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INDEFINITE TRANS-SASAKIAN MANIFOLD WITH A TRANSVERSAL HALF LIGHTLIKE SUBMANIFOLD

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Abstract. We study the geometry of indefinite trans-Sasakian manifold $\bar{M}$ admitting a half lightlike submanifold M such that the structure vector field of $\bar{M}$ belongs to the transversal vector bundle of M. We prove several classification theorems of such an indefinite trans-Sasakian manifold.

1. Introduction

Oubina [14] introduced the notion of a trans-Sasakian manifold, of type (α, β). Sasakian manifold is an important kind of trans-Sasakian manifold with $\alpha = 1$ and $\beta = 0$. Kenmotsu manifold is another kind of trans-Sasakian manifold such that $\alpha = \beta = 0$. Cosymplectic manifold is also an example with $\alpha = 0$ and $\beta = 1$. We say that a trans-Sasakian manifold $\bar{M}$ is an *indefinite trans-Sasakian manifold* if $\bar{M}$ is a semi-Riemannian manifold.

Alegre *et al.* [1] introduced generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$. Sasakian, Kenmotsu and cosymplectic space forms are important kinds of generalized Sasakian space forms such that

$$f_1 = \tfrac{c+3}{4}, f_2 = f_3 = \tfrac{c-1}{4}; \quad f_1 = \tfrac{c-3}{4}, f_2 = f_3 = \tfrac{c+1}{4}; \quad f_1 = f_2 = f_3 = \tfrac{c}{4},$$

respectively, where c is a constant J-sectional curvature of each space forms. We say that a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ is an *indefinite generalized Sasakian space form* if $\bar{M}$ is a semi-Riemannian manifold. The theory of lightlike submanifolds is an important topic of research in differential geometry due to its application in mathematical physics, especially in the general relativity. The study of such notion was initiated by Duggal and Bejancu [2] and later studied by many authors (see two books [3, 5]).

The class of codimension 2 lightlike submanifolds of semi-Riemannian manifolds is compose of two classes by virtue of the rank of its radical distribution, which are called half lightlike submanifold or coisotropic submanifold [3]. Half lightlike submanifold is a particular case of r-lightlike submanifold [2] such that

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$r = 1$ and its geometry is more general form than that of coisotrophic submanifolds or lightlike hypersurfaces. Much of the works on half lightlike submanifolds will be immediately generalized in a formal way to arbitrary r-lightlike submanifolds. For this reason, we study only half lightlike submanifolds.

The author studied half lightlike submanifolds M of indefinite Sasakian manifolds $\bar{M}$ [6] or indefinite cosymplectic manifolds $\bar{M}$ [11] subject such that the structure vector field of $\bar{M}$ belongs to the transversal vector bundle of M, such a M is called a *transversal half lightlike submanifold*. In this paper, we study the geometry of indefinite trans-Sasakian manifold $\bar{M}$ admitting a transversal half lightlike submanifold M. The main result is to prove several classification theorems of such an indefinite trans-Sasakian manifold.

2. Half lightlike submanifold

A codimension 2 lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a *half lightlike submanifold* of $\bar{M}$ if the rank of its radical distribution $Rad(TM) = TM \cap TM^{\perp}$ is 1. In this case, $Rad(TM)$ is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$ of M, of rank 1. Therefore, there exist complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, which are called the *screen distribution* and *co-screen distribution* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \quad (2.1)$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of (2.1). We use same notations for any others. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\bar{M}$. Certainly, $TM^{\perp}$ is a vector subbundle of $S(TM)^{\perp}$. As $S(TM^{\perp})$ is a non-degenerate subbundle of $S(TM)^{\perp}$, the orthogonal complementary distribution $S(TM^{\perp})^{\perp}$ of $S(TM^{\perp})$ in $S(TM)^{\perp}$ is also a non-degenerate vector bundle such that

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp}.$$

Clearly, $Rad(TM)$ is a subbundle of $S(TM^{\perp})^{\perp}$. Choose $L \in \Gamma(S(TM^{\perp}))$ as a unit spacelike vector field, without loss of generality. It is well known [3] that, for any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely defined null vector field $N \in \Gamma(S(TM^{\perp})^{\perp})$ satisfying

$$\bar{g}(\xi, N) = 1, \; \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0, \; \forall\, X \in \Gamma(S(TM)).$$

