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HALF LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE TRANS-SASAKIAN MANIFOLD WITH A QUARTER-SYMMETRIC METRIC CONNECTION

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Abstract. Jin [10] studied lightlike hypersurfaces of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection. We study further the geometry of this subject. The object of this paper is to study the geometry of half lightlike submanifolds of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection.

1. Introduction

A linear connection $\bar\nabla$ on a semi-Riemannian manifold $(\bar M, \bar g)$ is said to be a *quarter-symmetric connection* if its torsion tensor $\bar T$ satisfies

$$\bar T(\bar X, \bar Y) = \theta(\bar Y)J\bar X - \theta(\bar X)J\bar Y, \tag{1.1}$$

where J is a $(1,1)$-type tensor field and θ is a 1-form associated with a smooth vector field ζ by $\theta(X) = \bar g(X, \zeta)$. Moreover, if this connection $\bar\nabla$ is metric, *i.e.*, $\bar\nabla \bar g = 0$, then $\bar\nabla$ is called a *quarter-symmetric metric connection*. The notion of quarter-symmetric metric connection was introduced by Yano-Imai [14]. The geometry of lightlike hypersurface of an indefinite trans-Sasakian manifolds with a quarter-symmetric metric connection was studied by Jin [10]. Throughout this paper, denote by $\bar X$, $\bar Y$ and $\bar Z$ the smooth vector fields on $\bar M$.

Let M be a submanifold of a semi-Riemannian manifold $(\bar M, \bar g)$ of codimension 2 with the tangent bundle TM and the normal bundle $TM^\perp$. Denoted by $Rad(TM) = TM \cap TM^\perp$ the radical distribution. Then M is called

(1) *half lightlike submanifold* if $rank\{Rad(TM)\} = 1$,
(2) *coisotropic submanifold* if $rank\{Rad(TM)\} = 2$.

Half lightlike submanifold was introduced by Duggal-Bejancu [4] and later, studied by Duggal-Jin [5]. Its geometry is more general than that of lightlike hypersurface or coisotropic submanifold. Much of its theory will be immediately generalized in a formal way to general lightlike submanifolds.

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The notion of trans-Sasakian manifold, of type (α, β), was introduced by Oubina [13]. Sasakian, Kenmotsu and cosymplectic manifolds are important kinds of trans-Sasakian manifold such that

$$\alpha = 1,\ \ \beta = 0; \qquad \alpha = 0,\ \ \beta = 1; \qquad \alpha = \beta = 0,$$

respectively. We say that a trans-Sasakian manifold $\bar{M}$ is an *indefinite trans-Sasakian manifold* if $\bar{M}$ is a semi-Riemannian manifold.

In this paper, we study half lightlike submanifolds of an indefinite trans-Sasakian manifold $\bar{M} \equiv (\bar{M}, J, \zeta, \theta, \bar{g})$ with a quarter-symmetric metric connection, in which the tensor field J and the 1-form θ, defined by (1.1), are identical with the structure tensor field J and the structure 1-form θ of the indefinite trans-Sasakian structure $(J, \theta, \zeta, \bar{g})$ on $\bar{M}$, respectively.

Remark 1. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of $\bar{M}$ with respect to the semi-Riemannian metric $\bar{g}$. Due to [9], it is known that *a linear connection* $\bar{\nabla}$ *on* $\bar{M}$ *is a quarter-symmetric metric connection if and only if* $\bar{\nabla}$ *satisfies*

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} - \theta(\bar{X})J\bar{Y}. \tag{1.2}$$

2. Preliminaries

An odd-dimensional semi-Riemannian manifold $(\bar{M},\, \bar{g})$ is called an *indefinite trans-Sasakian manifold* if there exist a structure set $\{J,\, \zeta,\, \theta,\, \bar{g}\}$, a Levi-Civita connection $\widetilde{\nabla}$ and two smooth functions α and β, where J is a $(1,1)$-type tensor field, ζ is a vector field, and θ is a 1-form such that

$$\begin{aligned} &J^2\bar{X} = -\bar{X} + \theta(\bar{X})\zeta, \quad \theta(\zeta) = 1, \quad \theta(\bar{X}) = \epsilon\bar{g}(\bar{X}, \zeta), \\ &\theta \circ J = 0, \qquad \bar{g}(J\bar{X},\, J\bar{Y}) = \bar{g}(\bar{X},\, \bar{Y}) - \epsilon\theta(\bar{X})\theta(\bar{Y}), \\ &(\widetilde{\nabla}_{\bar{X}}J)\bar{Y} = \alpha\{\bar{g}(\bar{X}, \bar{Y})\zeta - \epsilon\theta(\bar{Y})\bar{X}\} \\ &\qquad\qquad + \beta\{\bar{g}(J\bar{X}, \bar{Y})\zeta - \epsilon\theta(\bar{Y})J\bar{X}\}, \end{aligned} \tag{2.1}$$

where ϵ denotes $\epsilon = 1$ or -1 according as ζ is spacelike or timelike, respectively. $\{J,\, \zeta,\, \theta,\, \bar{g}\}$ is called an *indefinite trans-Sasakian structure of type* (α, β).

In the entire discussion of this paper, we shall assume that the structure vector field ζ is a spacelike one, *i.e.*, $\epsilon = 1$, without loss of generality.

