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THE RELIABLE MODIFIED OF LAPLACE ADOMIAN DECOMPOSITION METHOD TO SOLVE NONLINEAR INTERVAL VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

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Abstract. In this paper, we propose a combined form for solving nonlinear interval Volterra-Fredholm integral equations of the second kind based on the modifying Laplace Adomian decomposition method. We find the exact solutions of nonlinear interval Volterra-Fredholm integral equations with less computation as compared with standard decomposition method. Finally, an illustrative example has been solved to show the efficiency of the proposed method.

1. Introduction

The topic of Volterra-Fredholm integral equations which have attracted growing interest in recent years. The Volterra-Fredholm integral equation appears from diverse biological and physical models. The primary features of these models are of wide usable [10]. There are many techniques both analytical and numerical approaches for solving nonlinear integral equations as decomposition method, variational iteration method, finite element method, homotopy perturbation method and homotopy analysis method, and its modification [1–3, 9, 10]. One of the efficient techniques for solving nonlinear Volterra-Fredholm integral

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equations is the decomposition method which was stated in [2, 10]. In this direction, Hussain and Khan [5] have used the modified Laplace decomposition method for giving the exact solutions of nonlinear partial or coupled partial differential equations, and provide the series solution of a Blasius flow equation [6]. Vinh et al. [7], studied the existence and uniqueness of solutions of the interval-valued Volterra integral equations. Also, Ahmed and Ghadle [4], carefully applied a reliable modification of Laplace Adomian decomposition method for solved the Volterra-Fredholm integro differential equations. Salahshour and Khan [8] have applied the modification of Laplace decomposition method to solve nonlinear interval Volterra integral equations of the form:

$$u(x) = f(x) + \mu \int_0^x k(x,t)u^n(t)dt.$$

But in this paper, we will study the modification of Laplace Adomian decomposition method to solve the nonlinear interval Volterra-Fredholm integral equation of the form:

$$u(x) = f(x) + \lambda \int_a^x k_1(x,t)u^n(t)dt + \mu \int_a^b k_2(x,t)u^m(t)dt,$$

Our aim in this work is to obtain the analytical solutions of the nonlinear interval Volterra-Fredholm integral equation by using the modified Laplace Adomian decomposition method. The remainder of the paper is organized as follows: In Section 2, some basic concepts about interval arithmetic are stated. In Section 3, a brief discussion of the nonlinear interval Volterra-Fredholm integral equations is introduced. In Section 4, the new method based on the modification of Laplace transform Adomian decomposition method is proposed. Section 5, contains illustrative example to demonstrate the accuracy and efficiency of the proposed method. Finally, we will give a report on our paper and a brief conclusion is given in Section 6.

2. Preliminaries

In this section, we state some basic concepts about interval computations [8]. Let $\mathbb{I}$ denote the family of all nonempty, compact and convex subsets of $\mathbb{R} \times \mathbb{R}$. If U and V are two intervals stated by

$$U = [U_1, U_2], \quad V = [V_1, V_2],$$

then,

$$U + V = [U_1 + V_1, U_2 + V_2],$$
$$\lambda U = [\lambda U_1, \lambda U_2], \quad \lambda \geq 0,$$

Note that the each $a \in \mathbb{R}$ can be stated as interval denoted by $[a, a]$. The Hausdorff metric $\mathcal{H}$ in $\mathbb{I}$ is stated by

$$\mathcal{H}(U, V) = \max\{|U_1 - V_1|, |U_2 - V_2|\}.$$

It is well-known that $(\mathbb{I}, \mathcal{H})$ is a complete, separable and locally compact metric space. Also, the following properties hold

$$\begin{aligned} \mathcal{H}(U + V, M + N) &\leq \mathcal{H}(U, M) + \mathcal{H}(V, N), \\ \mathcal{H}(\lambda U, \lambda V) &\leq |\lambda| \mathcal{H}(U, V), \end{aligned}$$

where U, V, M and N are intervals.

In this paper, we adopt the following operation for division

$$\frac{U}{V} = \left[\frac{U_1}{V_1}, \frac{U_2}{V_2} \right]. \tag{1}$$

Clearly, using this notation, $\frac{U}{V}$ is not always as interval. But, when we translate each interval system to two related real-valued systems, all these systems will solve distinctly. After obtaining solutions of each real-valued system, we finally check that the obtained solutions create an interval as output of original interval system or not. On the other hand, we should determine the domain the lower solution is less than or equal to upper solution for each independent argument of the solution. Our results demonstrate that using this king of division, some interesting results are derived. Note that, appearing such unusual computation is not new in the interval theory, for example, introducing the Hukhara difference and etc. Now, we state an interesting result using Eq.(1).