Denote by $ltr(TM)$ the subbundle of $S(TM^{\perp})^{\perp}$ locally spanned by N. Then we show that $S(TM^{\perp})^{\perp} = Rad(TM) \oplus ltr(TM)$. Let $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$. We call N, $ltr(TM)$ and $tr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to the screen distribution $S(TM)$, respectively. Then the tangent bundle

$T\bar{M}$ of $\bar{M}$ is decomposed as

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \quad (2.2)\\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned}$$

In the sequel, let X, Y and Z be the vector fields on M, unless otherwise specified. Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of TM on $S(TM)$ with respect to the decomposition (2.1). Then the local Gauss and Weingarten formulas of M and $S(TM)$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N + D(X,Y)L, \quad (2.3)$$
$$\bar{\nabla}_X N = -A_{N} X + \tau(X)N + \rho(X)L, \quad (2.4)$$
$$\bar{\nabla}_X L = -A_{L} X + \phi(X)N; \quad (2.5)$$
$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \quad (2.6)$$
$$\nabla_X \xi = -A^*_{\xi} X - \tau(X)\xi, \quad (2.7)$$

where ∇ and ∇^* are induced connections on TM and $S(TM)$, respectively, B and D are called the *local second fundamental forms* of M, C is called the *local second fundamental form* on $S(TM)$. A_{N}, A^*_{ξ} and A_{L} are called the *shape operators*, and τ, ρ and ϕ are 1-forms on TM. We say that $h(X,Y) = B(X,Y)N + D(X,Y)L$ is the *second fundamental form tensor* of M.

Since the connection $\bar{\nabla}$ on $\bar{M}$ is torsion-free, the induced connection ∇ on M is also torsion-free, and B and D are symmetric. The above three local second fundamental forms of M and $S(TM)$ are related to their shape operators by

$$B(X,Y) = g(A^*_{\xi}X,Y), \qquad \bar{g}(A^*_{\xi}X,N) = 0, \quad (2.8)$$
$$C(X,PY) = g(A_{N}X,PY), \qquad \bar{g}(A_{N}X,N) = 0, \quad (2.9)$$
$$D(X,Y) = g(A_{L}X,Y) - \phi(X)\eta(Y), \qquad \bar{g}(A_{L}X,N) = \rho(X), \quad (2.10)$$

where η is a 1-form on TM such that $\eta(X) = \bar{g}(X,N)$. From (2.8), (2.9) and (2.10), we see that B and D satisfy

$$B(X,\,\xi) = 0, \qquad D(X,\,\xi) = -\phi(X), \quad (2.11)$$

A^*_{ξ} and A_{N} are $S(TM)$-valued, and A^*_{ξ} is self-adjoint on TM such that

$$A^*_{\xi}\xi = 0. \quad (2.12)$$

The induced connection ∇ of $M$is not a metric one, and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y). \quad (2.13)$$

But the connection ∇^* on $S(TM)$ is a metric one.

Definition 1. A half lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ is called *totally geodesic* [3] if $h = 0$ on any coordinate neighborhood $\mathcal{U}$.

Remark 1. It is easy to see that M is totally geodesic if and only if the local second fundamental forms B and D of M satisfy $B = 0$ and $D = 0$.

Denote by $\bar{R}$, R and R^* the curvature tensors of the Lavi-Civita connection $\bar{\nabla}$ on $\bar{M}$ and the induced connections ∇ and ∇^* on M and $S(TM)$ respectively.

Definition 2. A lightlike submanifold $M = (M, g, \nabla)$ equipped with a degenerate metric g and a linear connection ∇ is said to be a *space of constant curvature* c if there exists a constant c such that R satisfies

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}. \tag{2.14}$$

3. Indefinite trans-Sasakian manifolds

An odd-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be an *indefinite almost contact metric manifold* ([6]$\sim$[13]) if there exist a structure set $\{J,\, \zeta,\, \theta,\, \bar{g}\}$, where J is a (1, 1)-type tensor field, ζ is a vector field which is called the *structure vector field* of $\bar{M}$ and θ is a 1-form such that

$$J^2X = -X + \theta(X)\zeta,\ \bar{g}(JX,\, JY) = \bar{g}(X,\, Y) - \epsilon\theta(X)\theta(Y),\ \theta(\zeta) = 1, \tag{3.1}$$

for any vector fields X and Y on $\bar{M}$, where $\epsilon = 1$ or -1 according as ζ is spacelike or timelike respectively. In this case, the structure set $\{J,\, \zeta,\, \theta,\, \bar{g}\}$ is called an *indefinite almost contact metric structure* of $\bar{M}$.