Replacing the Levi-Civita connection $\widetilde{\nabla}$ by the quarter-symmetric metric connection $\bar{\nabla}$ given by (1.2), the last equation of (2.1) is reduced to

$$(\bar{\nabla}_{\bar{X}}J)\bar{Y} = \alpha\{\bar{g}(\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})\bar{X}\} + \beta\{\bar{g}(J\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})J\bar{X}\}. \tag{2.2}$$

Replacing Y by ζ to (2.2) and using $J\zeta = 0$ and $\theta(\bar{\nabla}_X\zeta) = 0$, we obtain

$$\bar{\nabla}_X\zeta = -\alpha JX + \beta(X - \theta(X)\zeta). \tag{2.3}$$

Let (M, g) be a half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ equipped with the radical distribution $Rad(TM)$, a screen distribution $S(TM)$ and a coscreen distribution $S(TM^{\perp})$ such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \qquad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}).$$

Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of the six equations in (2.1). We use the same notations for any others. Let ξ be a section of $Rad(TM)$. Assume that L is a unit spacelike basis vector field of $S(TM^\perp)$, without loss of generality. Consider the orthogonal complementary distribution $S(TM)^\perp$ to $S(TM)$ in $T\bar{M}$. Certainly ξ and L belong to $\Gamma(S(TM)^\perp)$. Thus we have

$$S(TM)^\perp = S(TM^\perp) \ \oplus_{orth} \ S(TM^\perp)^\perp,$$

where $S(TM^\perp)^\perp$ is the orthogonal complementary to $S(TM^\perp)$ in $S(TM)^\perp$. It is known [5] that, for any null section ξ of $Rad(TM)$, there exists a uniquely defined null vector field $N \in \Gamma(S(TM^\perp)^\perp)$ satisfying

$$\bar{g}(\xi, N) = 1, \ \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0, \ \forall\, X \in \Gamma(S(TM)).$$

Denote by $ltr(TM)$ the vector subbundle of $S(TM^\perp)^\perp$ locally spanned by N. Then we show that $S(TM^\perp)^\perp = Rad(TM) \oplus ltr(TM)$. We call N, $ltr(TM)$ and $tr(TM) = S(TM^\perp) \oplus_{orth} ltr(TM)$ the *null transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to the screen distribution $S(TM)$, respectively.

Denote by X, Y and Z the vector fields on M, unless otherwise specified. As the tangent bundle $T\bar{M}$ of the ambient manifold $\bar{M}$ is satisfied

$$T\bar{M} = TM \oplus tr(TM) = TM \oplus ltr(TM) \oplus_{orth} S(TM^\perp),$$

the Gauss and Weingarten formulae of M are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L, \tag{2.4}$$
$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{2.5}$$
$$\bar{\nabla}_X L = -A_L X + \lambda(X)N, \tag{2.6}$$

where ∇ is the linear connection on M, B and D are the local second fundamental forms of M, A_N and A_L are the shape operators, and τ, ρ and λ are 1-forms on TM. Let P be the projection morphism of TM on $S(TM)$ and η a 1-form such that $\eta(X) = \bar{g}(X, N)$. As $TM = S(TM) \oplus_{orth} Rad(TM)$, the Gauss and Weingarten formulae of $S(TM)$ are given respectively by

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.7}$$
$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.8}$$

where ∇^* is the linear connection on $S(TM)$, C is the local screen second fundamental form of $S(TM)$, A^*_ξ is the shape operator.

From the facts that $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ and $D(X, Y) = \bar{g}(\bar{\nabla}_X Y, L)$, we show that B and D are independent of the choice of $S(TM)$ and satisfy

$$B(X, \xi) = 0, \qquad D(X, \xi) = -\lambda(X). \tag{2.9}$$

The local second fundamental forms are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.10}$$

$$C(X,PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0, \tag{2.11}$$

$$D(X,Y) = g(A_L X, Y) - \lambda(X)\eta(Y), \qquad \bar{g}(A_L X, N) = \rho(X). \tag{2.12}$$

3. Structure equations on M

Călin [2] proved that if ζ is tangent to M, then it belongs to $S(TM)$ which we assume. It is known [7] that, for any half lightlike submanifold M of an indefinite trans-Sasakian manifold $\bar{M}$, $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^\perp))$ are vector subbundles of $S(TM)$, of rank 1. There exist two non-degenerate almost complex distributions H_o and H with respect to J such that

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^\perp)) \oplus_{orth} H_o,$$
$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o, .$$

In this case, the tangent bundle TM is decomposed as follow:

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^\perp)). \tag{3.1}$$

Consider two local null vector fields U and V, a local unit spacelike vector field W on $S(TM)$, and their 1-forms u, v and w defined by

$$U = -JN, \qquad V = -J\xi, \qquad W = -JL, \tag{3.2}$$

$$u(X) = g(X,V), \quad v(X) = g(X,U), \quad w(X) = g(X,W). \tag{3.3}$$

Let S be the projection morphism of TM on H and F the tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$. Then JX is expressed as

$$JX = FX + u(X)N + w(X)L. \tag{3.4}$$

Applying J to (3.4) and using (2.1) and (3.2), we have

$$F^2X = -\,X \,+\, u(X)U \,+\, w(X)W + \theta(X)\zeta. \tag{3.5}$$

In the following, we say that F is the *structure tensor field* of M.

Substituting (3.4) into (2.3) and using (2.4), we see that

$$\nabla_X\zeta = -\alpha FX + \beta(X - \theta(X)\zeta), \tag{3.6}$$

$$B(X,\zeta) = -\alpha u(X), \qquad D(X,\,\zeta) = -\alpha w(X). \tag{3.7}$$

Applying $\bar{\nabla}_X$ to $\bar{g}(\zeta, N) = 0$ and using (2.3), (2.5) and (2.11), we have

$$C(X,\zeta) = -\,\alpha v(X) + \beta\eta(X). \tag{3.8}$$

Substituting (2.4) and (3.4) into (1.1) and then, comparing the tangent, lightlike transversal and co-screen components, we obtain

$$T(X,Y) = \theta(Y)FX - \theta(X)FY, \tag{3.9}$$

$$B(X,Y) - B(Y,X) = \theta(Y)u(X) - \theta(X)u(Y), \tag{3.10}$$

$$D(X,Y) - D(Y,X) = \theta(Y)w(X) - \theta(X)w(Y), \tag{3.11}$$

where T is the torsion tensor with respect to ∇. From (3.10) and (3.11), we see that B and D are never symmetric. Replacing Y by ξ to (2.10) and using $(2.9)_1$, (3.10) and the fact that $S(TM)$ is non-degenerate, we obtain