Theorem 2.1. *Let us consider the given interval U. Then, using the mentioned division formulation, we have $\frac{U}{U} = [1, 1]$.*

Proof. Using Eq.(1), and $U = [U_1, U_2]$ we have

$$\begin{aligned} \frac{U}{U} &= \frac{[U_1, U_2]}{[U_1, U_2]} \\ &= \left[\frac{U_1}{U_1}, \frac{U_2}{U_2} \right] \\ &= [1, 1] \end{aligned}$$

which completes the proof of the theorem. □

3. Interval Volterra-Fredholm Integral Equations

Let us consider the nonlinear interval Volterra-Fredholm integral equation of the form

$$u(x) = f(x) + \lambda \int_a^x k_1(x,t)u^n(t)dt + \mu \int_a^b k_2(x,t)u^m(t)dt, \tag{2}$$

where f is interval non-homogeneous term with lower-upper representation $f(x) = [f_1(x), f_2(x)]$, λ, μ are a parameters, $k_1(x,t)$ and $k_2(x,t)$ are the kernels of the equation belongs to $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, and u^n, u^m are the nonlinear interval terms, $n, m \in \mathbb{N}$, with $u^n = [u_1^n, u_2^n]$, ($\mathbb{R}, \mathbb{N}$ are the space of all real and natural numbers, respectively) and $x \in T = [a,b], a < b$. Moreover, we say that $\hat{u}$ is the solution of (2), if

$$\sup_{x \in T} \mathcal{H}\left(\hat{u}(x), f(x) + \lambda \int_a^x k_1(x,t)\hat{u}^n(t)dt + \mu \int_a^b k_2(x,t)\hat{u}^m(t)dt\right) = 0.$$

Now, we state a characterization theorem for Eq.(2).

Theorem 3.1. *The nonlinear interval Volterra-Fredholm integral equation* (2) *is equivalent to real-valued integral systems*

$$\begin{aligned} u_1(x) &= f_1(x) + \lambda \int_a^x k_1(x,t)(u^n)_1(t)dt + \mu \int_a^b k_2(x,t)(u^m)_1(t)dt, \\ u_2(x) &= f_2(x) + \lambda \int_a^x k_1(x,t)(u^n)_2(t)dt + \mu \int_a^b k_2(x,t)(u^m)_2(t)dt, \end{aligned}$$

where $n, m \in \mathbb{N}, x \in T$, $k_1(x,t)(u^n)_i(t)$ *and* $k_2(x,t)(u^m)_i(t), i = 1,2$ *are equicontinuous functions.*

Proof. It is straightforward. □

4. Description of the Method

Here, we state our proposed method to solve NIVIEs based on the two steps Laplace transform and Adomian decomposition method. Applying the Laplace transform $\mathcal{L}$ on the both sides of the equation yield

$$\mathcal{L}[u(x)] = \mathcal{L}[f(x)] + \mathcal{L}\left[\lambda \int_a^x k_1(x,t)u^n(t)dt + \mu \int_a^b k_2(x,t)u^m(t)dt\right], \tag{3}$$

The lower-upper representation (LU-representation) of Eq. (3) is

$$\begin{aligned}\mathcal{L}[u_1(x)] &= \mathcal{L}[f_1(x)] + \mathcal{L}\left[\lambda \int_a^x k_1(x,t)(u^n)_1(t)dt + \mu \int_a^b k_2(x,t)(u^m)_1(t)dt\right],\\ \mathcal{L}[u_2(x)] &= \mathcal{L}[f_2(x)] + \mathcal{L}\left[\lambda \int_a^x k_1(x,t)(u^n)_2(t)dt + \mu \int_a^b k_2(x,t)(u^m)_2(t)dt\right],\end{aligned}$$

Then, using the inverse Laplace transform leads to

$$\begin{aligned}u_1(x) &= f_1(x) + \mathcal{L}^{-1}\left[\mathcal{L}\left[\lambda \int_a^x k_1(x,t)(u^n)_1(t)dt + \mu \int_a^b k_2(x,t)(u^m)_1(t)dt\right]\right],\\ u_2(x) &= f_2(x) + \mathcal{L}^{-1}\left[\mathcal{L}\left[\lambda \int_a^x k_1(x,t)(u^n)_2(t)dt + \mu \int_a^b k_2(x,t)(u^m)_2(t)dt\right]\right],\end{aligned}$$

Using the assumption of LADM, let us consider the solution $u(x) = [u_1(x), u_2(x)]$ is expanded into infinite series as follows:

$$u_1(x) = \sum_{n=0}^{\infty}(u_n)_1, \quad u_2(x) = \sum_{n=0}^{\infty}(u_n)_2. \tag{4}$$

