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AN EXTRAPOLATED HIGHER ORDER CHARACTERISTIC FINITE ELEMENT METHOD FOR SOBOLEV EQUATIONS

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Abstract. We introduce an extrapolated higher order characteristic finite element method to construct approximate solutions of a Sobolev equation with a convection term. The higher order of convergence in both the temporal direction and the spatial direction in L^2 normed space is established and some computational results to support our theoretical results are presented.

1. Introduction

In this paper, we consider a Sobolev equation with a convection term: Find $u(\boldsymbol{x},t)$ defined on $\Omega\times[0,T]$ such that

$$\begin{aligned} c(\boldsymbol{x})u_t + \boldsymbol{d}(\boldsymbol{x})\cdot\nabla u - \nabla\cdot(a(u)\nabla u) - \nabla\cdot(b(u)\nabla u_t) \\ = f(\boldsymbol{x},t,u), \quad &\text{in } \Omega\times(0,T], \\ u(\boldsymbol{x},t) = 0, \quad &\text{on } \partial\Omega\times(0,T], \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \quad &\text{in } \Omega, \end{aligned} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^m$, $1\le m\le 3$, is a bounded convex domain with its boundary $\partial\Omega$ and $c, \boldsymbol{d}, a, b$, and f are known functions. For the existence, uniqueness, regularity results, and physical applications of Sobolev equations, we refer to [2, 3, 4, 20, 23] and the papers cited therein.

To obtain the numerical results for Sobolev equations without a convection term, we apply many numerical methods, such as classical finite element methods [1, 6, 10, 11, 12], least-squares methods [9, 17, 18, 24, 25], H^1-Galerkin mixed finite element method [8], or discontinuous finite element methods [13, 14, 21, 22]. But in many cases, a convection term $\boldsymbol{d}(\boldsymbol{x})\cdot\nabla u$ exists and $\boldsymbol{d}(\boldsymbol{x})$ is large. To discretize both the time derivative term and the convection term effectively, we use a characteristic finite element method which is naturally derived from the physical point of view. And the effectiveness of this method are

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shown in [5, 7]. Gu in [7] and Shi et al [19] introduce a characteristic finite element method for a Sobolev equation and establish the higher order convergence in the spatial variable and the first order convergence in the temporal variable for approximate solutions. But the first order convergence in the temporal variable deteriorates the higher order convergence in the spatial variable. So, Ohm and Shin in [15, 16] introduce a Crank-Nicolson or an extrapolated Crank-Nicolson characteristic finite element method for the Sobolev equation to obtain the higher order of convergence in both the spatial direction and the temporal direction in L^2 normed space.

In this paper, we introduce an extrapolated characteristic finite element method for the Sobolev equation to get the higher order of convergence both in the temporal variable and in the spatial variable. This method is based on a backward three-point formula to approximate simultaneously the time derivative and the convection term and on an extrapolation technique to avoid the difficulty in solving the nonlinear systems. Our paper is organized as follows: In Section 2, some assumptions of $u(\boldsymbol{x},t)$, the conditions of the coefficients of (1.1) and basic notations are given. In Section 3, finite element spaces and basic approximation properties are given. In Section 4, we construct characteristic finite element approximations of $u(\boldsymbol{x},t)$ and establish the higher order of convergence in L^2 and H^1 normed spaces. In Section 5, we provide the computational results to confirm the theoretical results obtained in Section 4.

2. Assumptions and notations

Now we introduce some notations for Sobolev spaces. For an $s \geq 0$ and $1 \leq p \leq \infty$, $W^{s,p}(\Omega)$ denote a usual Sobolev space equipped with its norm $\|\cdot\|_{s,p}$. For our convenience, denote $H^s(\Omega)$ instead of $W^{s,2}(\Omega)$, and $\|\cdot\|$, $\|\cdot\|_\infty$, and $\|\cdot\|_s$ instead of $\|\cdot\|_{0,2}$, $\|\cdot\|_{0,\infty}$, and $\|\cdot\|_{s,2}$, respectively. Let $\boldsymbol{H}^s(\Omega) = \{\boldsymbol{w} = (w_1, w_2, \ldots, w_m) \mid w_i \in H^s(\Omega), 1 \leq i \leq m\}$ be a Sobolev space equipped with its norm $\|\boldsymbol{w}\|_s^2 = \sum\limits_{i=1}^{m} \|w_i\|_s^2$ and let $H_0^1(\Omega) = \{w \in H^1(\Omega) \mid w(\boldsymbol{x}) = 0 \text{ on } \partial\Omega\}$. For a given Banach space X and $t_1, t_2 \in [0,T]$, we introduce the the following Sobolev spaces with the corresponding norms:

$$W^{s,p}(t_1,t_2;X) = \Big\{w(\boldsymbol{x},t) \mid \|\frac{\partial^\beta w}{\partial t^\beta}(\cdot,t)\|_X \in L^p(t_1,t_2), 0 \leq \beta \leq s\Big\},$$

where

$$\|w\|_{W^{s,p}(t_1,t_2;X)} = \begin{cases} \Big(\sum_{\beta=0}^{s} \int_{t_1}^{t_2} \|\frac{\partial^\beta w}{\partial t^\beta}(\cdot,t)\|_X^p dt\Big)^{1/p}, & 1 \leq p < \infty \\ max_{0\leq\beta\leq s}\ \text{esssup}_{t\in(t_1,t_2)} \|\frac{\partial^\beta w}{\partial t^\beta}(\cdot,t)\|_X, & p = \infty. \end{cases}$$

And denote $L^p(X)$ and $W^{s,p}(X)$ instead of $L^{0,p}(0,T;X)$ and $W^{s,p}(0,T;X)$, respectively.

For the Sobolev equation (1.1), let the coefficient functions $c(\boldsymbol{x}), \boldsymbol{d}(\boldsymbol{x}) = (d_1(\boldsymbol{x}), d_2(\boldsymbol{x}), \cdots, d_m(\boldsymbol{x}))^T$, $a(p), b(p)$ and $f(\boldsymbol{x}, t, p)$ satisfy the following assumptions:

(A1) There exist constants $c_*, c^*, d^*, a_*, a^*, b_*$, and b^*, such that $0 < c_* \leq c(\boldsymbol{x}) \leq c^*$, $0 < |\boldsymbol{d}(\boldsymbol{x})| \leq d^*$ for all $\boldsymbol{x} \in \Omega$ and $0 < a_* \leq a(p) \leq a^*$, $0 < b_* \leq b(p) \leq b^*$, for all $p \in \mathbb{R}$, where $|\boldsymbol{d}(\boldsymbol{x})| = \sum_{i=1}^{m} d_i^2(\boldsymbol{x})$.

(A2) $a_p(p), a_{pp}(p), a_{ppp}(p), b_p(p)$, $b_{pp}(p)$, and $b_{ppp}(p)$ are bounded for all $p \in \mathbb{R}$.