$$A^*_\xi \xi = 0. \tag{3.12}$$

Applying $\bar{\nabla}_X$ to (3.2) ∼ (3.4) by turns and using (2.4), (2.5), (2.6), (2.9) ∼ (2.10), (2.12) and (3.2) ∼ (3.4), we have

$$B(X,U) = C(X,V),\ B(X,W) = D(X,V),\ C(X,W) = D(X,U), \tag{3.13}$$
$$\nabla_X U = F(A_{_N}X) + \tau(X)U + \rho(X)W - \{\alpha\eta(X) + \beta v(X)\}\zeta, \tag{3.14}$$
$$\nabla_X V = F(A^*_\xi X) - \tau(X)V - \lambda(X)W - \beta u(X)\zeta, \tag{3.15}$$
$$\nabla_X W = F(A_{_L}X) + \lambda(X)U - \beta w(X)\zeta, \tag{3.16}$$
$$\begin{aligned}(\nabla_X F)(Y) = {} & u(Y)A_{_N}X + w(Y)A_{_L}X - B(X,Y)U - D(X,Y)W \\ & + \alpha\{g(X,Y)\zeta - \theta(Y)X\} + \beta\{\bar{g}(JX,Y)\zeta - \theta(Y)FX\},\end{aligned} \tag{3.17}$$
$$(\nabla_X u)(Y) = -u(Y)\tau(X) - w(Y)\lambda(X) - \beta\theta(Y)u(X) - B(X,FY), \tag{3.18}$$
$$\begin{aligned}(\nabla_X v)(Y) = {} & v(Y)\tau(X) + w(Y)\rho(X) - \theta(Y)\{\alpha\eta(X) + \beta v(X)\} \\ & - g(A_{_N}X, FY).\end{aligned} \tag{3.19}$$

4. Recurrent and Lie recurrent structure tensors

Definition 1. The structure tensor field F of M is said to be *recurrent* [8] if there exists a smooth 1-form ϖ on M such that

$$(\nabla_X F)Y = \varpi(X)FY.$$

Definition 2. A half lightlike submanifold M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be *statical* [6] if $\bar{\nabla}_X L \in \Gamma(S(TM))$ for any $X \in \Gamma(TM)$.

Remark 2. From (2.6) and $(2.12)_2$, we show that Definition 2 is equivalent to the conditions: $\lambda = 0$ and $\rho = 0$. The condition $\lambda = 0$ is equivalent to the conception: M is *irrotational*, *i.e.*, $\bar{\nabla}_X\xi \in \Gamma(TM)$ [12]. The condition $\rho = 0$ is equivalent to the conception: M is *solenoidal*, *i.e.*, $A_{_L}X \in \Gamma(S(TM))$ [11].

Theorem 4.1. *Let M be a half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. If F is recurrent, then the following six statements are satisfied:*

(1) *F is parallel with respect to the induced connection ∇ on M,*
(2) *$\bar{M}$ is an indefinite cosymplectic manifold, i.e., $\alpha = \beta = 0$,*
(3) *M is statical, i.e., $\lambda = 0$ and $\rho = 0$,*
(4) *W is parallel vector field with respect to the connection ∇,*
(5) *H, $J(ltr(TM))$ and $J(S(TM^\perp))$ are parallel distributions on M,*
(6) *M is locally a product manifold $\mathcal{C}_{_U} \times \mathcal{C}_{_W} \times M^\sharp$, where $\mathcal{C}_{_U}$ is a null curve tangent to $J(ltr(TM))$, $\mathcal{C}_{_W}$ is a spacelike curve tangent to $J(S(TM^\perp))$, and $M^\sharp$ is a leaf of the distributions H.*

Proof. Denote by μ, ν and σ the 1-forms on M such that

$$\mu(X) = B(X,U) = C(X,V), \qquad \sigma(X) = D(X,W),$$
$$\nu(X) = B(X,W) = D(X,V).$$

(1) As F is recurrent, from the above definition and (3.17), we get

$$\begin{aligned}\varpi(X)FY &= u(Y)A_N X + w(Y)A_L X - B(X,Y)U - D(X,Y)W \\ &\quad + \alpha\{g(X,Y)\zeta - \theta(Y)X\} + \beta\{\bar{g}(JX,Y)\zeta - \theta(Y)FX\}.\end{aligned} \tag{4.1}$$

Replacing Y by ξ and using (2.9) and the fact that $F\xi = -V$, we get

$$-\varpi(X)V = \lambda(X)W + \beta u(X)\zeta. \tag{4.2}$$

Taking the scalar product with U to (4.2), we obtain $\varpi = 0$. Thus F is parallel with respect to the connection ∇.

(2) Taking the scalar product with ζ to (4.2), we get $\beta = 0$. Taking the scalar product with U to (4.1) satisfying $\varpi = \beta = 0$, we get

$$u(Y)g(A_N X,U) + w(Y)g(A_L X,U) - \alpha\theta(Y)v(X) = 0. \tag{4.3}$$

Replacing Y by ζ to this equation, we have $\alpha = 0$. As $\alpha = \beta = 0$, $\bar{M}$ is an indefinite cosymplectic manifold.

(3) Taking the scalar product with W to (4.2) and with N to (4.1), we have

$$\lambda(X) = 0, \qquad \rho(X) = \bar{g}(A_L X,N) = 0. \tag{4.4}$$

As $\lambda = 0$, M is irrotational. As $\rho = 0$, M is solenoidal. Thus M is statical.