Also, the nonlinear term $u^n(x) = [(u^n)_1(x), (u^n)_2(x)]$ and $u^m(x) = [(u^m)_1(x), (u^m)_2(x)]$ where

$$\begin{aligned}(u^n)_1(x) &= \sum_{n=0}^{\infty}(A_n)_1(x),\\ (u^n)_2(x) &= \sum_{n=0}^{\infty}(A_n)_2(x),\end{aligned} \tag{5}$$

such that $(A_n)_1$ and $(A_n)_2$ are Adomian polynomials. Then, substituting Eqs.(5) and Eq. (4) in Eqs. (4) yield

$$\begin{aligned}\sum_{n=0}^{\infty}(u_n)_1 = f_1(x) + \mathcal{L}^{-1}\Bigg[\mathcal{L}\Bigg[\lambda \int_a^x k_1(x,t)\sum_{n=0}^{\infty}(A_n)_1(t)dt\\ + \mu \int_a^b k_2(x,t)\sum_{n=0}^{\infty}((u_n)_1)^m(t)dt\Bigg]\Bigg],\\ \sum_{n=0}^{\infty}(u_n)_2 = f_2(x) + \mathcal{L}^{-1}\Bigg[\mathcal{L}\Bigg[\lambda \int_a^x k_1(x,t)\sum_{n=0}^{\infty}(A_n)_2(t)dt\\ + \mu \int_a^b k_2(x,t)\sum_{n=0}^{\infty}((u_n)_2)^m(t)dt\Bigg]\Bigg],\end{aligned}$$

By using the following Adomian polynomials

$$(A_n)_1 = \frac{1}{n!}[\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty}\lambda^i y_i)_1]]_{\lambda=0}.$$

$$(A_n)_2 = \frac{1}{n!}[\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty}\lambda^i y_i)_2]]_{\lambda=0}.$$

we derive the recursive relation given by

$$(u_0)_1 = f_1(x),$$

$$(u_{n+1})_1 = \mathcal{L}^{-1}\left[\mathcal{L}\left[\lambda\int_a^x k_1(x,t)(A_n)_1(t)dt + \mu\int_a^b k_2(x,t)((u_n)_1)^m(t)dt\right]\right], \quad n \geq 0,$$

and

$$(u_0)_2 = f_2(x),$$

$$(u_{n+1})_2 = \mathcal{L}^{-1}\left[\mathcal{L}\left[\lambda\int_a^x k_1(x,t)(A_n)_2(t)dt + \mu\int_a^b k_2(x,t)((u_n)_2)^m(t)dt\right]\right], \quad n \geq 0,$$

the LU-representation of solution will be determined.

5. Illustrative Example

In this section, we apply our proposed method to obtain the exact solution of the nonlinear interval Volterra-Fredholm integral equation. Indeed, this modification for Laplace transform Adomian decomposition method is about some ideas to choose the initial values.

Example 5.1. Let us consider the nonlinear interval Volterra-Fredholm integral equation

$$(6)\quad u(x) = \hat{c}(2x - \frac{x^4}{12} - \frac{5}{3}) + \frac{1}{4\hat{c}}\int_0^x (x-t)u^2(t)dt + \int_0^1 (1+t)u(t)dt,$$

where $\hat{c} = [\hat{c}_1, \hat{c}_2]$, $\hat{c}_1 > 0$. The LU-representation of (6) is as follows:

$$\begin{aligned} u_1(x) &= \hat{c}_1(2x - \frac{x^4}{12} - \frac{5}{3}) + \frac{1}{4\hat{c}_1}\int_0^x (x-t)(u^2)_1(t)dt + \int_0^1 (1+t)u_1(t)dt, \\ u_2(x) &= \hat{c}_2(2x - \frac{x^4}{12} - \frac{5}{3}) + \frac{1}{4\hat{c}_2}\int_0^x (x-t)(u^2)_2(t)dt + \int_0^1 (1+t)u_2(t)dt, \end{aligned}$$

Applying Laplace transform yields

$$\begin{aligned} \mathcal{L}[u_1(x)] &= \mathcal{L}\left[\hat{c}_1(2x - \frac{x^4}{12} - \frac{5}{3}) + \frac{1}{4\hat{c}_1}\int_0^x (x-t)(u^2)_1(t)dt + \int_0^1 (1+t)u_1(t)dt\right], \\ \mathcal{L}[u_2(x)] &= \mathcal{L}\left[\hat{c}_2(2x - \frac{x^4}{12} - \frac{5}{3}) + \frac{1}{4\hat{c}_2}\int_0^x (x-t)(u^2)_2(t)dt + \int_0^1 (1+t)u_2(t)dt\right], \end{aligned}$$