(A3) $f(\boldsymbol{x}, t, p)$ is locally Lipschitz continuous in the third variable p, i.e., if $|p - p^*| \leq \tilde{K}$ then $|f(\boldsymbol{x}, t, p) - f(\boldsymbol{x}, t, p^*)| \leq K(p, \tilde{K})|p - p^*|$. And also $a(p)$ and $b(p)$ are locally Lipschitz continuous.

Let $\boldsymbol{\nu} = \boldsymbol{\nu}(\boldsymbol{x}, t)$ be the unit vector in the direction of $(\boldsymbol{d}(\boldsymbol{x}), c(\boldsymbol{x}))$ and $\psi(\boldsymbol{x}) = [c(\boldsymbol{x})^2 + |\boldsymbol{d}(\boldsymbol{x})|^2]^{\frac{1}{2}}$. Then, we get $\frac{\partial u}{\partial \boldsymbol{\nu}} = \frac{c(\boldsymbol{x})}{\psi(\boldsymbol{x})} u_t + \frac{\boldsymbol{d}(\boldsymbol{x})}{\psi(\boldsymbol{x})} \cdot \nabla u$. Therefore the Sobolev equation (1.1) can be transformed into

$$
\begin{aligned}
\psi(\boldsymbol{x}) \frac{\partial u}{\partial \boldsymbol{\nu}} - \nabla \cdot (a(u)\nabla u) - \nabla \cdot (b(u)\nabla u_t) = f(\boldsymbol{x}, t, u), &\qquad \text{in } \Omega \times (0, T], \\
u(\boldsymbol{x}, t) = 0, &\qquad \text{on } \partial\Omega \times (0, T], \\
u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), &\qquad \text{in } \Omega.
\end{aligned} \tag{2.1}
$$

Now we have the following variational formulation of (2.1): Find $u(\boldsymbol{x}, t) \in H_0^1(\Omega)$ such that

$$
\begin{aligned}
(\psi(\boldsymbol{x}) \frac{\partial u}{\partial \boldsymbol{\nu}}, \tau) + (a(u)\nabla u, \nabla \tau) + (b(u)\nabla u_t, \nabla \tau) & \\
= (f(x, t, u), \tau), &\qquad \forall \tau \in H_0^1(\Omega), \\
u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}). &
\end{aligned} \tag{2.2}
$$

3. Finite element spaces and an elliptic projection

Let $\{S_h^r\}$ be a finite dimensional subspaces of $H_0^1(\Omega)$ satisfying the following approximation and inverse properties: for $\phi \in H_0^1(\Omega) \cap W^{s,p}(\Omega)$, there exist a positive constant K_1 independent of h, ϕ and r, and a sequence $P_h\phi \in S_h^r$ such that for any $0 \leq q \leq s$ and $1 \leq p \leq \infty$

$$
\|\phi - P_h\phi\|_{q,p} \leq K_1 h^{\mu - q} \|\phi\|_{s,p},
$$

where $\mu = \min(r+1, s)$ and also there exist a positive constant K_2 independent of h and r, such that

$$
\|\varphi\|_1 \leq K_2 h^{-1} \|\varphi\| \text{ and } \|\varphi\|_\infty \leq K_2 h^{-\frac{m}{2}} \|\varphi\|, \ \forall \varphi \in S_h^r.
$$

Now we define bilinear forms A and B on $H_0^1(\Omega) \times H_0^1(\Omega)$ by

$$
A(u : v, w) = (a(u)\nabla v, \nabla w), \quad B(u : v, w) = (b(u)\nabla v, \nabla w). \tag{3.1}
$$

By following the idea in [10, 14] and the assumption (A1), we define a differentiable function $\tilde{u} : [0,T] \to S_h^r$ satisfying

$$\begin{aligned} A(u : u - \tilde{u}, \chi) + B(u : u_t - \tilde{u}_t, \chi) = 0, \qquad & \forall \chi \in S_h^r, \\ (\tilde{u}(0), \chi) = (u_0, \chi), \quad & \forall \chi \in S_h^r. \end{aligned} \tag{3.2}$$

Now let $\eta = u - \tilde{u}$. The following estimates for η, η_t, η_{tt} and η_{ttt} are given in [15, 16].

Lemma 3.1. *Let $u_0 \in H^s(\Omega)$, $u_t, u_{tt}, u_{ttt} \in H^s(\Omega)$, and $u_t \in L^2(H^s(\Omega))$. Then there exists a constant K, independent of h, such that*

(**i**) $\|\eta\| + h\|\eta\|_1 \le K h^{\mu}(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s)$,

(**ii**) $\|\eta_t\| + h\|\eta_t\|_1 \le K h^{\mu}(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s + \|u_t\|_s)$,

(**iii**) $\|\eta_{tt}\|_1 \le K h^{\mu-1}(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s + \|u_t\|_s + \|u_{tt}\|_s)$,

(**iv**) $\|\eta_{ttt}\|_1 \le K h^{\mu-1}(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s + \|u_t\|_s + \|u_{tt}\|_s + \|u_{ttt}\|_s)$,

where $\mu = \min(r+1, s)$ and $s \ge 2$.

Lemma 3.2. *Let $u_0 \in H^s(\Omega)$, $u, u_t, u_{tt}, u_{ttt} \in L^\infty(H^s(\Omega)) \cap L^\infty(W^{1,\infty}(\Omega))$, $u_t \in L^2(H^s(\Omega))$ and $s \ge 2$. If $\mu. \ge 1 + \frac{m}{2}$, then there exists a constant K, independent of h, such that*

$$\max\{\|\eta\|_\infty,\ \|\nabla\eta\|_\infty,\ \|\nabla\eta_t\|_\infty,\ \|\nabla\eta_{tt}\|_\infty,\ \|\nabla\eta_{ttt}\|_\infty\} \le K,$$

where $\mu = \min(r+1, s)$.

Throughout this paper, a generic positive constant K depends on the domain $\Omega, \tilde{K}$, and $u(\boldsymbol{x}, t)$, but is independent of the discretization magnitudes of the spatial and the temporal directions. So any K in the different places does not need to be the same.