(4) Taking $Y = U$ and $Y = W$ to (4.3) by turns, we have

$$g(A_N X,U) = C(X,U) = 0, \qquad g(A_L X,U) = 0. \tag{4.5}$$

Taking the scalar product with V and W to (4.1) by turns, we have

$$B(X,Y) = u(Y)\mu(X) + w(Y)\nu(X), \qquad D(X,Y) = w(Y)\sigma(X), \tag{4.6}$$

due to $(4.5)_2$. Replacing Y by V to the two equations of (4.6), we have

$$B(X,V) = 0, \qquad \nu(X) = B(X,W) = D(X,V) = 0. \tag{4.7}$$

Taking $Y = U$ and $Y = W$ to (4.1) and using $(4.5)_2$ and $(4.7)_2$, we get

$$A_N X = \mu(X)U, \qquad A_L X = \sigma(X)W. \tag{4.8}$$

Using $(4.7)_2$ and the fact that $S(TM)$ is non-degenerate, $(4.6)_1$ reduces

$$A^*_\xi X = \mu(X)V. \tag{4.9}$$

Substituting $(4.8)_1$ into (3.14) and $(4.8)_2$ into (3.16), and using the facts that $\lambda = \rho = \alpha = \beta = 0$ and $FU = FW = 0$, we have

$$\nabla_X U = \tau(X)U, \qquad \nabla_X W = 0. \tag{4.10}$$

From $(4.10)_2$, we see that W is parallel vector field with respect to ∇.

(5) From (4.10), we see that both $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are parallel distributions on M with respect to the connection ∇, that is,

$$\nabla_X U \in \Gamma(J(ltr(TM))), \qquad \nabla_X W \in \Gamma(J(S(TM^{\perp}))).$$

On the other hand, taking $Y \in \Gamma(H)$ to (4.1), we have

$$B(X,Y) = 0, \quad D(X,Y) = 0, \quad \forall\, X \in \Gamma(TM), \quad \forall\, Y \in \Gamma(H). \tag{4.11}$$

By straightforward calculations from (2.8), (2.10), (3.4), (3.15), (3.16), (4.7), (4.11) and the facts that $\lambda = 0$ and $FZ \in \Gamma(H_o)$ for $Z \in \Gamma(H_o)$, we have

$$\begin{aligned} &g(\nabla_X \xi,\, V) = -B(X,V) = 0, \quad && g(\nabla_X \xi,\, W) = -\nu(X) = 0,\\ &g(\nabla_X V,\, V) = 0, && g(\nabla_X V,\, W) = -\lambda(X) = 0,\\ &g(\nabla_X Z,\, V) = B(X,FZ) = 0, && g(\nabla_X Z,\, W) = D(X,FZ) = 0. \end{aligned}$$

for all $X \in \Gamma(TM)$ and $Z \in \Gamma(H_o)$, or equivalently, we get

$$\nabla_X Y \in \Gamma(H), \quad \forall\, X \in \Gamma(TM), \quad \forall\, Y \in \Gamma(H).$$

Thus H is a parallel distribution on M with respect to ∇.

(6) As $J(ltr(TM))$, $J(S(TM^{\perp}))$ and H are parallel distributions and satisfed (3.1), by the decomposition theorem of de Rham [3], M is locally a product manifold $\mathcal{C}_U \times \mathcal{C}_W \times M^{\sharp}$, where $\mathcal{C}_U$ is a null curve tangent to $J(ltr(TM))$, $\mathcal{C}_W$ is a spacelike curve tangent to $J(S(TM^{\perp}))$, and $M^{\sharp}$ is a leaf of H. □

Definition 3. The structure tensor field F of M is said to be *Lie recurrent* [8] if there exists a smooth 1-form ϑ on M such that

$$(\mathcal{L}_X F)Y = \vartheta(X)FY,$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X. The structure tensor field F is called *Lie parallel* if $\mathcal{L}_X F = 0$.

Theorem 4.2. *Let M be a half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. If F is Lie recurrent, then the following four statements are satisfied:*

1. *F is Lie parallel,*
2. *$\alpha = 0$, i.e., $\bar{M}$ is not an indefinite Sasakian manifold,*
3. *the 1-forms θ and τ satisfy $d\theta = 0$ and $\tau = -\beta\theta$ on M,*
4. *the shape operator A^*_ξ satisfies*

$$A^*_\xi V = 0, \qquad A^*_\xi U = 0.$$

Proof. (1) As $(\mathcal{L}_X F)Y = [X, FY] - F[X,Y]$, using (3.9) and (3.17), we get

$$\begin{aligned} \vartheta(X)FY = &-\nabla_{FY}X + F\nabla_Y X - \theta(Y)\{X - \theta(X)\zeta\} \\ &+ u(Y)A_N X + w(Y)A_L X \\ &- \{B(X,Y) - \theta(Y)u(X)\}U - \{D(X,Y) - \theta(Y)w(X)\}W \\ &+ \alpha\{g(X,Y)\zeta - \theta(Y)X\} + \beta\{\bar{g}(JX,Y)\zeta - \theta(Y)FX\}, \end{aligned} \tag{4.12}$$

by (3.5). Taking $Y=\xi$ to this equation and using (2.9), we have

$$-\vartheta(X)V=\nabla_V X+F\nabla_\xi X+\lambda(X)W+\beta u(X)\zeta. \tag{4.13}$$

Taking the scalar product with V, W and ζ to (4.13) by turns, we have

$$u(\nabla_V X)=0,\quad w(\nabla_V X)=-\lambda(X),\quad \theta(\nabla_V X)=-\beta u(X). \tag{4.14}$$

Replacing Y by V to (4.12) and using the fact that $\theta(V)=0$, we have

$$\vartheta(X)\xi=-\nabla_\xi X+F\nabla_V X-B(X,V)U-D(X,V)W+\alpha u(X)\zeta. \tag{4.15}$$

Applying F to this equation and using (3.5) and (4.14), we obtain

$$\vartheta(X)V=\nabla_V X+F\nabla_\xi X+\lambda(X)W+\beta u(X)\zeta.$$

Comparing this equation with (4.13), we get $\vartheta=0$. Thus F is Lie parallel.

(2) Taking the scalar product with ζ to (4.15) with $\vartheta=0$, we have

$$g(\nabla_\xi X,\zeta)=\alpha u(X).$$

Replacing X by U to this equation and using (3.14), we obtain $\alpha=0$.

(3) Applying $\bar\nabla_{\bar X}$ to $\theta(\bar Y)=\bar g(\bar Y,\zeta)$ and using (1.1) and (2.3), we obtain

$$d\theta(\bar X,\bar Y)=\alpha\bar g(\bar X,J\bar Y),$$

due to the fact $\bar\nabla$ is metric. As $\alpha=0$, we see that $d\theta=0$.