Then, applying $\mathcal{L}^{-1}$ and some simplification yield

$$\begin{aligned} u_1(x) = \hat{c}_1(2x - \frac{x^4}{12} - \frac{5}{3}) + \mathcal{L}^{-1}\Bigg[\mathcal{L}\Bigg[\frac{1}{4\hat{c}_1}\int_0^x (x-t)(u^2)_1(t)dt \\ + \int_0^1 (1+t)u_1(t)dt\Bigg]\Bigg], \\ u_2(x) = \hat{c}_2(2x - \frac{x^4}{12} - \frac{5}{3}) + \mathcal{L}^{-1}\Bigg[\mathcal{L}\Bigg[\frac{1}{4\hat{c}_2}\int_0^x (x-t)(u^2)_2(t)dt \\ + \int_0^1 (1+t)u_2(t)dt\Bigg]\Bigg], \end{aligned}$$

Using the fact

$$u_1(x) = \sum_{n=0}^{\infty}(u_n)_1(x), \quad u_2(x) = \sum_{n=0}^{\infty}(u_n)_2(x),$$

we get:

$$\begin{aligned} \sum_{n=0}^{\infty}(u_n)_1(x) = \hat{c}_1(2x - \frac{x^4}{12} - \frac{5}{3}) + \mathcal{L}^{-1}\Bigg[\mathcal{L}\Bigg[\frac{1}{4\hat{c}_1}\int_0^x (x-t)(u^2)_1(t)dt \\ + \int_0^1 (1+t)u_1(t)dt\Bigg]\Bigg], \\ \sum_{n=0}^{\infty}(u_n)_2(x) = \hat{c}_2(2x - \frac{x^4}{12} - \frac{5}{3}) + \mathcal{L}^{-1}\Bigg[\mathcal{L}\Bigg[\frac{1}{4\hat{c}_2}\int_0^x (x-t)(u^2)_2(t)dt \\ + \int_0^1 (1+t)u_2(t)dt\Bigg]\Bigg], \end{aligned}$$

By setting

$$\sum_{n=0}^{\infty}(A_n)_1(x) = (u^2)_1(x),$$
$$\sum_{n=0}^{\infty}(A_n)_2(x) = (u^2)_2(x),$$

As a consequence, some components of the Adomian polynomials are given by:

$$\begin{aligned}
(A_0)_1(t) &= (u_0^2)_1(t),\\
(A_1)_1(t) &= 2(u_0)_1(t)(u_1)_1(t),\\
&\;\;\vdots\\
(A_n)_1(t) &= \sum_{i=0}^{\infty}(u_{n-i})_1(t)(u_i)_1(t),
\end{aligned}$$

and

$$\begin{aligned}
(A_0)_2(t) &= (u_0^2)_2(t),\\
(A_1)_2(t) &= 2(u_0)_2(t)(u_1)_2(t),\\
&\;\;\vdots\\
(A_n)_2(t) &= \sum_{i=0}^{\infty}(u_{n-i})_2(t)(u_i)_2(t),
\end{aligned}$$

So, we obtain

$$\begin{aligned}
(u_0)_1(x) &= \hat{c}_1(2x - \frac{x^4}{12} - \frac{5}{3})\\
(u_{n+1})_1(x) &= \mathcal{L}^{-1}\left[\mathcal{L}\left[\frac{1}{4\hat{c}_1}\int_0^x (x-t)(A_n)_1(t)dt + \int_0^1 (1+t)(u_n)_1(t)dt\right]\right],\\
&\quad n \geq 0,
\end{aligned}$$

and

$$\begin{aligned}(u_0)_2(x) &= \hat{c}_2(2x - \frac{x^4}{12} - \frac{5}{3})\\ (u_{n+1})_2(x) &= \mathcal{L}^{-1}\left[\mathcal{L}\left[\frac{1}{4\hat{c}_2}\int_0^x (x-t)(A_n)_2(t)dt + \int_0^1 (1+t)(u_n)_2(t)dt\right]\right],\\ & n \geq 0,\end{aligned}$$