4. The optimal $L^\infty(L^2)$ and $L^\infty(H^1)$ error estimates

Let N be a positive integer, $\Delta t = T/N$, $t^n = n\Delta t$, and $u^n = u(\boldsymbol{x}, t^n)$ for $0 \le n \le N$. For $1 \le n \le N-1$, from (2.2) and the definitions of bilinear forms A and B, we obtain

$$\begin{aligned} &\Big(\psi(\boldsymbol{x})\frac{\partial u^{n+1}}{\partial \boldsymbol{\nu}}, \chi\Big) + A(u^{n+1} : u^{n+1}, \chi) + B(u^{n+1} : u_t^{n+1}, \chi) \\ &= (f(\boldsymbol{x}, t^{n+1}, u^{n+1}), \chi), \ \ \forall \chi \in S_h^r, \end{aligned} \tag{4.1}$$

and similarly, we have

$$\begin{aligned} &\Big(\psi(\boldsymbol{x})\frac{\partial u(t^{\frac{1}{2}})}{\partial \boldsymbol{\nu}}, \chi\Big) + A(u(t^{\frac{1}{2}}) : u(t^{\frac{1}{2}}), \chi) + B(u(t^{\frac{1}{2}}) : u_t(t^{\frac{1}{2}}), \chi) \\ &= (f(\boldsymbol{x}, t^{\frac{1}{2}}, u(t^{\frac{1}{2}})), \chi), \ \ \forall \chi \in S_h^r. \end{aligned} \tag{4.2}$$

Let $\tilde{\boldsymbol{d}}(\boldsymbol{x}) = \frac{\boldsymbol{d}(\boldsymbol{x})}{c(\boldsymbol{x})}$, $\check{\boldsymbol{x}} = \boldsymbol{x} - \tilde{\boldsymbol{d}}(\boldsymbol{x})\Delta t$, $\hat{\boldsymbol{x}} = \boldsymbol{x} - 2\tilde{\boldsymbol{d}}(\boldsymbol{x})\Delta t$, $\check{\check{\boldsymbol{x}}} = \boldsymbol{x} + \frac{\tilde{\boldsymbol{d}}(\boldsymbol{x})}{2}\Delta t$, and $\hat{\hat{\boldsymbol{x}}} = \boldsymbol{x} - \frac{\tilde{\boldsymbol{d}}(\boldsymbol{x})}{2}\Delta t$. Now an extrapolated higher order characteristic finite element scheme can be introduced as follows: Find $\{u_h^n\}_{n=0}^N \in S_h^r$ such that for $1 \leq n \leq N-1$

$$\begin{aligned} &\Big(c(\boldsymbol{x})\frac{\frac{3}{2}u_h^{n+1} - 2\check{u}_h^n + \frac{1}{2}\hat{u}_h^{n-1}}{\Delta t}, \chi\Big) + A(Eu_h^n\,; u_h^{n+1}, \chi) \\ &+ B(Eu_h^n\,; \frac{\frac{3}{2}u_h^{n+1} - 2u_h^n + \frac{1}{2}u_h^{n-1}}{\Delta t}, \chi) \\ &= (f(\boldsymbol{x}, t^{n+1}, Eu_h^n), \chi), \quad \forall \chi \in S_h^r, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\Big(c(\boldsymbol{x})\frac{\check{u}_h^1 - \hat{u}_h^0}{\Delta t}, \chi\Big) + A(u_h^{\frac{1}{2}}\,; u_h^{\frac{1}{2}}, \chi) + B(u_h^{\frac{1}{2}}\,; \frac{u_h^1 - u_h^0}{\Delta t}, \chi) \\ &= (f(\boldsymbol{x}, t^{\frac{1}{2}}, u_h^{\frac{1}{2}}), \chi), \quad \forall \chi \in S_h^r, \end{aligned} \tag{4.4}$$

and

$$u_h^0(\boldsymbol{x}) = \tilde{u}(\boldsymbol{x}, 0), \tag{4.5}$$

where $u_h^n = u_h^n(\boldsymbol{x})$ for $0 \leq n \leq N$, $\check{u}_h^n = u_h^n(\check{\boldsymbol{x}})$, $\hat{u}_h^{n-1} = u_h^{n-1}(\hat{\boldsymbol{x}})$, $Eu_h^n = 2u_h^n - u_h^{n-1}$ for $1 \leq n \leq N-1$, $u_h^{\frac{1}{2}} = \frac{1}{2}(u_h^1 + u_h^0)$, $\check{u}_h^1 = u_h^1(\check{\boldsymbol{x}})$, and $\hat{u}_h^0 = u_h^0(\hat{\boldsymbol{x}})$. Note that (4.3) is based on a three-point backward difference formula to approximate both the directional and the temporal derivatives and on an extrapolation technique to avoid the difficulty in solving the nonlinear systems and (4.4) is based on a Crank-Nicolson difference formula to approximate both the directional and the temporal derivatives.

For the error analysis, we denote $\xi^n = u_h^n - \tilde{u}^n$ and $\partial_t \xi^n = \frac{\xi^n - \xi^{n-1}}{\Delta t}$. The following theorem is the same as Theorem 4.1 in [15].

Theorem 4.1. *Let u and $\{u_h^n\}$ be solutions of (2.2) and (4.3)-(4.5), respectively. In addition to the assumptions of Lemma 3.2, if $\mu \geq 1 + \frac{m}{2}$, $u \in L^\infty(H^3(\Omega))$, and $\Delta t = O(h)$, then*

$$\|\nabla \xi^1\|^2 + \Delta t(\|\partial_t \xi^1\|^2 + \|\nabla \partial_t \xi^1\|^2) \leq K \Delta t (h^{2\mu} + (\Delta t)^4),$$

where $\mu = \min(r+1, s)$.

Theorem 4.2. *Under the same assumptions of Theorem 4.1, we have*

$$\max_{0 \leq n \leq N} \Big[\|u^n - u_h^n\| + h\|\nabla(u^n - u_h^n)\|\Big] \leq K(h^\mu + (\Delta t)^2),$$

where $\mu = \min(r+1, s)$.

Proof. To establish this theorem, we prove the following statement by mathematical induction: There exist $0 < \tilde{h} < 1$ and $0 < \tilde{\Delta t} < 1$ such that

$$\|\nabla \xi^n\|^2 + \Delta t(\|\partial_t \xi^n\|^2 + \|\nabla \partial_t \xi^n\|^2) \leq K(h^{2\mu} + (\Delta t)^4) \tag{4.6}$$

for all $0 < h < \tilde{h}$, $0 < \Delta t < \tilde{\Delta t}$, and $n = 0, 1, \cdots, N$. For our convenience, we abuse the notations such as $Eu_h^0 = 0$ and $\xi^{-1} = 0$. Since $\xi^0 = 0$, (4.6) trivially holds for $n = 0$. And by Theorem 4.1, (4.6) holds for $n = 1$. Now we assume that (4.6) holds with $n \le l-1$. Notice that $\|\xi^n\|_\infty \le K,\ 0 \le n \le l-1$. For $1 \le n \le l-1$, subtract (4.3) from (4.1) to get