Definition 3. An indefinite almost contact metric manifold $(\bar{M}, \bar{g})$, with the Levi-Civita connection $\bar{\nabla}$ with respect to $\bar{g}$, is called an *indefinite trans-Sasakian manifold* if there exist two smooth functions α and β on $\bar{M}$ such that

$$(\bar{\nabla}_X J)Y = \alpha\{\bar{g}(X,Y)\zeta - \epsilon\theta(Y)X\} + \beta\{\bar{g}(JX,Y)\zeta - \epsilon\theta(Y)JX\}, \tag{3.2}$$

for any vector fields X and Y on $\bar{M}$. In this case, we say that $\{J, \zeta, \theta, \bar{g}\}$ is an *indefinite trans-Sasakian structure of type* (α, β) [1, 12, 14].

By replacing Y by ζ in (3.2) and using (3.1), we get

$$\bar{\nabla}_X\zeta = -\epsilon\alpha JX + \epsilon\beta(X - \theta(X)\zeta). \tag{3.3}$$

Remark 2. If $\beta = 0$, then $\bar{M}$ is said to be an *indefinite α-Sasakian manifold.* Indefinite Sasakian manifolds [6, 7, 11] appear as examples of indefinite α-Sasakian manifolds, with $\alpha = 1$. Another important kind of indefinite trans-Sasakian manifold is that of indefinite cosymplectic manifolds [8, 13] obtained for $\alpha = \beta = 0$. If $\alpha = 0$, then $\bar{M}$ is said to be an *indefinite β-Kenmotsu manifold.* Indefinite Kenmotsu manifolds [9, 10] are particular examples of indefinite β-Kenmotsu manifold, with $\beta = 1$.

It is known [7, 8] that, for any half lightlike submanifold M of an indefinite almost contact metric manifold $\bar{M}$, $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are subbundles of $S(TM)$, of rank 1. In the sequel, we shall assume that ζ is a unit spacelike vector field, *i.e.*, $\epsilon = 1$, without loss of generality.

Definition 4. A half lightlike submanifold M of an indefinite almost contact metric manifold $\bar{M}$ is called a *transversal half lightlike submanifold* [6, 11] if ζ belongs to the transversal vector bundle $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$.

In case M is a transversal half lightlike submanifold, the structure vector field ζ is decomposed as $\zeta = eL + bN$, where e and b are smooth functions given by $e = \theta(L)$ and $b = \theta(\xi)$. As $\bar{g}(\zeta, \zeta) = 1$, we have $e^2 = 1$. Thus, we may choose $e = 1$, without loss of generality. Therefore, we have

$$\zeta = L + bN. \tag{3.4}$$

Theorem 3.1. *Any indefinite trans-Sasakian manifold $\bar{M}$, of type (α, β), admitting a transversal half lightlike submanifold M such that $b = 0$ satisfies $\alpha = 0$ and $\phi = 0$. Thus $\bar{M}$ is never indefinite Sasakian manifold. Moreover, if $D = 0$, then $\bar{M}$ is an indefinite cosymplectic manifold.*

Proof. As $b = 0$, we get $\zeta = L$ by (3.4). Using (2.5) and (3.3), we obtain

$$-\alpha JX + \beta(X - \theta(X)\zeta) = -A_{_L}X + \phi(X)N.$$

Taking the scalar product with ξ and JN to this by turns, we have

$$\alpha g(X, J\xi) = \phi(X), \qquad \alpha\eta(X) - \beta g(X, JN) = D(X, JN), \tag{3.5}$$

respectively. From these equations and $(2.11)_2$, we get

$$\alpha = \alpha\eta(\xi) - \beta g(\xi, JN) = D(\xi, JN) = -\phi(JN) = -\alpha g(JN, J\xi) = -\alpha.$$

Thus $\alpha = 0$. As $\alpha = 0$, we get $\phi = 0$ by (3.5).

