East Asian Math. J.
Vol. 33 (2017), No. 5, pp. 527–531
http://dx.doi.org/10.7858/eamj.2017.036

REMARK ON A SUMMATION FORMULA FOR THE SERIES ${}_4F_3(1)$

Junesang Choi*, Yashoverdhan Vyas and Arjun K. Rathie

Abstract. We aim to prove a known summation formula for the series ${}_4F_3(1)$ by mainly using a similar method as in [2], which is different from that in [3]. The method of proof here as well as that in [2] is potentially useful in getting some other summation formulas for ${}_pF_q$.

1. Introduction

Throughout this paper, let ${}_pF_q$ denote the generalized hypergeometric series (see, for details, e.g., [6], [7], [8, Section 1.5]). We begin by recalling the following two summation formulas for the series ${}_3F_2$ and ${}_4F_3$ (see, e.g., [7, p. 245])

$$ {}_3F_2\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ b\,; \\ \frac{1}{2}a,\ 1+a-b\,; \end{matrix}\ -1\right] = \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma(1+a-b)}{\Gamma(1+a)\,\Gamma\left(\frac{1}{2}+\frac{1}{2}a-b\right)} \tag{1.1} $$

and

$$ \begin{aligned} &{}_4F_3\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ b,\ c\,; \\ \frac{1}{2}a,\ 1+a-b,\ 1+a-c\,; \end{matrix}\ 1\right] \\ &= \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma\left(\frac{1}{2}+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\,\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a-c+\frac{1}{2}\right)\,\Gamma(1+a-b-c)}. \end{aligned} \tag{1.2} $$

For our present investigation, we also recall the following two summation formulas due to Kim et al. [3]:

$$ \begin{aligned} &{}_3F_2\left[\begin{matrix} a,\ b,\ 1+d\,; \\ 1+a-b,\ d\,; \end{matrix}\ -1\right] \\ &= \left(1-\frac{a}{2d}\right)\frac{\Gamma\left(1+\frac{1}{2}a\right)\,\Gamma(1+a-b)}{\Gamma(1+a)\,\Gamma\left(1+\frac{1}{2}a-b\right)} + \frac{a}{2d}\cdot\frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma(1+a-b)}{\Gamma(1+a)\,\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}\right)} \end{aligned} \tag{1.3} $$

Received August 29, 2017; Accepted September 19, 2017.
2010 *Mathematics Subject Classification.* 33C20.
Key words and phrases. Gamma function; Pochhammer symbol; Generalized hypergeometric function ${}_pF_q$; Summation formulas for ${}_pF_q$.
* Corresponding author.

(pISSN 1226-6973, eISSN 2287-2833)

and

$$
{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1\,; \\ 1+a-b,\ 1+a-c,\ d\,; \end{matrix}\ 1\right]
$$
$$
=\left(1-\frac{a}{2d}\right)\frac{\Gamma\left(1+\frac{1}{2}a\right)\ \Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\,\Gamma(1+a-b-c)\,\Gamma\left(1+\frac{1}{2}a-b\right)\ \Gamma\left(1+\frac{1}{2}a-c\right)}
$$
$$
+\frac{a}{2d}\cdot\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\ \Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma\left(\frac{1}{2}+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\,\Gamma(1+a-b-c)\,\Gamma\left(\frac{1}{2}+\frac{1}{2}a-b\right)\ \Gamma\left(\frac{1}{2}+\frac{1}{2}a-c\right)}. \tag{1.4}
$$

Remark 1. The identities (1.1) and (1.2) are obvious special cases of (1.3) and (1.4), respectively. Taking the limit in (1.4) as $c\to\infty$ yields (1.3).