$$\begin{aligned}
&\Big(c(\boldsymbol{x})\frac{\frac{3}{2}\xi^{n+1} - 2\check{\xi}^n + \frac{1}{2}\hat{\xi}^{n-1}}{\Delta t}, \chi\Big) + A(Eu_h^n:\ \xi^{n+1}, \chi) \\
&\quad + B(Eu_h^n:\ \frac{\frac{3}{2}\xi^{n+1} - 2\xi^n + \frac{1}{2}\xi^{n-1}}{\Delta t}, \chi) \\
&= \Big(c(\boldsymbol{x})\frac{\frac{3}{2}\eta^{n+1} - 2\check{\eta}^n + \frac{1}{2}\hat{\eta}^{n-1}}{\Delta t}, \chi\Big) \\
&\quad - \Big(c(\boldsymbol{x})\frac{\frac{3}{2}u^{n+1} - 2\check{u}^n + \frac{1}{2}\hat{u}^{n-1}}{\Delta t}, \chi\Big) \\
&\quad + A(Eu_h^n:\ \eta^{n+1}, \chi) - A(Eu_h^n:\ u^{n+1}, \chi) \\
&\quad + B(Eu_h^n:\ \frac{\frac{3}{2}\eta^{n+1} - 2\eta^n + \frac{1}{2}\eta^{n-1}}{\Delta t}, \chi) \\
&\quad - B(Eu_h^n:\ \frac{\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1}}{\Delta t}, \chi) \\
&\quad + (f(\boldsymbol{x}, t^{n+1}, Eu_h^n) - f(\boldsymbol{x}, t^{n+1}, u^{n+1}), \chi) \\
&\quad + \Big(\psi(\boldsymbol{x})\frac{\partial u^{n+1}}{\partial \boldsymbol{\nu}}, \chi\Big) + A(u^{n+1} : u^{n+1}, \chi) + B(u^{n+1} : u_t^{n+1}, \chi).
\end{aligned} \tag{4.7}$$

Notice that

$$\begin{aligned}
&\frac{3}{2}\xi^{n+1} - 2\check{\xi}^n + \frac{1}{2}\hat{\xi}^{n-1} \\
&= \frac{3}{2}(\xi^{n+1} - \xi^n) - \frac{1}{2}(\xi^n - \xi^{n-1}) - \frac{1}{2}(\xi^{n-1} - \hat{\xi}^{n-1}) - 2(\check{\xi}^n - \xi^n).
\end{aligned} \tag{4.8}$$

By applying (4.8) to (4.7), we get

$$\begin{aligned}
&\Big(c(\boldsymbol{x})\frac{\frac{3}{2}(\xi^{n+1} - \xi^n) - \frac{1}{2}(\xi^n - \xi^{n-1})}{\Delta t}, \chi\Big) + A(Eu_h^n : \xi^{n+1}, \chi) \\
&\quad + B(Eu_h^n : \frac{(\xi^{n+1} - \xi^n) + \frac{1}{2}[(\xi^{n+1} - \xi^n) - (\xi^n - \xi^{n-1})]}{\Delta t}, \chi) \\
&= \Big(c(\boldsymbol{x})\frac{2(\check{\xi}^n - \xi^n) + \frac{1}{2}(\xi^{n-1} - \hat{\xi}^{n-1})}{\Delta t}, \chi\Big) \\
&\quad + \Big(c(\boldsymbol{x})\frac{2(\eta^{n+1} - \check{\eta}^n) - \frac{1}{2}(\eta^{n+1} - \hat{\eta}^{n-1})}{\Delta t}, \chi\Big) \\
&\quad + \Big(\psi(\boldsymbol{x})\frac{\partial u^{n+1}}{\partial \boldsymbol{\nu}} - c(\boldsymbol{x})\frac{\frac{3}{2}u^{n+1} - 2\check{u}^n + \frac{1}{2}\hat{u}^{n-1}}{\Delta t}, \chi\Big) \\
&\quad + \Big[A(Eu_h^n : \eta^{n+1}, \chi) - A(u^{n+1} : \eta^{n+1}, \chi)\Big] + A(u^{n+1} : \eta^{n+1}, \chi) \\
&\quad + \Big[A(u^{n+1} : u^{n+1}, \chi) - A(Eu_h^n : u^{n+1}, \chi)\Big]
\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
&+\Big[B(Eu_h^n : \frac{\frac{3}{2}\eta^{n+1} - 2\eta^n + \frac{1}{2}\eta^{n-1}}{\Delta t}, \chi) \\
&\quad - B(u^{n+1} : \frac{\frac{3}{2}\eta^{n+1} - 2\eta^n + \frac{1}{2}\eta^{n-1}}{\Delta t}, \chi)\Big] \\
&+ B(u^{n+1} : \frac{\frac{3}{2}\eta^{n+1} - 2\eta^n + \frac{1}{2}\eta^{n-1}}{\Delta t} - \eta_t^{n+1}, \chi) + B(u^{n+1} : \eta_t^{n+1}, \chi) \\
&+ B(u^{n+1} : u_t^{n+1} - \frac{\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1}}{\Delta t}, \chi) \\
&+ \Big[B(u^{n+1} : \frac{\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1}}{\Delta t}, \chi) \\
&\quad - B(Eu_h^n : \frac{\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1}}{\Delta t}, \chi)\Big] \\
&+ (f(\boldsymbol{x}, t^{n+1}, Eu_h^n) - f(\boldsymbol{x}, t^{n+1}, u^{n+1}), \chi) \\
\equiv\ & \Sigma_{i=1}^{12} R_i.
\end{aligned}
$$

Now denote three terms of the left-hand side of (4.9) by L_1, L_2 and L_3, respectively and choose $\chi = \partial_t \xi^{n+1}$ in (4.9). First the lower bounds of L_1, L_2 and L_3 can be estimated as follows:

$$
\begin{aligned}
L_1 \geq & c_* \|\partial_t \xi^{n+1}\|^2 + \frac{1}{4}(\|\sqrt{c(\boldsymbol{x})}\partial_t \xi^{n+1}\|^2 - \|\sqrt{c(\boldsymbol{x})}\partial_t \xi^n\|^2), \\
L_2 \geq & \frac{1}{2\Delta t}(\|\sqrt{a(Eu_h^n)}\nabla \xi^{n+1}\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla \xi^n\|^2) \\
& + \frac{1}{2\Delta t}(\|\sqrt{a(Eu_h^{n-1})}\nabla \xi^n\|^2 - \|\sqrt{a(Eu_h^n)}\nabla \xi^n\|^2), \\
L_3 \geq & b_* \|\nabla \partial_t \xi^{n+1}\|^2 + \frac{1}{4}(\|\sqrt{b(Eu_h^n)}\nabla \partial_t \xi^{n+1}\|^2 - \|\sqrt{b(Eu_h^{n-1})}\nabla \partial_t \xi^n\|^2) \\
& + \frac{1}{4}(\|\sqrt{b(Eu_h^{n-1})}\nabla \partial_t \xi^n\|^2 - \|\sqrt{b(Eu_h^n)}\nabla \partial_t \xi^n\|^2).
\end{aligned}
$$