Taking $X=W$ to (4.12) and using (2.12), (3.5), (3.10) and (3.11), we get

$$\begin{aligned}u(Y)A_N W+w(Y)A_L W-A_L Y-F(A_L FY)&\\ -\lambda(FY)U-\theta(Y)W=0.&\end{aligned} \tag{4.16}$$

Taking the scalar product with N and using $(2.11)_2$ and $(2.12)_{1,2}$, we have

$$D(FY,U)=w(Y)\rho(W)-\rho(Y). \tag{4.17}$$

Replacing Y by V and using $(2.9)_2$, we get $\rho(V)=\lambda(U)$, while taking $X=U$ to $(4.14)_2$ and using (3.14), we have $\rho(V)=-\lambda(U)$. Thus, $\rho(V)=\lambda(U)=0$. Taking $Y=\xi$ to (4.16), we have $A_L\xi=F(A_L V)+\lambda(V)U$. Multiplying this by V and using (2.9), (2.12) and (3.11), we get $\lambda(V)=0$. Therefore,

$$\rho(V)=0,\qquad \lambda(U)=0,\qquad \lambda(V)=0. \tag{4.18}$$

Taking the scalar product with N to (4.12) and using $(2.12)_2$, we have

$$\begin{aligned}-\bar g(\nabla_{FY}X,N)+g(\nabla_Y X,U)+w(Y)\rho(X)&\\ -\theta(Y)\{\eta(X)+\beta v(X)\}=0.&\end{aligned} \tag{4.19}$$

Replacing X by ξ to (4.19) and using (2.8) and $(2.10)_{1,2}$, we have

$$B(X,U)+\theta(X)-w(X)\rho(\xi)=\tau(FX). \tag{4.20}$$

Replacing X by U and using $(3.13)_1$ and the fact that $FU=0$, we get

$$C(U,V)=B(U,U)=0. \tag{4.21}$$

Replacing X by V to (4.19) and using (2.10), (3.15) and $\rho(V)=0$, we have

$$B(FX,U)+\tau(X)+\beta\theta(X)=0.$$

Taking $X=U$, $X=W$ and $X=\zeta$ to this equation by turns, we get

$$\tau(U)=0, \qquad \tau(W)=0, \qquad \tau(\zeta)=-\beta. \tag{4.22}$$

Replacing Y by ξ to (4.17) and using (3.11), we obtain

$$D(U,V)=\rho(\xi). \tag{4.23}$$

Taking $X=U$ to (4.12) and using (2.11), (3.5) and (3.10) $\sim$ (3.14), we get

$$\begin{aligned} &u(Y)A_{_N}U+w(Y)A_{_L}U-\theta(Y)U \\ &-F(A_{_N}FY)-A_{_N}Y-\tau(FY)U-\rho(FY)W=0. \end{aligned} \tag{4.24}$$

Taking the scalar product with V and using (3.13), (4.21) and (4.23), we get

$$B(X,U)+\theta(X)-w(X)\rho(\xi)=-\tau(FX).$$

Comparing this equation with (4.20), we obtain $\tau(FX)=0$. Replacing X by FY and using (3.5) and (4.22), we have $\tau=-\beta\theta$ on M.

(4) Replacing Y by W to (4.24) and using $FW=0$, we have $A_{_L}U=A_{_N}W$. Taking the scalar product with U and using $(3.13)_3$, we have

$$C(W,U)=C(U,W).$$

Taking the scalar product with W to (4.24), we have

$$\rho(FY)=-C(Y,W)+u(Y)C(U,W)+w(Y)D(U,W).$$

Taking the scalar product with U to (4.16) and using $(3.13)_3$, we have

$$\rho(FY)=C(Y,W)-u(Y)C(U,W)-w(Y)D(U,W).$$

From the last two equations, we obtain $\rho(FY)=0$. It follows that $\rho(\xi)=0$.

As $\tau(X)=\beta\theta(X)$, we have $\tau(V)=\tau(\xi)=0$. Taking $X=\xi$ to (4.13) and using (3.12), we obtain $A^*_\xi V=0$. From (3.10) and (4.20), we have $B(U,X)=0$, *i.e.*, $g(A^*_\xi U,X)=0$. As $S(TM)$ is non-degenerate, we obtain $A^*_\xi U=0$. □

5. Indefinite generalized Sasakian space forms

Definition 4. An indefinite trans-Sasakian manifold $(\bar M,J,\zeta,\theta,\bar g)$ is called an *indefinite generalized Sasakian space form*, denote it by $\bar M(f_1,f_2,f_3)$, if there exist three smooth functions f_1, f_2 and f_3 on $\bar M$ such that

$$\begin{aligned} \widetilde R(\bar X,\bar Y)\bar Z \;=\; & f_1\{\bar g(\bar Y,\bar Z)\bar X-\bar g(\bar X,\bar Z)\bar Y\} \\ &+f_2\{\bar g(\bar X,J\bar Z)J\bar Y-\bar g(\bar Y,J\bar Z)J\bar X+2\bar g(\bar X,J\bar Y)J\bar Z\} \\ &+f_3\{\theta(\bar X)\theta(\bar Z)\bar Y-\theta(\bar Y)\theta(\bar Z)\bar X \\ &\qquad +\bar g(\bar X,\bar Z)\theta(\bar Y)\zeta-\bar g(\bar Y,\bar Z)\theta(\bar X)\zeta\}, \end{aligned} \tag{5.1}$$

where $\widetilde R$ is the curvature tensor of the Levi-Civita connection $\widetilde\nabla$ on $\bar M$.

Remark 3. The notion of generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ was introduced by Alegre *et. al*. [1]. Indefinite Sasakian, Kenmotsu and cosymplectic space forms are important kinds of generalized Sasakian space forms such that

$$f_1 = \tfrac{c+3}{4}, f_2 = f_3 = \tfrac{c-1}{4}; \quad f_1 = \tfrac{c-3}{4}, f_2 = f_3 = \tfrac{c+1}{4}; \quad f_1 = f_2 = f_3 = \tfrac{c}{4}$$

respectively, where c is a constant J-sectional curvature of each space forms.