Consequently, some of the first few components of u_n are given:

$$\begin{aligned}u_1(x) = \mathcal{L}^{-1}\Bigg[\mathcal{L}\Bigg[\frac{1}{4\hat{c}_1}\int_0^x (x-t)(\hat{c}_1(2t - \frac{t^4}{12} - \frac{5}{3}))^2 dt\\ + \int_0^1 (1+t)(\hat{c}_1(2x - \frac{x^4}{12} - \frac{5}{3}))dt\Bigg]\Bigg],\\ = \hat{c}_1(-\frac{311}{360} + \frac{25}{72}x^2 - \frac{5}{18}x^3 + \frac{1}{12}x^4 + ...).\end{aligned}$$

$$\begin{aligned}u_2(x) = \mathcal{L}^{-1}\Bigg[\mathcal{L}\Bigg[\frac{1}{4\hat{c}_2}\int_0^x (x-t)(\hat{c}_2(2t - \frac{t^4}{12} - \frac{5}{3}))^2 dt\\ + \int_0^1 (1+t)(\hat{c}_2(2x - \frac{x^4}{12} - \frac{5}{3}))dt\Bigg]\Bigg],\\ = \hat{c}_2(-\frac{311}{360} + \frac{25}{72}x^2 - \frac{5}{18}x^3 + \frac{1}{12}x^4 + ...).\end{aligned}$$

Now, we apply our new proposed approach. Let

$$\begin{aligned}f_1(x) &= f_1^0(x) + f_1^1(x),\\ f_2(x) &= f_2^0(x) + f_2^1(x),\end{aligned}$$

where

$$\begin{aligned}f_1^0(x) &= \hat{c}_1(2x),\\ f_1^1(x) &= \hat{c}_1(-\frac{x^4}{12} - \frac{5}{3})\\ f_2^0(x) &= \hat{c}_2(2x)\\ f_2^1(x) &= \hat{c}_2(-\frac{x^4}{12} - \frac{5}{3}),\end{aligned}$$

Since f_1^1 and f_2^1 do not satisfy Eq.(6), choosing

$$(u_0)_1(x) = f_1^0(x),$$

and

$$(u_0)_2(x) = f_2^0(x),$$

the exact solution is derived as follows:

$$\begin{aligned}
(u_0)_1(x) &= \hat{c}_1(2x),\\
(u_1)_1(x) &= \hat{c}_1(-\frac{x^4}{12}-\frac{5}{3})+\mathcal{L}^{-1}\Big[\mathcal{L}\Big[\frac{1}{4\hat{c}_1}\int_0^x (x-t)(A_0)_1(t)dt\\
&\qquad + \int_0^1 (1+t)(u_0)_1(t)dt\Big]\Big]=0\\
&\vdots\\
(u_{n+1})_1(x) &= 0, \quad n\geq 1,
\end{aligned}$$

and

$$\begin{aligned}
(u_0)_2(x) &= \hat{c}_2(2x),\\
(u_1)_2(x) &= \hat{c}_2(-\frac{x^4}{12}-\frac{5}{3})+\mathcal{L}^{-1}\Big[\mathcal{L}\Big[\frac{1}{4\hat{c}_2}\int_0^x (x-t)(A_0)_2(t)dt\\
&\qquad + \int_0^1 (1+t)(u_0)_2(t)dt\Big]\Big]=0\\
&\vdots\\
(u_{n+1})_2(x) &= 0, \quad n\geq 1.
\end{aligned}$$

Hence, the LU-representation of solution is obtained as follows:

$$u_1(x) = \sum_{n=0}^{\infty}(u_n)_1(x) = \hat{c}_1(2x),$$

$$u_2(x) = \sum_{n=0}^{\infty}(u_n)_2(x) = \hat{c}_2(2x),$$

or in the closed form, we obtain:

$$u(x) = \sum_{n=0}^{\infty}(u_n)(x) = \hat{c}(2x). \tag{7}$$

Indeed, in this case, we obtain asymptotic solution, i.e., it is easy to verify that for each $x \in \mathbb{R}$, u stated in Eq.(7) provides an interval-valued function. Also,

$$\mathcal{H}\Bigg(\hat{c}(2x), \hat{c}(2x) + \hat{c}(-\frac{x^4}{12} - \frac{5}{3}) + \frac{1}{4\hat{c}} \int_0^x (x-t)(\hat{c}(2t))^2(t)dt + \int_0^1 (1+t)(\hat{c}(2t))(t)dt\Bigg) = 0.$$

6. Conclusion

In this paper, we carefully applied a reliable modification of Laplace Adomian decomposition method to solve the nonlinear interval Volterra-Fredholm integral equations of the second kind (NIVFIEs). The main advantage of this method is the fact that it gives the analytical solution. Also, this method is combining two powerful methods for obtaining exact solutions of (NIVFIEs). However, in order to convert the original problem to some related deterministic equations, we have used some new changes in the interval arithmetic to achieve the correct extension of the original problem in the interval framework. At the end, our results show the enough efficiency of the proposed approach.

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