By applying these lower bounds of $L_1 \sim L_3$ to (4.9), we get

$$
\begin{aligned}
& c_* \|\partial_t \xi^{n+1}\|^2 + \frac{1}{4}(\|\sqrt{c(\boldsymbol{x})}\partial_t \xi^{n+1}\|^2 - \|\sqrt{c(\boldsymbol{x})}\partial_t \xi^n\|^2) \\
& + \frac{1}{2\Delta t}(\|\sqrt{a(Eu_h^n)}\nabla \xi^{n+1}\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla \xi^n\|^2) \\
& + \frac{1}{4}(\|\sqrt{b(Eu_h^n)}\nabla \partial_t \xi^{n+1}\|^2 - \|\sqrt{b(Eu_h^{n-1})}\nabla \partial_t \xi^n\|^2) \\
& + b_* \|\nabla \partial_t \xi^{n+1}\|^2 \\
\leq & \frac{1}{2\Delta t}((a(Eu_h^n) - a(Eu_h^{n-1}))\nabla \xi^n, \nabla \xi^n) \\
& + \frac{1}{4}((b(Eu_h^n) - b(Eu_h^{n-1}))\nabla \partial_t \xi^n, \nabla \partial_t \xi^n) + \sum_{i=1}^{12} R_i.
\end{aligned}
\tag{4.10}
$$

By the induction hypothesis and the fact that $\Delta t = O(h)$, we have

$$\begin{aligned}
&\|Eu_h^n - Eu_h^{n-1}\|_\infty \\
=&\|E(u_h^n - \tilde{u}^n) - E(u_h^{n-1} - \tilde{u}^{n-1}) + E\tilde{u}^n - E\tilde{u}^{n-1}\|_\infty \\
\le&\Delta t(2\|\partial_t\xi^n\|_\infty + \|\partial_t\xi^{n-1}\|_\infty) + K\Delta t \\
\le&K\Delta t.
\end{aligned} \tag{4.11}$$

Hence, by (A4) and (4.11), (4.10) can be estimated as follows:

$$\begin{aligned}
&c_*\|\partial_t\xi^{n+1}\|^2 + \frac{1}{4}(\|\sqrt{c(\boldsymbol{x})}\partial_t\xi^{n+1}\|^2 - \|\sqrt{c(\boldsymbol{x})}\partial_t\xi^n\|^2) \\
&+ \frac{1}{2\Delta t}(\|\sqrt{a(Eu_h^n)}\nabla\xi^{n+1}\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla\xi^n\|^2) \\
&+ \frac{1}{4}(\|\sqrt{b(Eu_h^n)}\nabla\partial_t\xi^{n+1}\|^2 - \|\sqrt{b(Eu_h^{n-1})}\nabla\partial_t\xi^n\|^2) \\
&+ b_*\|\nabla\partial_t\xi^{n+1}\|^2 \\
&\le K[\Delta t\|\nabla\partial_t\xi^n\|^2 + \|\nabla\xi^n\|^2] + \sum_{i=1}^{12} R_i.
\end{aligned} \tag{4.12}$$

By (A2) and the Taylor expansion, we obtain

$$\begin{aligned}
R_1 &= \Big(c(\boldsymbol{x})\frac{2(\check{\xi}^n - \xi^n) + \frac{1}{2}(\xi^{n-1} - \hat{\xi}^{n-1})}{\Delta t}, \partial_t\xi^{n+1}\Big) \\
&\le \epsilon\|\partial_t\xi^{n+1}\|^2 + K\Big(\|\frac{\xi^n - \check{\xi}^n}{\Delta t}\|^2 + \|\frac{\xi^{n-1} - \hat{\xi}^{n-1}}{\Delta t}\|^2\Big) \\
&\le \epsilon\|\partial_t\xi^{n+1}\|^2 + K(\|\nabla\xi^n\|^2 + \|\nabla\xi^{n-1}\|^2).
\end{aligned}$$

Since

$$\eta^{n+1} - \check{\eta}^n = \Delta t[\eta_t(t_\theta^n) + \nabla\eta(\tilde{\boldsymbol{x}}_1, t^n)\cdot\tilde{\boldsymbol{d}}]$$

and

$$\eta^{n+1} - \hat{\eta}^{n-1} = \Delta t[\eta_t(t_\theta^{n-1}) + \nabla\eta(\tilde{\boldsymbol{x}}_2, t^{n-1})\cdot\tilde{\boldsymbol{d}}]$$

for some $t_\theta^n \in (t^n, t^{n+1})$, $t_\theta^{n-1} \in (t^{n-1}, t^{n+1})$, $\tilde{\boldsymbol{x}}_1 \in (\check{\boldsymbol{x}}, \boldsymbol{x})$ and $\tilde{\boldsymbol{x}}_2 \in (\hat{\boldsymbol{x}}, \boldsymbol{x})$, by integration by parts, we have

$$\begin{aligned}
R_2 &= \Big(c(\boldsymbol{x})\frac{2(\eta^{n+1} - \check{\eta}^n) - \frac{1}{2}(\eta^{n+1} - \hat{\eta}^{n-1})}{\Delta t}, \partial_t\xi^{n+1}\Big) \\
&\le \epsilon\|\nabla\partial_t\xi^{n+1}\|^2 + \epsilon\|\partial_t\xi^{n+1}\|^2 + K(\|\eta_t\|^2_{L^\infty(L^2)} + \|\eta^n\|^2 + \|\eta^{n-1}\|^2).
\end{aligned}$$

By the Taylor expansion, there exist $t_\theta^n \in (t^n, t^{n+1}), t_\theta^{n-1} \in (t^{n-1}, t^{n+1}), \check{\boldsymbol{x}}_{\theta i} \in (\check{\boldsymbol{x}}, \boldsymbol{x})$, and $\hat{\boldsymbol{x}}_{\theta i} \in (\hat{\boldsymbol{x}}, \boldsymbol{x})$, $1 \le i \le 3$, satisfying

