^2(\mathrm{tr}(2A)-2\lambda_i)(x_i+\frac{b_i}{2\lambda_i})^2)(\sum_{i=1}^k\lambda_i x_i^2-d)+4c(\sum_{i=1}^k\lambda_i^2(x_i+\frac{b_i}{2\lambda_i})^2)^2=0. \quad (31)$$

Suppose $b_1\neq 0$. Locally we may consider M as the graph of the function $x_1=\pm\sqrt{\frac{1}{\lambda_1}(e-\sum_{i=2}^k\lambda_i(x_i+\frac{b_i}{2\lambda_i})^2}-\frac{b_1}{2\lambda_1}$. Substituting this function into (31) we find

$$g(x_2,\cdots,x_k)(e-d+\frac{b_1^2}{4\lambda_1}-\sum_{i=2}^{k}\frac{b_i^2}{4\lambda_i}-\sum_{i=2}^{k}b_ix_i \quad \pm \quad b_1\sqrt{\frac{1}{\lambda_1}(e-\sum_{i=2}^{k}\lambda_i(x_i+\frac{b_i}{2\lambda_i})^2)}$$
$$+ \quad 4ch(x_2,\cdots,x_k)^2=0, \qquad (32)$$

where

$$g(x_2,\cdots,x_k)=\sum_{i=2}^{k}\lambda_i^2(\mathrm{tr}(2A)-2\lambda_i)(x_i+\frac{b_i}{2\lambda_i})^2+\lambda_1(\mathrm{tr}(2A)-2\lambda_1)(e-\sum_{i=2}^{k}\lambda_i(x_i+\frac{b_i}{2\lambda_i})^2)$$

and

$$h(x_2,\cdots,x_k)=\sum_{i=2}^{k}\lambda_i^2(x_i+\frac{b_i}{2\lambda_i})^2+\lambda_1(e-\sum_{i=2}^{k}\lambda_i(x_i+\frac{b_i}{2\lambda_i})^2).$$

If $g(x_2,\cdots,x_k)$ is not identically zero, then a rational function is equal to a irrational function because of (32). So we have $h(x_2,\cdots,x_k)=0$, which implies that $\lambda_i=\lambda_1,\ i=2,\cdots,k$ and $e=0$. This implies that $\sum_{i=1}^{k}\lambda_i(x+\frac{b_i}{2\lambda_i})^2=e=0$ or $\lambda_1\sum_{i=1}^{k}(x+\frac{b_i}{2\lambda_1})^2=0$. It is a contradiction. So we may conclude that $b_i=0, i=1,\cdots,k$. Thus equation (26) can be written as

$$-\mathrm{tr}(2A)\langle 2Ax,2Ax\rangle(2d)+\langle(2A)^2x,2Ax\rangle(2d)+c\langle 2Ax,2Ax\rangle^2=0$$

or

$$-\mathrm{tr}(A)\langle Ax,Ax\rangle d+\langle A^2x,Ax\rangle d+c\langle Ax,Ax\rangle^2=0.$$

By this and similar arguments we have $\lambda_i=\lambda_1,\ i=2,\cdots,k$. This implies that if $k=n+1$, then M is a hypersphere and if $k<n+1$, then M is a spherical cylinder. Combining results in Case 1 and Case 2, we have the following proposition.

Proposition 4.5. *If a quadric hypersurface M described by (16) in E^{n+1} satisfies $\langle\Delta x,x\rangle=c$ for a constant c, then it is one of the followings:*

(1) *a minmal quadric hypersurface.*
(2) *a nonminimal quadric hypersurface described by $\langle Ax,x\rangle=0$ for a diagonal matrix A.*
(3) *a hypersphere.*
(4) *a spherical cylinder.*

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Changrim Jang
Dept. of Mathematcs, College of Natural Science
University of Ulsan 44610, Ulsan, Korea
E-mail address: crjang@mail.ulsan.ac.kr

Haerae Jo
Ulsan Joongang High School 44434, Ulsan, Korea
E-mail address: haerae@hanmail.net