Moreover, if $D = 0$, then we have $\beta g(X, JN) = 0$ for all $X \in \Gamma(TM)$. Thus $\beta = 0$. Therefore, $\bar{M}$ is an indefinite cosymplectic manifold. $\square$

In the sequel, by saying that *transversal half lightlike submanifolds* we shall mean *half lightlike submanifolds satisfying* (3.4) *such that* $b \neq 0$.

Theorem 3.2. *Any indefinite trans-Sasakian manifold $\bar{M}$ admitting a transversal half lightlike submanifold M satisfies $\alpha = 1$. Therefore $\bar{M}$ is neither indefinite β-Kenmotsu manifold nor indefinite cosymplectic manifold.*

Proof. Applying J to (3.4) and using $J\zeta = 0$, we have $JL = -bJN$. Thus we see that $J(ltr(TM)) = J(S(TM^{\perp}))$. If $J(Rad(TM)) \cap J(ltr(TM)) \neq \{0\}$, then there exists a non-vanishing smooth function f such that $J\xi = fJN$. Then we have $-b^2 = g(J\xi, J\xi) = f^2 g(JN, JN) = 0$, *i.e.*, $b = 0$. It is a contradiction to $b \neq 0$. Thus $J(Rad(TM)) \cap J(ltr(TM)) = \{0\}$. Therefore, there exists a non-degenerate almost complex distribution H with respect to J such that

$$TM = Rad(TM) \oplus_{orth} \{J(Rad(TM)) \oplus_{orth} J(ltr(TM)) \oplus_{orth} H\}.$$

Consider the mutually orthonormal local timelike and spacelike vector fields V and U on $S(TM)$ respectively, and their 1-forms v and u defined by

$$V = -b^{-1}J\xi, \qquad U = -b^{-1}\{J\xi + b^2 JN\}, \tag{3.6}$$

$$v(X) = -g(X, V), \qquad u(X) = g(X, U). \tag{3.7}$$

As $Span\{J\xi,\, JN\} = Span\{V,\, U\}$, we show that $\{V, U\}$ is an orthonormal frame field of $J(Rad(TM)) \oplus J(ltr(TM))$ and $J(Rad(TM)) \oplus J(ltr(TM)) = J(Rad(TM)) \oplus_{orth} Span\{U\}$. Thus TM is decomposed as follow:

$$TM = Rad(TM) \oplus_{orth} \{J(Rad(TM)) \oplus_{orth} Span\{U\} \oplus_{orth} H\}. \tag{3.8}$$

Denote by S the projection morphism of TM on H with respect to the decomposition (3.8). Using (3.8), any vector field X on M is expressed as

$$X = SX + v(X)V + u(X)U + \eta(X)\xi. \tag{3.9}$$

Using (3.6) and (3.9), the action JX of X by J is expressed as

$$JX = FX - b\eta(X)V + b^{-1}\omega(X)\xi - bv(X)N - \omega(X)L, \tag{3.10}$$

where F is a tensor field on TM of type $(1,1)$ defined by

$$FX = JSX, \quad \text{and} \quad \omega(X) = u(X) + v(X).$$

Applying J to (3.10) and using (3.1), (3.4), (3.6), and (3.9), we have

$$F^2X = -X + v(X)V + u(X)U + \eta(X)\xi = -SX. \tag{3.11}$$

Applying $\bar{\nabla}_X$ to (3.4) and using (2.4), (2.5), (3.3), (3.9) and (3.10), we have

$$\begin{aligned} A_{_L}X + bA_{_N}X = \alpha FX - \beta SX - \{\alpha\theta(X) + \beta v(X)\}V \quad & (3.12)\\ - \beta u(X)U + b^{-1}\{\alpha\omega(X) - \beta\theta(X)\}\xi, &\\ X[b] + b\tau(X) + \phi(X) = b\{\alpha v(X) - \beta\theta(X)\}. \quad & (3.13)\end{aligned}$$

Taking the scalar product with V to (3.12), we have

$$D(X,V) + bC(X,V) = \alpha\theta(X) + \beta v(X). \tag{3.14}$$

Replacing X by ξ to (3.14) and using $(2.11)_2$, we have $bC(\xi, V) = b\alpha + \phi(V)$.