$$\begin{aligned}
&\psi(\boldsymbol{x})\frac{\partial u^{n+1}}{\partial \boldsymbol{\nu}} - c(\boldsymbol{x})\frac{\frac{3}{2}u^{n+1} - 2\check{u}^n + \frac{1}{2}\hat{u}^{n-1}}{\Delta t} \\
&= c(\boldsymbol{x})u_t^{n+1} + \boldsymbol{d}(\boldsymbol{x})\cdot\nabla u^{n+1} \\
&\quad - c(\boldsymbol{x})\frac{1}{\Delta t}\Big[\frac{3}{2}u^{n+1} - 2\Big(\check{u}^{n+1} - \Delta t\check{u}_t^{n+1} + \frac{(\Delta t)^2}{2}\check{u}_{tt}^{n+1} - \frac{1}{6}(\Delta t)^3\check{u}_{ttt}(t_\theta^n)\Big) \\
&\quad + \frac{1}{2}\Big(\hat{u}^{n+1} - 2\Delta t\hat{u}_t^{n+1} + \frac{(2\Delta t)^2}{2}\hat{u}_{tt}^{n+1} - \frac{1}{6}(2\Delta t)^3\hat{u}_{ttt}(t_\theta^{n-1})\Big)\Big] \\
&= c(\boldsymbol{x})u_t^{n+1} + \boldsymbol{d}(\boldsymbol{x})\cdot\nabla u^{n+1} \\
&\quad - c(\boldsymbol{x})\frac{1}{\Delta t}\Big[\frac{3}{2}u^{n+1} - 2\Big(u^{n+1} - \Delta t\tilde{\boldsymbol{d}}\cdot\nabla u^{n+1} + \frac{1}{2}(\tilde{\boldsymbol{d}}\Delta t)^2\cdot\nabla^2 u^{n+1} \\
&\quad - \frac{1}{6}(\tilde{\boldsymbol{d}}\Delta t)^3\cdot\nabla^3 u^{n+1}(\check{\boldsymbol{x}}_{\theta 1})\Big) \\
&\quad + 2\Delta t\Big(u_t^{n+1} - \tilde{\boldsymbol{d}}\Delta t\cdot\nabla u_t^{n+1} + \frac{1}{2}(\tilde{\boldsymbol{d}}\Delta t)^2\cdot\nabla^2 u_t^{n+1}(\check{\boldsymbol{x}}_{\theta 2})\Big) \\
&\quad - (\Delta t)^2\Big(u_{tt}^{n+1} - \tilde{\boldsymbol{d}}\Delta t\cdot\nabla u_{tt}^{n+1}(\check{\boldsymbol{x}}_{\theta 3})\Big) + \frac{1}{3}(\Delta t)^3\check{u}_{ttt}(t_\theta^n) \\
&\quad + \frac{1}{2}\Big(u^{n+1} - 2\tilde{\boldsymbol{d}}\Delta t\cdot\nabla u^{n+1} + \frac{1}{2}(2\tilde{\boldsymbol{d}}\Delta t)^2\cdot\nabla^2 u^{n+1} \\
&\qquad - \frac{1}{6}(2\tilde{\boldsymbol{d}}\Delta t)^3\cdot\nabla^3 u^{n+1}(\hat{\boldsymbol{x}}_{\theta 1})\Big) \\
&\quad - \Delta t\Big(u_t^{n+1} - 2\tilde{\boldsymbol{d}}\Delta t\cdot\nabla u_t^{n+1} + \frac{1}{2}(2\tilde{\boldsymbol{d}}\Delta t)^2\cdot\nabla^2 u_t^{n+1}(\hat{\boldsymbol{x}}_{\theta 2})\Big) \\
&\quad + (\Delta t)^2\Big(u_{tt}^{n+1} - 2\Delta t\tilde{\boldsymbol{d}}\cdot\nabla u_{tt}^{n+1}(\hat{\boldsymbol{x}}_{\theta 3})\Big) - \frac{2}{3}(\Delta t)^3\hat{u}_{ttt}(t_\theta^{n-1})\Big] \\
&= c(\boldsymbol{x})(\Delta t)^2\Big[\frac{1}{3}\tilde{\boldsymbol{d}}^3\cdot\nabla^3 u^{n+1}(\check{\boldsymbol{x}}_{\theta 1}) + \tilde{\boldsymbol{d}}^2\cdot\nabla^2 u_t^{n+1}(\check{\boldsymbol{x}}_{\theta 2}) - 2\tilde{\boldsymbol{d}}\cdot\nabla u_{tt}^{n+1}(\check{\boldsymbol{x}}_{\theta 3}) \\
&\quad - \frac{1}{3}\check{u}_{ttt}(t_\theta^n) + \frac{2}{3}\tilde{\boldsymbol{d}}^3\cdot\nabla^3 u^{n+1}(\hat{\boldsymbol{x}}_{\theta 1}) + 2\tilde{\boldsymbol{d}}^2\cdot\nabla^2 u_t^{n+1}(\hat{\boldsymbol{x}}_{\theta 2}) \\
&\quad + 2\tilde{\boldsymbol{d}}\cdot\nabla u_{tt}^{n+1}(\hat{\boldsymbol{x}}_{\theta 3}) + \frac{2}{3}\hat{u}_{ttt}(t_\theta^{n-1})\Big],
\end{aligned}$$

where $\boldsymbol{d}^j\cdot(\nabla^j u^{n+1}) = \sum_{l=0}^{j}\binom{j}{l}d_1^{j-l}d_2^l\frac{\partial^j u^{n+1}}{\partial x_1^{j-l}\partial x_2^l}$ for $j=2$ and $j=3$ when $m=2$ and we use similar notations when $m=3$. Since $u(t) \in L^\infty(H^3(\Omega)) \cap W^{1,\infty}(H^2(\Omega)) \cap W^{2,\infty}(H^1(\Omega)) \cap W^{3,\infty}(L^2(\Omega))$, we get

$$|R_3| \le K(\Delta t)^4 + \epsilon\|\partial_t\xi^{n+1}\|^2.$$

Note that

$$\tilde{u}^{n+1} - E\tilde{u}^n = \tilde{u}^{n+1} - 2\tilde{u}^n + \tilde{u}^{n-1} = \frac{1}{2}(\Delta t)^2(\tilde{u}_{tt}(t_1^n) + \tilde{u}_{tt}(t_1^{n-1}))$$

holds for some $t_1^n \in (t^n, t^{n+1})$ and $t_1^{n-1} \in (t^{n-1}, t^n)$. Hence we have

$$\begin{aligned} \|u^{n+1} - Eu_h^n\| &= \|u^{n+1} - \tilde{u}^{n+1} + \tilde{u}^{n+1} - E\tilde{u}^n + E\tilde{u}^n - Eu_h^n\| \\ &\le \|\eta^{n+1}\| + \|\tilde{u}^{n+1} - E\tilde{u}^n\| + \|E\xi^n\| \\ &\le \|\eta^{n+1}\| + K(\Delta t)^2 + 2\|\xi^n\| + \|\xi^{n-1}\|. \end{aligned} \tag{4.13}$$

By (4.13) and Lemma 3.2, the estimate for R_4 is given as follows:

$$\begin{aligned} R_4 &= ((a(Eu_h^n) - a(u^{n+1}))\nabla\eta^{n+1}, \nabla\partial_t\xi^{n+1}) \\ &\le K\|\nabla\eta^{n+1}\|_\infty(\|\eta^{n+1}\| + \|\xi^n\| + \|\xi^{n-1}\| + (\Delta t)^2)\|\nabla\partial_t\xi^{n+1}\| \\ &\le \epsilon\|\nabla\partial_t\xi^{n+1}\|^2 + K(\|\eta^{n+1}\|^2 + \|\xi^n\|^2 + \|\xi^{n-1}\|^2 + (\Delta t)^4). \end{aligned}$$

By (3.2), $R_5 + R_9 = 0$ holds. By (4.13), R_6 can be estimated as follows:

$$\begin{aligned} R_6 &= ((a(u^{n+1}) - a(Eu_h^n))\nabla u^{n+1}, \nabla\partial_t\xi^{n+1}) \\ &\le \epsilon\|\nabla\partial_t\xi^{n+1}\|^2 + K(\|\eta^{n+1}\|^2 + \|\xi^n\|^2 + \|\xi^{n-1}\|^2 + (\Delta t)^4). \end{aligned}$$