Let $\bar{R}$ be the curvature tensor of the quarter-symmetric metric connection $\bar{\nabla}$ on $\bar{M}$. By directed calculations from (1.1) and (1.2), we see that

$$\bar{R}(\bar{X}, \bar{Y})\bar{Z} = \widetilde{R}(\bar{X}, \bar{Y})\bar{Z} - \{(\bar{\nabla}_X\theta)(Y) - (\bar{\nabla}_Y\theta)(X)\}JZ. \tag{5.2}$$

Denote by R and R^* the curvature tensors of the induced connections ∇ and ∇^* on M and $S(TM)$ respectively. Using the local Gauss-Weingarten formulae, we have the Gauss-Codazzi equations for M and $S(TM)$ such that

$$\begin{aligned} \bar{R}(X, Y)Z = R(X, Y)Z &+ B(X, Z)A_N Y - B(Y, Z)A_N X \\ &+ D(X, Z)A_L Y - D(Y, Z)A_L X \\ &+ \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ &\quad + \tau(X)B(Y, Z) - \tau(Y)B(X, Z) \\ &\quad + \lambda(X)D(Y, Z) - \lambda(Y)D(X, Z) \\ &\quad - \theta(X)B(FY, Z) + \theta(Y)B(FX, Z)\}N, \\ &+ \{(\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) \\ &\quad + \rho(X)B(Y, Z) - \rho(Y)B(X, Z) \\ &\quad - \theta(X)D(FY, Z) + \theta(Y)D(FX, Z)\}L, \end{aligned} \tag{5.3}$$

$$\begin{aligned} R(X, Y)PZ = R^*(X, Y)PZ &+ C(X, PZ)A^*_\xi Y - C(Y, PZ)A_\xi X \\ &+ \{(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ &\quad - \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ) \\ &\quad - \theta(X)C(FY, PZ) + \theta(Y)C(FX, PZ)\}\xi, \end{aligned} \tag{5.4}$$

$$\begin{aligned} R(X, Y)\xi = -\nabla^*_X(A^*_\xi Y) &+ \nabla^*_Y(A^*_\xi X) + A^*_\xi[X, Y] \\ &- \tau(X)A^*_\xi Y + \tau(Y)A^*_\xi X \\ &+ \{C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\tau(X, Y)\}\xi, \end{aligned} \tag{5.5}$$

Comparing the tangential and lightlike transversal components of two equations of (5.3) and (5.2) and using (3.4), we obtain

$$\begin{aligned} R(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} \\ &+ f_2\{\bar{g}(X, JZ)FY - \bar{g}(Y, JZ)FX + 2\bar{g}(X, JY)FZ\} \\ &+ f_3\{[\theta(X)Y - \theta(Y)X]\theta(Z) + [g(X, Z)\theta(Y) - g(Y, Z)\theta(X)]\zeta\} \\ &- \{(\bar{\nabla}_X\theta)(Y) - (\bar{\nabla}_Y\theta)(X)\}FZ \\ &+ B(Y, Z)A_N X - B(X, Z)A_N Y + D(Y, Z)A_L X - D(X, Z)A_L Y, \end{aligned} \tag{5.6}$$

$$\begin{aligned}&(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ &+ \lambda(X)D(Y,Z) - \lambda(Y)D(X,Z) - \theta(X)B(FY,Z) + \theta(Y)B(FX,Z) \\ &+ \{(\bar\nabla_X\theta)(Y) - (\bar\nabla_Y\theta)(X)\}u(Z) \\ &= f_2\{u(Y)\bar g(X,JZ) - u(X)\bar g(Y,JZ) + 2u(Z)\bar g(X,JY)\},\end{aligned} \tag{5.7}$$

Taking the scalar product with N to (5.3) and then, substituting (5.4) and (5.2) into the left and right terms and using $(2.12)_4$, we obtain

$$\begin{aligned}&(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) - \tau(X)C(Y,PZ) \\ &+ \tau(Y)C(X,PZ) - \rho(X)D(Y,PZ) + \rho(Y)D(X,PZ) \\ &- \theta(X)C(FY,PZ) + \theta(Y)C(FX,PZ) \\ &+ \{(\bar\nabla_X\theta)(Y) - (\bar\nabla_Y\theta)(X)\}v(PZ) \\ &= f_1\{g(Y,PZ)\eta(X) - g(X,PZ)\eta(Y)\} \\ &+ f_2\{v(Y)\bar g(X,JPZ) - v(X)\bar g(Y,JPZ) + 2v(PZ)\bar g(X,JY)\} \\ &+ f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ).\end{aligned} \tag{5.8}$$

Theorem 5.1. *Let M be a half lightlike submanifold of an indefinite generalized Sasakian space form $\bar M(f_1,f_2,f_3)$ with a quarter-symmetric metric connection. Then α, β, f_1, f_2 and f_3 are satisfied $\beta = 0$, α is a constant on M and*

$$f_1 - f_2 = \alpha^2, \qquad f_1 - f_3 = \alpha(\alpha+1).$$

Proof. Applying ∇_Y to $(3.13)_1$: $B(X,U) = C(X,V)$ and using (2.1), $(2.10)_{1,2}$, $(2.11)_{1,2}$, (3.4), $(3.7)_1$, (3.8), (3.14) and (3.15), we have

$$\begin{aligned}&(\nabla_X B)(Y,U) \\ &= (\nabla_X C)(Y,V) - 2\tau(X)C(Y,V) - \lambda(X)C(Y,W) - \rho(X)B(Y,W) \\ &- \alpha^2 u(Y)\eta(X) - \beta^2 u(X)\eta(Y) + \alpha\beta\{u(X)v(Y) - u(Y)v(X)\} \\ &- g(A^*_\xi X, F(A_N Y)) - g(A^*_\xi Y, F(A_N X)).\end{aligned}$$