From the facts that $\eta(Y) = \bar{g}(Y,N)$ and $\theta(Y) = b\eta(Y)$, we obtain

$$\begin{aligned} 2d\eta(X,Y) &= g(X, A_{_N}Y) - g(A_{_N}X, Y) + \tau(X)\eta(Y) - \tau(Y)\eta(X), \quad (3.15)\\ 2d\theta(X,Y) &= 2b\, d\eta(X,Y) + X[b]\eta(Y) - Y[b]\eta(X).\end{aligned}$$

Using these equations, (3.13) and the fact that $d\theta(X,Y) = \bar{g}(X, JY)$, we get

$$\begin{aligned} 2\bar{g}(X,JY) = b\{&g(X, A_{_N}Y) - g(A_{_N}X, Y)\}\\ &+ \{b\alpha v(X) - b\beta\theta(X) - \phi(X)\}\eta(Y)\\ &- \{b\alpha v(Y) - b\beta\theta(Y) - \phi(Y)\}\eta(X).\end{aligned}$$

Taking $X = V$ and $Y = \xi$ to this equation and using the fact that $bg(A_{_N}\xi, V) = bC(\xi, V) = b\alpha + \phi(V)$, we obtain $\alpha = 1$. □

Applying $\bar{\nabla}_X$ to $bV = -J\xi$ for all $X \in \Gamma(TM)$ and using (2.3), (2.6), (2.7), (2.11), (3.2), (3.4), (3.6), (3.9), (3.10) and (3.13), we get

$$b^2C(X,V) = B(X, U - V) + b\{\alpha\theta(X) + \beta\omega(X)\}, \tag{3.16}$$

$$bD(X,V) = B(X, V - U) - b\beta u(X), \tag{3.17}$$

$$\nabla^*_X V = \alpha SX + \beta FX + b^{-1}F(A^*_\xi X) + \{\alpha u(X) + b^{-1}\phi(X)\}U. \tag{3.18}$$

Theorem 3.3. *Any indefinite trans-Sasakian manifold $\bar{M}$, of type (α, β), admitting a totally geodesic transversal half lightlike submanifold M satisfies $\alpha = 1$ and $\beta = 0$. Therefore, $\bar{M}$ is an indefinite Sasakian manifold.*

Proof. Assume that M is totally geodesic, *i.e.*, $B = D = 0$. Using (3.17) and the fact that $b \neq 0$, we have $\beta u(X) = 0$ for all $X \in \Gamma(TM)$. Taking $X = U$ to this result, we get $\beta = 0$. Therefore, from this result and Theorem 3.3, we obtain $\alpha = 1$ and $\beta = 0$. Thus we have our theorem. □

Definition 5. We say that M is *locally symmetric* [6, 7, 9] if its curvature tensor R be parallel, *i.e.*, have vanishing covariant differential, $\nabla R = 0$.

Theorem 3.4. *Let M be a totally geodesic transversal half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$. If M is locally symmetric, then M is a space of constant positive curvature* 1*.*

Proof. As M is totally geodesic, *i.e.*, $B = D = 0$, we show that $A^*_\xi = 0$ by (2.8), and $\phi = 0$ by $(2.11)_2$. As $\alpha = 1$ and $\beta = 0$, (3.18) is reduced to

$$\nabla^*_X V = SX + u(X)U.$$

From (3.14) and the fact that $\theta(X) = b\eta(X)$, we get $C(X, V) = \eta(X)$. Thus, from (2.6) with $PY = V$ and (3.9), we obtain

$$\nabla_X V = X - v(X)V. \tag{3.19}$$

Applying ∇_Y to (3.19) and using (3.19), we have

$$\nabla_X \nabla_Y V = \nabla_X Y - v(Y)X - \{X(v(Y)) - v(X)v(Y)\}V.$$

From the last two equation, we obtain

$$R(X, Y)V = v(X)Y - v(Y)X - 2dv(X, Y)V.$$

As M is totally geodesic, $\bar{R}(X, Y)V = R(X, Y)V$. From this result and the fact that $\bar{g}(\bar{R}(X, Y)V,\, V) = 0$, we obtain $dv = 0$. Thus

$$R(X, Y)V = v(X)Y - v(Y)X. \tag{3.20}$$

Applying $\bar{\nabla}_X$ to $v(Y) = -g(Y, V)$ and using (3.7) and (3.19), we have

$$(\nabla_X v)(Y) = -g(X, Y) - v(X)v(Y). \tag{3.21}$$

Applying ∇_Z to (3.20) and using (3.19) $\sim$ (3.21) and the fact that M is locally symmetric, *i.e.*, $\nabla_Z R = 0$ for any $Z \in \Gamma(TM)$, we have