Since

$$\|\nabla\Big(\frac{3}{2}\eta^{n+1} - 2\eta^n + \frac{1}{2}\eta^{n-1}\Big)\|_\infty \le \frac{3}{2}\Delta t\|\nabla\eta_t\|_{L^\infty(L^\infty)} + \frac{1}{2}\Delta t\|\nabla\eta_t\|_{L^\infty(L^\infty)},$$

by (4.13) and Lemma 3.2, we get

$$\begin{aligned} R_7 &= \Big((b(Eu_h^n) - b(u^{n+1}))\nabla\frac{\frac{3}{2}\eta^{n+1} - 2\eta^n + \frac{1}{2}\eta^{n-1}}{\Delta t}, \nabla\partial_t\xi^{n+1}\Big) \\ &\le K(\|\eta^{n+1}\| + \|\xi^n\| + \|\xi^{n-1}\| + (\Delta t)^2)\|\nabla\eta_t\|_{L^\infty(L^\infty)}\|\nabla\partial_t\xi^{n+1}\| \\ &\le K(\|\eta^{n+1}\|^2 + \|\xi^n\|^2 + \|\xi^{n-1}\|^2 + (\Delta t)^4) + \epsilon\|\nabla\partial_t\xi^{n+1}\|^2. \end{aligned}$$

By the Taylor expansion, we have

$$\frac{\frac{3}{2}\eta^{n+1} - 2\eta^n + \frac{1}{2}\eta^{n-1}}{\Delta t} - \eta_t^{n+1} = \frac{1}{3}(\Delta t)^2\eta_{ttt}(t_\theta^n) - \frac{2}{3}(\Delta t)^2\eta_{ttt}(t_\theta^{n-1})$$

for some $t_\theta^n \in (t^n, t^{n+1})$ and $t_\theta^{n-1} \in (t^{n-1}, t^{n+1})$. So, by Lemma 3.1, we get

$$\begin{aligned} R_8 &= B(u^{n+1} : \frac{\frac{3}{2}\eta^{n+1} - 2\eta^n + \frac{1}{2}\eta^{n-1}}{\Delta t} - \eta_t^{n+1}, \partial_t\xi^{n+1}) \\ &\le \epsilon\|\nabla\partial_t\xi^{n+1}\|^2 + K(\Delta t)^4. \end{aligned}$$

By the Taylor expansion and (4.13), the estimates for R_{10} and R_{11} can be obtained as follows:

$$\begin{aligned} R_{10} &= B(u^{n+1} : u_t^{n+1} - \frac{\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1}}{\Delta t}, \partial_t\xi^{n+1}) \\ &\le \epsilon\|\nabla\partial_t\xi^{n+1}\|^2 + K(\Delta t)^4, \\ R_{11} &= \Big((b(u^{n+1}) - b(Eu_h^n))\nabla\frac{\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1}}{\Delta t}, \nabla\partial_t\xi^{n+1}\Big) \\ &\le \epsilon\|\nabla\partial_t\xi^{n+1}\|^2 + K(\|\eta^{n+1}\|^2 + \|\xi^n\|^2 + \|\xi^{n-1}\|^2 + (\Delta t)^4). \end{aligned}$$

Since f is locally Lipchitz continuous in u, by (4.13), we have

$$\begin{aligned} R_{12} &= (f(\boldsymbol{x}, t^{n+1}, Eu_h^n) - f(\boldsymbol{x}, t^{n+1}, u^{n+1}), \partial_t \xi^{n+1}) \\ &\le \epsilon \|\partial_t \xi^{n+1}\|^2 + K(\|\eta^{n+1}\|^2 + \|\xi^n\|^2 + \|\xi^{n-1}\|^2 + (\Delta t)^4). \end{aligned}$$

By applying the estimates for $R_1 \sim R_{12}$ to (4.12), we get

$$\begin{aligned} & c_* \|\partial_t \xi^{n+1}\|^2 + \frac{1}{4}(\|\sqrt{c(\boldsymbol{x})}\partial_t \xi^{n+1}\|^2 - \|\sqrt{c(\boldsymbol{x})}\partial_t \xi^n\|^2) \\ & + \frac{1}{2\Delta t}(\|\sqrt{a(Eu_h^n)}\nabla \xi^{n+1}\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla \xi^n\|^2) + b_* \|\nabla \partial_t \xi^{n+1}\|^2 \\ & + \frac{1}{4}(\|\sqrt{b(Eu_h^n)}\nabla \partial_t \xi^{n+1}\|^2 - \|\sqrt{b(Eu_h^n)}\nabla \partial_t \xi^n\|^2) \\ \le & K\Big[\Delta t(\|\partial_t \xi^n\|^2 + \|\nabla \partial_t \xi^n\|^2) + \|\nabla \xi^n\|^2 + \|\nabla \xi^{n-1}\|^2 + \|\eta_t\|^2_{L^\infty(L^\infty)} \\ & + \|\eta^n\|^2 + \|\eta^{n-1}\|^2 + \|\eta^{n+1}\|^2 + \|\xi^n\|^2 + \|\xi^{n-1}\|^2 + (\Delta t)^4\Big] \\ & + 4\epsilon \|\partial_t \xi^{n+1}\|^2 + 7\epsilon \|\nabla \partial_t \xi^{n+1}\|^2. \end{aligned}$$

Since ϵ is sufficiently small, we obtain

$$\begin{aligned} & \frac{c^*}{2}\Delta t \|\partial_t \xi^{n+1}\|^2 + \frac{1}{4}\Delta t(\|\sqrt{c(\boldsymbol{x})}\partial_t \xi^{n+1}\|^2 - \|\sqrt{c(\boldsymbol{x})}\partial_t \xi^n\|^2) \\ & + \frac{1}{2}(\|\sqrt{a(Eu_h^n)}\nabla \xi^{n+1}\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla \xi^n\|^2) \\ & + \frac{\Delta t}{4}(\|\sqrt{b(Eu_h^n)}\nabla \partial_t \xi^{n+1}\|^2 - \|\sqrt{b(Eu_h^{n-1})}\nabla \partial_t \xi^n\|^2) \\ & + \frac{b_*}{2}\Delta t \|\nabla \partial_t \xi^{n+1}\|^2 \\ & \le K\Delta t\Big[\Delta t(\|\partial_t \xi^n\|^2 + \|\nabla \partial_t \xi^n\|^2) + \|\eta^{n+1}\|^2 + \|\eta^{n-1}\|^2 \\ & \qquad + \|\eta^n\|^2 + \|\eta_t\|^2_{L^\infty(L^\infty)} + \|\nabla \xi^n\|^2 + \|\nabla \xi^{n-1}\|^2 \\ & \qquad + \|\xi^{n-1}\|^2 + \|\xi^n\|^2 + (\Delta t)^4\Big]. \end{aligned} \tag{4.14}$$