Setting $b=-n$ $(n\in\mathbb{N}_0)$ in (1.3) and (1.4), respectively, we obtain the following interesting identities:

$$
{}_3F_2\left[\begin{matrix} -n,\ b,\ 1+d\,; \\ 1+a+n,\ d\,; \end{matrix}\ -1\right]
=\left(1-\frac{a}{2d}\right)\frac{(1+a)_n}{\left(1+\frac{1}{2}a\right)_n}+\frac{a}{2d}\cdot\frac{(1+a)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n} \tag{1.5}
$$

and

$$
{}_4F_3\left[\begin{matrix} -n,\ a,\ b,\ d+1\,; \\ 1+a+n,\ 1+a-b,\ d\,; \end{matrix}\ 1\right]
=\left(1-\frac{a}{2d}\right)\frac{(1+a)_n\,\left(1+\frac{1}{2}a-c\right)_n}{\left(1+\frac{1}{2}a\right)_n(1+a-c)_n}
+\frac{a}{2d}\cdot\frac{(1+a)_n\left(\frac{1}{2}+\frac{1}{2}a-c\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n(1+a-c)_n}. \tag{1.6}
$$

Here and in the following, let $\mathbb{C}$, $\mathbb{N}$ and $\mathbb{Z}_0^-$ be the sets of complex numbers, positive integers and non-positive integers, respectively, and let $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$.

Kim et al. [3] established the result (1.3) with the help of classical Kummer's summation theorem and its contiguous results in [5] and established the result (1.4) with the help of classical Dixon's summation theorem and its contiguous result in [4]. Very recently, Choi et al. [2] have proved an extended Watson's summation theorem for the series ${}_4F_3(1)$ in [3] by mainly using a known summation formula for ${}_3F_2(1/2)$. Here, similarly as in [2], we aim to prove (1.4) by mainly using (1.3).

2. Derivation of (1.4)

Let $\mathcal{L}$ be the left side of (1.4). Expressing ${}_4F_3$ as the series, we obtain

$$
\mathcal{L}=\sum_{k=0}^{\infty}\frac{(-1)^k\,(a)_k\,(b)_k\,(1+d)_k}{(1+a-b)_k\,(d)_k\,k!}\ \left\{\frac{(-1)^k\,(c)_k}{(1+a-c)_k}\right\}, \tag{2.1}
$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [8, p. 2 and pp. 4-6]):

$$\begin{aligned}(\lambda)_n :&= \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\dots(\lambda+n-1) & (n\in\mathbb{N}) \end{cases} \\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad \left(\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-\right),\end{aligned} \tag{2.2}$$

where Γ is the familiar Gamma function.

Using the following identity (cf., [6, p. 69, Exercise 5])

$${}_2F_1\left[\begin{matrix} -k,\ a+k\,; \\ 1+a-c\,; \end{matrix}\ 1\right] = \frac{(-1)^k\,(c)_k}{(1+a-c)_k} \quad (k\in\mathbb{N}_0)$$

in (2.1), we have

$$\mathcal{L} = \sum_{k=0}^{\infty} \frac{(-1)^k\,(a)_k\,(b)_k\,(1+d)_k}{(1+a-b)_k\,(d)_k\,k!}\ {}_2F_1\left[\begin{matrix} -k,\ a+k\,; \\ 1+a-c\,; \end{matrix}\ 1\right]. \tag{2.3}$$

Expressing ${}_2F_1$ in (2.3) as the series, we get

$$\mathcal{L} = \sum_{k=0}^{\infty}\sum_{m=0}^{k} \frac{(-1)^k\,(a)_k\,(b)_k\,(1+d)_k\,(-k)_m\,(a+k)_m}{(1+a-b)_k\,(d)_k\,(1+a-c)_m\,k!\,m!},$$

which, upon using the identities

$$(\alpha)_k\,(\alpha+k)_m = (\alpha)_{k+m} \quad (\alpha\in\mathbb{C};\ k,\,m\in\mathbb{N}_0) \tag{2.4}$$

and

$$(-k)_m = \frac{(-1)^m\,k!}{(k-m)!},$$

yields

$$\mathcal{L} = \sum_{k=0}^{\infty}\sum_{m=0}^{k} \frac{(-1)^{k+m}\,(a)_{k+m}\,(b)_k\,(1+d)_k}{(1+a-b)_k\,(1+a-c)_m\,(d)_k\,m!\,(k-m)!}. \tag{2.5}$$