Substituting this equation into (5.7) with $Z = U$ and using $(3.13)_{2,3}$, we get

$$\begin{aligned}&(\nabla_X C)(Y,V) - (\nabla_Y C)(X,V) - \tau(X)C(Y,V) \\ &+ \tau(Y)C(X,V) - \rho(X)D(Y,V) + \rho(Y)D(X,V) \\ &- \theta(X)C(FY,V) + \theta(Y)C(FX,V) \\ &+ (\bar\nabla_X\theta)(Y) - (\bar\nabla_Y\theta)(X) \\ &+ (\alpha^2 - \beta^2)\{u(X)\eta(Y) - u(Y)\eta(X)\} \\ &+ 2\alpha\beta\{u(X)v(Y) - u(Y)v(X)\} \\ &= f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar g(X,JY)\}.\end{aligned}$$

Comparing this equation with (5.8) such that $PZ = V$, we obtain

$$\begin{aligned}&\{f_1 - f_2 - \alpha^2 + \beta^2\}[u(Y)\eta(X) - u(X)\eta(Y)] \\ &= 2\alpha\beta\{u(Y)v(X) - u(X)v(Y)\}.\end{aligned}$$

Taking $X = \xi, Y = U$ and $X = V, Y = U$ to this equation by turns, we get

$$f_1 - f_2 = \alpha^2 - \beta^2, \qquad \alpha\beta = 0.$$

Applying $\bar{\nabla}_X$ to $\eta(Y) = \bar{g}(Y, N)$ and using (2.4) and (2.5) we have

$$(\nabla_X \eta)(Y) = -g(A_N X, Y) + \tau(X)\eta(Y).$$

Applying ∇_Y to (3.8) and using (2.11), (3.6), (3.8), (3.19) and $\alpha\beta = 0$, we have

$$\begin{aligned}(\nabla_X C)(Y, \zeta) = &-(X\alpha)v(Y) + (X\beta)\eta(Y) + \alpha^2\theta(Y)\eta(X) + \beta^2\theta(X)\eta(Y)\\ &+ \alpha\{g(A_N X, FY) + g(A_N Y, FX) - v(Y)\tau(X) - w(Y)\rho(X)\}\\ &- \beta\{g(A_N X, Y) + g(A_N Y, X) - \tau(X)\eta(Y)\}.\end{aligned}$$

Substituting this equation and (3.8) into (5.8) such that $PZ = \zeta$, we get

$$\begin{aligned}&\{X\beta + [f_1 - f_3 - (\alpha^2 - \beta^2) - \alpha]\theta(X)\}\eta(Y)\\ &- \{Y\beta + [f_1 - f_3 - (\alpha^2 - \beta^2) - \alpha]\theta(Y)\}\eta(X)\\ &= \{X\alpha + \beta\theta(X)\}v(Y) - \{Y\alpha + \beta\theta(Y)\}v(X).\end{aligned}$$

Taking $X = \zeta$, $Y = \xi$ and $X = U$, $Y = V$ to this by turns, we obtain

$$f_1 - f_3 = (\alpha^2 - \beta^2) + \alpha - \zeta\beta, \qquad U\alpha = 0.$$

Applying ∇_Y to $(3.7)_1$ and using (3.6) and (3.18), we have

$$\begin{aligned}(\nabla_X B)(Y, \zeta) = &-(X\alpha)u(Y) - \beta B(Y, X)\\ &+ \alpha\{u(Y)\tau(X) + w(Y)\lambda(X) + B(X, FY) + B(Y, FX)\}.\end{aligned}$$

Substituting this into (5.7) such that $Z = \zeta$ and using (3.7) and (3.10), we get

$$\{X\alpha + \beta\theta(X)\}u(Y) = \{Y\alpha + \beta\theta(X)\}u(X).$$

Taking $Y = U$ and using the fact that $U\alpha = 0$, we have $X\alpha + \beta\theta(X) = 0$.

Assume that $\beta \neq 0$. Then $X\alpha \neq 0$ due to $X\alpha = -\beta\theta(X)$. Applying $\bar{\nabla}_X$ to $\alpha\beta = 0$ and using the fact that $X\alpha = -\beta\theta(X)$, we obtain

$$\alpha X\beta = \beta^2\theta(X).$$

Multiplying β to this result, we get $\beta = 0$. It is a contradiction to $\beta \neq 0$. Thus $\beta = 0$. Therefore, α is a constant, $f_1 - f_2 = \alpha^2$ and $f_1 - f_3 = \alpha(\alpha + 1)$. □

Definition 5. (1) A screen distribution $S(TM)$ is called *totally umbilical* [5] in M if there exists smooth function γ such that $A_N = \gamma P$, or equivalently,

$$C(X, PY) = \gamma g(X, Y).$$

In case $\gamma = 0$, we say that $S(TM)$ is *totally geodesic* in M.

(2) A lightlike submanifold M is called *screen conformal* [6] if there exists non-vanishing smooth function φ on $\mathcal{U}$ such that $A_N = \varphi A^*_\xi$, or equivalently,

$$C(X, PY) = \varphi B(X, PY). \tag{5.9}$$

Theorem 5.2. *Let M be a half lightlike submanifold of $\bar{M}(f_1, f_2, f_3)$ with a quarter-symmetric metric connection. If one of the following four statements*

(1) *F is recurrent,*
(2) *F is Lie recurrent,*
(3) *$S(TM)$ is totally umbilical,*
(4) *M is screen conformal,*

is satisfied, then $\bar{M}(f_1, f_2, f_3)$ is a flat manifold with an indefinite cosymplectic structure. In case (1)*, M is also flat. In case* (3)*, $S(TM)$ is totally geodesic.*

Proof. (1) By Theorem 4.1, we get (4.8), (4.10) and the results: $\alpha = \beta = 0$ and $\lambda = \rho = 0$. Since $\alpha = \beta = 0$, we have $f_1 = f_2 = f_3$ by Theorem 5.1.