Now we add both sides of (4.14) from $n = 1$ to $l - 1$ to get

$$\begin{aligned} & \frac{c^*}{2}\Delta t \sum_{n=1}^{l-1} \|\partial_t \xi^{n+1}\|^2 + \frac{\Delta t}{4}\|\sqrt{c(\boldsymbol{x})}\partial_t \xi^l\|^2 + \frac{1}{2}\|\sqrt{a(Eu_h^{l-1})}\nabla \xi^l\|^2 \\ & + \frac{b^*}{2}\Delta t \sum_{n=1}^{l-1} \|\nabla \partial_t \xi^{n+1}\|^2 + \frac{\Delta t}{4}\|\sqrt{b(Eu_h^{l-1})}\nabla \partial_t \xi^l\|^2 \end{aligned}$$

$$\begin{aligned}&\leq \frac{\Delta t}{4}\|\sqrt{c(\boldsymbol{x})}\partial_t\xi^1\|^2 + \frac{1}{2}\|\sqrt{a(Eu_h^0)}\nabla\xi^1\|^2 + \frac{\Delta t}{4}\|\sqrt{b(Eu_h^0)}\nabla\partial_t\xi^1\|^2 \\ &\quad + K\Delta t\sum_{n=1}^{l-1}\Big\{\Delta t(\|\partial_t\xi^n\|^2 + \|\nabla\partial_t\xi^n\|^2) + \|\xi^n\|^2 + \|\nabla\xi^n\|^2\Big\} \\ &\quad + K\Delta t\sum_{n=0}^{l}\Big\{\|\eta^n\|^2 + \|\eta_t\|^2_{L^\infty(L^2)} + (\Delta t)^4\Big\}.\end{aligned}$$

So, by Lemma 3.1 and Theorem 4.1, we have

$$\begin{aligned}&\|\nabla\xi^l\|^2 + \Delta t\{\|\partial_t\xi^l\|^2 + \|\nabla\partial_t\xi^l\|^2\} \\ \leq &K\Big[\Delta t\sum_{n=1}^{l-1}\{\Delta t(\|\partial_t\xi^n\|^2 + \|\nabla\partial_t\xi^n\|^2) + \|\nabla\xi^n\|^2\} + \Delta t\sum_{n=0}^{l}\{h^{2\mu} + (\Delta t)^4\}\Big].\end{aligned}$$

Hence by Gronwall's inequality, we obtain

$$\|\nabla\xi^l\|^2 + \Delta t\{\|\partial_t\xi^l\|^2 + \|\nabla\partial_t\xi^l\|^2\} \leq K[h^{2\mu} + (\Delta t)^4],$$

which completes the proof of the statement (4.6). By the triangle inequality and the Poincare's inequality, we finally obtain $\|u^l - u_h^l\| \leq K[h^\mu + (\Delta t)^2]$ and $\|\nabla(u^l - u_h^l)\| \leq K[h^{\mu-1} + (\Delta t)^2]$. Thus the result of this theorem is true. □

5. Computational results

In this section, we will present some numerical results to verify the convergence order of the higher order extrapolated characteristic FEM proposed in (4.3)-(4.5). For the sake of simplicity, we consider the Sobolev equation (1.1) with $c(\boldsymbol{x}) = d(\boldsymbol{x}) = 1$, $a(u) = b(u) = 0.001$ and $\Omega = [0, 1]$ so that (1.1) can be a convection dominated problem. We construct the approximation of $u(x, t)$ on the finite element space consisting of the piecewise linear polynomials. For the sake of convenience, we choose the exact solution $u(x, t)$ first and then compute $f(x, t)$ or $f(x, t, u)$ satisfying (1.1).

First we will show the convergence order of $u_h(x)$ at $T = 0.4$ to $u(x, 0.4)$ in the case that f depends only on x and t. Choose the exact solution $u(x, t)$ as follows:

$$u(x,t) = \begin{cases}(25(x-t-0.2)(0.6+t-x))^3, & 0.2 \leq x-t \leq 0.6 \\ 0 & \text{, otherwise,}\end{cases} \tag{5.1}$$

and compute $f(x, t) = u_t + u_x - 10^{-3}u_{xx} - 10^{-3}u_{txx}$ by substituting $u(x, t)$ defined in (5.1). As we expect from the conclusions of Theorem 4.2, Table 1 shows that the approximations of u at $T = 0.4$ converge with the order 2 in the temporal direction as well as the spatial direction. To show the order of convergence, we choose $h = \frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}$ consecutively ($N = 20, 40, 80, 160, 320$ consecutively) and also we choose $\Delta t = h$.

TABLE 1. The rate of convergence for the approximate values u_h of $u(\cdot, 0.4)$ when $f_1(x,t)$ and $\Delta t = h$

$h = \Delta t$	$\|u(\cdot,0.4) - u_h(\cdot,0.4)\|$	convergence order
1/20	$0.562766e-1$	
1/40	$0.191283e-1$	1.56
1/80	$0.529704e-2$	1.85
1/160	$0.138938e-2$	1.93
1/320	$0.341329e-3$	2.03

Next we will show the convergence order of $u_h(x)$ in the case that f depends on u as well as x and t. Now we choose the same exact solution $u(x,t)$ defined in (5.1), compute $f_1(x,t) = u_t + u_x - 10^{-3}u_{xx} - 10^{-3}u_{txx} - u^2$ by substituting $u(x,t)$, and let $f(x,t,u) = f_1(x,t) + u^2$. Then $u(x,t)$ is the solution of the following problem:

$$\begin{cases} u_t + u_x - 10^{-3}u_{txx} - 10^{-3}u_{xx} = f(x,t,u), & (x,t) \in \Omega \times (0, 0.4) \\ u(x,t) = 0, \quad (x,t) = \{0,1\} \times (0, 0.4) \\ u(x,0) = (25(x-0.2)(0.6-x))^3, \quad x \in \Omega. \end{cases}$$

We provide the order of convergence at $T = 0.4$ in Table 2 which verifies our theoretical result in Theorem 4.2.

TABLE 2. The rate of convergence for the approximate values u_h of $u(\cdot, 0.4)$ when $f(x,t,u)$ and $\Delta t = h$

$h = \Delta t$	$\|u(\cdot,0.4) - u_h(\cdot,0.4)\|$	convergence order
1/20	$0.778250e-1$	
1/40	$0.260070e-1$	1.58
1/80	$0.717642e-2$	1.86
1/160	$0.188200e-2$	1.93
1/320	$0.465709e-3$	2.01

Conclusions. We have introduced an extrapolated higher order characteristic FEM to approximate the solution $u(x,t)$ of the problem (1.1). We derive the higher order of convergence in both temporal direction and spatial direction in L^2 normed space. And some numerical results are given to verify the theoretical analysis.

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