Applying the following formal manipulation of double series (see, e.g., [1], [6, p. 57, Lemma 10(2)])

$$\sum_{k=0}^{\infty}\sum_{m=0}^{k} A(m,k) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} A(m,k+m),$$

we obtain

$$\mathcal{L} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^k\,(a)_{k+2m}\,(b)_{k+m}\,(1+d)_{k+m}}{(1+a-b)_{k+m}\,(d)_{k+m}(1+a-c)_m\,m!\,k!}. \tag{2.6}$$

Using (2.4) in (2.6), we get

$$\mathcal{L} = \sum_{m=0}^{\infty} \frac{(a)_{2m}\,(b)_m\,(1+d)_m}{(1+a-b)_m\,(1+a-c)_m\,(d)_m\,m!} \times \sum_{k=0}^{\infty} \frac{(-1)^k\,(a+2m)_k\,(b+m)_k\,(1+d+m)_k}{(1+a-b+m)_k\,(d+m)_k\,k!},$$

which, upon expressing the inner series as ${}_3F_2$, gives

$$\mathcal{L} = \sum_{m=0}^{\infty} \frac{(a)_{2m}\,(b)_m\,(1+d)_m}{(1+a-b)_m\,(1+a-c)_m\,(d)_m\,m!} \times {}_3F_2\left[\begin{matrix} a+2m,\ b+m,\ 1+d+m\,; \\ 1+a-b+m,\ d+m\,; \end{matrix}\ -1\right]. \tag{2.7}$$

Finally, using (1.3) to evaluate the ${}_3F_2$ in (2.7), after some simplification, we find that the resulting right side of (2.7) leads to the right side of (1.4).

Acknowledgments. The authors would like to express their deep-felt thanks for the reviewers' helpful comments.

References

[1] J. Choi, Notes on formal manipulations of double series, *Commun. Korean Math. Soc.* **18**(4) (2003), 781–789.

[2] J. Choi, V. Rohira and A. K. Rathie, Note on the extended Watson's summation theorem for the series ${}_4F_3(1)$, (2017), submitted.

[3] Y. S. Kim, M. A. Rakha and A. K. Rathie, Extensions of certain classical summation theorems for the series ${}_2F_1$, ${}_3F_2$ and ${}_4F_3$ with applications in Ramanujan's summations, *Int. J. Math. Math. Sci.* **2010** (2010), Article ID 309503, 26 pages.

[4] J. L. Lavoie, F. Grondin, A. K. Rathie, and K. Arora, Generalizations of Dixon's theorem on the sum of a ${}_3F_2$, *Math. Comput.* **62** (1994), 267–276.

[5] J. L. Lavoie, F. Grondin, and A. K. Rathie, Generalizations of Whipple's theorem on the sum of a ${}_3F_2$, *J. Comput. Appl. Math.* **72** (1996), 293–300.

[6] E. D. Rainville, *Special Functions*, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[7] L.J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.

[8] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

Junesang Choi
Department of Mathematics, Dongguk University, Gyeongju 38066, Republic of Korea
E-mail address: junesang@mail.dongguk.ac.kr

Yashoverdhan Vyas
Department of Mathematics, School of Engineering, Sir Padampat Singhania University, Bhatewar, Udaipur, 313601, Rajasthan State, India
E-mail address: yashoverdhan.vyas@spsu.ac.in

Arjun K. Rathie
Department of Mathematics, School of Physical sciences, Central University of Kerala, Periye P.O., Kasaragod-671316, Kerala, India
E-mail address: akrathie@cukerala.ac.in