Taking the scalar product with U to $(4.8)_{1,\,2}$, we get

$$C(X, U) = 0, \qquad\qquad D(X, U) = 0.$$

Applying ∇_X to $C(Y, U) = 0$ and using $(4.10)_1$, we obtain

$$(\nabla_X C)(Y, U) = 0.$$

Substituting the last equations into (5.8) with $PZ = U$, we have

$$(f_1 + f_2)\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.$$

Taking $X = V$ and $Y = \xi$ to this result, we obtain $f_1 + f_2 = 0$. Therefore, we see that $f_1 = f_2 = f_3 = 0$. Thus $\bar{M}(f_1, f_2, f_3)$ is flat.

As $f_1 = f_2 = f_3 = 0$, (5.6) is reduced to

$$\begin{aligned} R(X, Y)Z = {} & B(Y, Z)A_N X - B(X, Z)A_N Y \\ & + D(Y, Z)A_L X - D(X, Z)A_L Y. \end{aligned}$$

Using this, (2.10), (2.12), (4.8), (4.9) and the fact that $\lambda = 0$, we obtain

$$\begin{aligned} R(X, Y)Z = {} & \{\mu(Y)\mu(X) - \mu(X)\mu(Y)\}u(Z)U \\ & + \{\sigma(Y)\sigma(X) - \sigma(X)\sigma(Y)\}w(Z)W = 0, \end{aligned}$$

for all X, Y, $Z \in \Gamma(TM)$. Therefore $R = 0$ and M is also flat.

(2) By Theorem 4.2 and 5.1, we get $\alpha = 0$ and $\beta = 0$. Thus $\bar{M}$ is an indefinite cosymplectic manifold. Since $\alpha = 0$, we have $f_1 = f_2 = f_3$ by Theorem 5.1. Also, since $\beta = 0$, by (3) of Theorem 4.2, we see that $\tau = 0$. Taking the scaler product with N to (5.6) with $Z = \xi$ and then, comparing this result with the radical component of (5.5) and using (2.9) and (2.12), we have

$$\begin{aligned} & C(Y, A^*_\xi X) - C(X, A^*_\xi Y) \\ = {} & f_2\{u(Y)v(X) - u(X)v(Y)\} + \lambda(X)\rho(Y) - \lambda(Y)\rho(X). \end{aligned}$$

Taking $X = U$ and $Y = V$ to this and using (4.18) and the result (4) in Theorem 4.2, we get $f_2 = 0$. Thus $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat.

(3) Assume that $S(TM)$ is totally umbilical. Then (3.8) is reduced to $\gamma\theta(X) = -\alpha v(X) + \beta\eta(X)$. Replacing X by V, ξ and ζ to this equation by turns, we have $\alpha = 0$, $\beta = 0$ and $\gamma = 0$ respectively. Since $\alpha = \beta = 0$, $\bar{M}$ is an indefinite cosymplectic manifold. As $\gamma = 0$, $S(TM)$ is totally geodesic.

As $\alpha = 0$, $f_1 = f_2 = f_3$ by Theorem 5.1. Taking $PZ = U$ to (5.8) with $C = 0$ and using the facts that $D(X, Uk) = C(X, W) = 0$, we get

$$(f_1 + f_2)\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.$$

Taking $X = \xi$ and $Y = V$ to this equation, we get $f_1 + f_2 = 0$. Thus $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat.

(4) Replacing Y by ζ to (5.9) and using $(3.7)_1$ and (3.8), we have

$$\alpha v(X) - \beta\eta(X) = \alpha\varphi u(X).$$

Taking $X = V$ and $X = \xi$ to this equation by turns, we obtain $\alpha = 0$ and $\beta = 0$ respectively. As $\alpha = \beta = 0$, $\bar{M}$ is an indefinite cosymplectic manifold. Since $\alpha = 0$, we have $f_1 = f_2 = f_3$ by Theorem 5.1.

Applying ∇_X to $C(Y, PZ) = \varphi B(Y, PZ)$, we have

$$(\nabla_X C)(Y, PZ) = (X\varphi)B(Y, PZ) + \varphi(\nabla_X B(Y, PZ).$$

Substituting this equation into (5.8) and using (5.7), we have

$$\begin{aligned} &\{X\varphi - 2\varphi\tau(X)\}B(Y, PZ) - \{Y\varphi - 2\varphi\tau(Y)\}B(X, PZ) \\ &- \{\rho(X) + \varphi\lambda(X)\}D(Y, PZ) + \{\rho(Y) + \varphi\lambda(Y)\}D(X, PZ) \\ &+ \{(\bar{\nabla}_X\theta)(Y) - (\bar{\nabla}_Y\theta)(X)\}g(\omega, PZ) \\ &= f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} \\ &+ f_2\{g(\omega, Y)\bar{g}(X, JPZ) - g(\omega, X)\bar{g}(Y, JPZ) + 2g(\omega, PZ)\bar{g}(X, JY) \\ &+ f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ). \end{aligned} \tag{5.10}$$

where $\omega = U - \varphi V$. From $(3.13)_1$ and (5.9); $(3.13)_{2,3}$ and (5.9), we get

$$B(X, \omega) = 0, \qquad D(X, \omega) = 0. \tag{5.11}$$

Applying $\bar{\nabla}_X$ to $\theta(\xi) = 0$ and $\theta(V) = 0$ by turns and using (2.4), (2.8), (2.10), (3.15) and the fact that $\alpha = \beta = 0$, we have

$$(\bar{\nabla}_X\theta)(\xi) = B(X, \zeta) = 0, \qquad (\bar{\nabla}_X\theta)(V) = \beta u(X) = 0. \tag{5.12}$$

Replacing PZ by ω to (5.10) and using (5.11), we obtain

$$\begin{aligned} &- 2\varphi\{(\bar{\nabla}_X\theta)(Y) - (\bar{\nabla}_Y\theta)(X)\} \\ &= (f_1 + f_2)\{g(\omega, Y)\eta(X) - g(\omega, X)\eta(Y)\} - 4\varphi f_2\bar{g}(X, JY)\} \end{aligned}$$

Taking $X = \xi$ and $Y = V$ to this equation and using (5.12), we get $f_1 + f_2 = 0$. Therefore, $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat. □

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