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y.$$

Due to (2.14), we show that M is a space of constant positive curvature 1. □

4. Indefinite generalized Sasakian space form

Definition 6. An indefinite almost contact metric manifold $\bar{M}$ is called an *indefinite generalized Sasakian space form* [1] and denote it by $\bar{M}(f_1, f_2, f_3)$ if there exist three smooth functions f_1, f_2 and f_3 on $\bar{M}$ such that

$$\begin{aligned} \bar{R}(X,Y)Z \;&=\; f_1\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \\ &\quad + f_2\{\bar{g}(X,JZ)JY - \bar{g}(Y,JZ)JX + 2\bar{g}(X,JY)JZ\} \\ &\quad + f_3\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X \\ &\qquad\qquad + \bar{g}(X,Z)\theta(Y)\zeta - \bar{g}(Y,Z)\theta(X)\zeta\}, \end{aligned} \tag{4.1}$$

for any vector fields X, Y and Z on $\bar{M}$.

Example 1. *An indefinite Sasakian space form, i.e., an indefinite Sasakian manifold with constant J-sectional curvature c, such that the structure vector field ζ is spacelike, is an indefinite generalized Sasakian space form with*

$$f_1 = \tfrac{c+3}{4}, \qquad f_2 = f_3 = \tfrac{c-1}{4}.$$

Example 2. *An indefinite Kenmotsu space form, i.e., an indefinite Kenmotsu manifold with constant J-sectional curvature c, such that the structure vector field ζ is spacelike, is an indefinite generalized Sasakian space form with*

$$f_1 = \tfrac{c-3}{4}, \qquad f_2 = f_3 = \tfrac{c+1}{4}.$$

Example 3. *An indefinite cosymplectic space form, i.e., an indefinite cosymplectic manifold with constant J-sectional curvature c, such that the structure vector field ζ is spacelike, is an indefinite generalized Sasakian space form with*

$$f_1 = f_2 = f_3 = \tfrac{c}{4}.$$

We need the following three Gauss-Codazzi equations for M and $S(TM)$ (for a full set of these equations see [3]): For all X, Y, $Z \in \Gamma(TM)$,

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z,\xi) &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &\quad + B(Y,Z)\tau(X) - B(X,Z)\tau(Y) \\ &\quad + D(Y,Z)\phi(X) - D(X,Z)\phi(Y), \end{aligned} \tag{4.2}$$

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z,N) &= \bar{g}(R(X,Y)Z,N) \\ &\quad + D(X,Z)\rho(Y) - D(Y,Z)\rho(X), \end{aligned} \tag{4.3}$$

$$\begin{aligned} g(R(X,Y)PZ,N) &= (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ &\quad + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X). \end{aligned} \tag{4.4}$$

Theorem 4.1. *Any indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$, equipped with indefinite trans-Sasakian sutucture of type (α,β), admitting a totally geodesic transversal half lightlike submanifold M satisfies*

$$\alpha = 1, \quad \beta = 0, \quad f_1 = 1 \quad \textit{and} \quad f_2 = f_3 = 0.$$

Thus, $\bar{M}(f_1, f_2, f_3)$ is an indefinite Sasakian manifold of constant curvature 1.

Proof. As M is totally geodesic, we get $B = D = \phi = A^*_\xi = 0$ and $g(A_L X, Y) = 0$ for all X, $Y \in \Gamma(TM)$. By Theorem 3.3 and Theorem 3.4, we show that $\alpha = 1$ and $\beta = 0$. Substituting (4.1) into (4.2), we obtain

$$\begin{aligned} &bf_2\{v(X)\bar{g}(Y, JZ) - v(Y)\bar{g}(X, JZ) - 2v(Z)\bar{g}(X, JY)\} \\ &+ bf_3\{g(X, Z)\theta(Y) - g(Y, Z)\theta(X)\} = 0. \end{aligned}$$

Taking $X = Z = U$ and $Y = \xi$ and using the fact that $b \neq 0$, we have $f_3 = 0$. Also, taking $X = Z = V$ and $Y = \xi$ and using (3.6) and (3.7), we get $f_2 = 0$.

From (2.14) and (3.14) and the facts that $\alpha = 1$ and $\beta = 0$, we have

$$C(X, V) = \eta(X), \qquad \nabla_X V = X - v(X)V. \tag{4.5}$$

Substituting (4.1) and (4.4) into (4.3) and using $f_2 = f_3 = 0$, we have

$$\begin{aligned} &f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} \\ &= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X). \end{aligned}$$

Replacing PZ by V to the last equation and using $(4.5)_1$, we have

$$\begin{aligned} &f_1\{v(X)\eta(Y) - v(Y)\eta(X)\} \\ &= (\nabla_X C)(Y, V) - (\nabla_Y C)(X, V) + \eta(X)\tau(Y) - \eta(Y)\tau(X). \end{aligned} \tag{4.6}$$

Applying ∇_X to $C(Y, V) = \eta(Y)$ and using $(4.5)_2$, we have

$$(\nabla_X C)(Y, V) = X(\eta(Y)) - \eta(\nabla_X Y) - g(A_N Y, X) + v(X)\eta(Y).$$

Substituting this equation into (4.6) and using (3.15), we get

$$(f_1 - 1)\{v(X)\eta(Y) - v(Y)\eta(X)\} = 0.$$

Taking $X = V$ and $Y = \xi$ to this, we have $f_1 = 1$. Thus $\bar{M}(f_1, f_2, f_3)$ is an indefinite Sasakian manifold of constant curvature 1. □

Definition 7. A half lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ is called *screen totally geodesic* [3] if $A_N X = 0$, or equivalently, $C = 0$.

Theorem 4.2. *There exists no indefinite generalized Sasakian space forms $\bar{M}(f_1, f_2, f_3)$, equipped with indefinite trans-Sasakian sutucture of type (α, β), admitting a screen totally geodesic transversal half lightlike submanifold M.*

Proof. As $A_N = C = 0$ and $\alpha = 1$, from (3.12) and (3.14), we show that

$$D(X, U) = -\beta u(X), \qquad D(X, V) = \theta(X) + \beta v(X). \tag{4.7}$$

Taking the scalar product with N to (3.12) and using $(2.10)_2$, we have

$$b\rho(X) = \omega(X) - \beta\theta(X). \tag{4.8}$$

Substituting (4.1) and (4.4) into (4.3) with $Z = PZ$ and using the facts that $V - U = bJN$ and $\theta(X) = b\eta(X)$, we have

$$\begin{aligned} &f_1\{\bar{g}(Y, PZ)\eta(X) - \bar{g}(X, PZ)\eta(Y)\} \\ &+ b^{-1}f_2\{\bar{g}(X, JPZ)\omega(Y) - \bar{g}(Y, JPZ)\omega(X) + 2\bar{g}(X, JY)\omega(PZ)\} \\ &= D(X, PZ)\rho(Y) - D(Y, PZ)\rho(X). \end{aligned} \tag{4.9}$$

Replacing PZ by U and using $(4.7)_1$ and the fact that $JU = b^{-1}\xi - L$, we get

$$f_1\{u(Y)\eta(X) - u(X)\eta(Y)\} + 2b^{-1}f_2\bar{g}(X, JY) \qquad (4.10)$$
$$= \beta\{u(Y)\rho(X) - u(X)\rho(Y)\}.$$

Taking $X = V$ and $Y = \xi$ to this and using $(3.6)_1$, we have $f_2 = 0$. Also, taking $X = U$ and $Y = \xi$ to (4.10) with $f_2 = 0$, we obtain $f_1 = \beta\rho(\xi)$. Replacing X by ξ to (4.8), we get $\rho(\xi) = -\beta$. Thus we have $f_1 = -\beta^2$. Replacing PZ by V to (4.9) and using the facts that $f_1 = -\beta^2$ and $f_2 = 0$, we have

$$\beta^2\{v(Y)\eta(X) - v(X)\eta(Y)\} = D(X, V)\rho(Y) - D(Y, V)\rho(X).$$

Taking $X = V$ and $Y = \xi$ to this equation and using $(4.7)_2$ and (4.8), we have $\beta^2 = \beta^2 + 1$. It is a contradiction. Thus we have our theorem. □

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