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A NOTE ON q-ANALOGUE OF POLY-BERNOULLI NUMBERS AND POLYNOMIALS†

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ABSTRACT. In this paper, we define a q-analogue of the poly-Bernoulli numbers and polynomials which is generalization of the poly Bernoulli numbers and polynomials including q-polylogarithm function. We also give the relations between generalized poly-Bernoulli polynomials. We derive some relations that are connected with the Stirling numbers of second kind. By using special functions, we investigate some symmetric identities involving q-poly-Bernoulli polynomials.

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1. Introduction

Many mathematicians are interested in the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials and their applications. They possess many interesting properties and are treated in many areas of mathematics and physics. Due to these reasons, many applications of Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials have been studied, and recently various analogues for the above numbers and polynomials was introduced(see [1-14]).

In this paper, we use the following notations. $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ denotes the set of nonnegative integer, $\mathbb{Z}$ denotes the set of integers, and $\mathbb{C}$ denotes the set of complex numbers, respectively.

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The ordinary Bernoulli polynomials $B_n(x)$ are given by the generating functions:

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad (see[1,2,6]). \tag{1.1}$$

When $x=0, B_{n,q}^{(k)}=B_{n,q}^{(k)}(0)$ are called poly-Bernoulli numbers.

The polylogarithm function Li_k is defined by

$$Li_k(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^k},\quad (k\in\mathbb{Z})(see[1,2,3,6,7,8,14]) \tag{1.2}$$

For $k\leq 1$, the polylogarithm functions are as follows

$$Li_1(x)=-log(1-x),\quad Li_0(x)=\frac{x}{1-x},\quad Li_{-1}(x)=\frac{x}{(1-x)^2},\quad \cdots .$$

By using polylogarithm function, Kaneko defined a sequence of rational numbers, which is refered to as poly-Bernoulli numbers,

In [3] and [13], the k-th q-analogue of polylogarithm function $Li_{k,q}$ is introduced by

$$Li_{k,q}(x)=\sum_{n=1}^{\infty}\frac{x^n}{[n]_q^k},\quad (k\in\mathbb{Z}). \tag{1.3}$$

The q-analogue of polylogarithm function for $k\leq 1$ is represented by a rational function,

$$Li_{k,q}(x)=\frac{1}{(1-q)^k}\sum_{l=0}^{k}(-1)^l\binom{k}{l}\frac{q^lx}{1-q^lx}.$$

In [4] and [12], the Stirling number of the first kind is given by

$$(x)_n=\sum_{m=0}^{n}S_1(n,m)x^m,(n\geq 0)$$

and

$$\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!}=\frac{(log(1+t))^m}{m!}$$

where

$$(x)_n=x(x-1)(x-2)\cdots(x-n+1)=\Pi_{k=1}^{n}(x-(k-1)) \tag{1.4}$$

is falling factorial. The Stirling numbers of the second kind is defined by

$$\sum_{n=m}^{\infty}S_2(n,m)\frac{t^n}{n!}=\frac{(e^t-1)^m}{m!}. \tag{1.5}$$

In this paper, we consider a q-analogue of the poly Bernoulli polynomials containing Equation(1.3). We also find some relations between q-poly-Bernoulli polynomials and ordinary Bernoulli polynomials. And we derive several properties that are connected with the Stirling numbers of the second kind. Finally, we find some symmetric identities of the q-degenerate poly Bernoulli polynomials by using special functions.

2. A q-analogue of the poly-Bernoulli polynomials

In this section, we define a q-analogue of poly-Bernoulli numbers $B_{n,q}^{(k)}$ and polynomials $B_{n,q}^{(k)}(x)$ by the generating functions. From the definition, we get some identities that is similar to the ordinary Bernoulli polynomials.

Definition 2.1. For $n \geq 0, n, k \in \mathbb{Z}, 0 \leq p < 1$, we introduce a q-analogue of poly-Bernoulli polynomials by:

$$\frac{Li_{k,q}(1-e^{-t})}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x) \frac{t^n}{n!} \tag{2.1}$$

where

$$Li_{k,q}(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q^k}$$

is the k-th q-polylogarithm function.
When $x = 0$, $B_{n,q}^{(k)} = B_{n,q}^{(k)}(0)$ are called a q-analogue of poly-Bernoulli numbers. Note that $lim_{q \to 1}[n]_q = n$, and $lim_{q \to 1} B_{n,q}^{(k)}(x) = B_n^{(k)}(x)$.

From Equation(2.1), we have the following relation between q-poly-Bernoulli numbers and q-poly-polynomials.

Theorem 2.2. *Let $n \geq 0, n, k \in \mathbb{Z}, 0 < p < 1$. We have*

$$B_{n,q}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{l,q}^{(k)} x^{n-l}. \tag{2.2}$$

Proof. For $n \geq 0, n, k \in \mathbb{Z}, 0 < p < 1$, we easily get:

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x) \frac{t^n}{n!} &= \frac{Li_{k,q}(1-e^{-t})}{e^t - 1} e^{xt} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} B_{l,q}^{(k)} x^{n-l} \right) \frac{t^n}{n!}. \end{aligned}$$

Therefore, we have

$$B_{n,q}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{l,q}^{(k)} x^{n-l}.$$

□

By Equation(2.2), we obtain an addition theorem.

Theorem 2.3. *For $n \geq 0, n, k \in \mathbb{Z}, 0 < p < 1$, we have*

$$B_{n,q}^{(k)}(x+y) = \sum_{l=0}^{n} \binom{n}{l} B_{l,q}^{(k)}(x) y^{n-l}.$$

Proof. Let $n \geq 0, n, k \in \mathbb{Z}$. Then we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,q}^{(k)}(x+y)\frac{t^n}{n!} &= \frac{Li_{k,q}(1-e^{-t})}{e^t-1}e^{(x+y)t} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}B_{l,q}^{(k)}(x)y^{n-l}\right)\frac{t^n}{n!}.\end{aligned}$$

Thus, we get

$$B_{n,q}^{(k)}(x+y) = \sum_{l=0}^{n}\binom{n}{l}B_{l,q}^{(k)}(x)y^{n-l}.$$

□

By using the definition of the q-analogue of polylogarithm function $Li_{k,q}$ in Equation(1.3), we have next relation which is connected with the ordinary Bernoulli polynomials.

Theorem 2.4. *Let $n \geq 0, n, k \in \mathbb{Z}, 0 \leq p < 1$. We obtain*

$$B_{n,q}^{(k)}(x) = \sum_{l=0}^{\infty}\frac{1}{[l+1]_q^k}\sum_{a=0}^{l+1}\binom{l+1}{a}(-1)^a B_n(x-a).$$

Proof. For $n \geq 0, n, k \in \mathbb{Z}, 0 \leq p < 1$,

$$\begin{aligned}\frac{Li_{k,q}(1-e^{-t})}{e^t-1}e^{xt} &= \sum_{l=1}^{\infty}\frac{(1-e^{-t})^l}{[l]_q^k}\frac{e^{xt}}{e^t-1} \\ &= \sum_{n=0}^{\infty}\frac{1}{[l+1]_q^k}\sum_{a=0}^{l+1}\binom{l+1}{a}(-1)^a\frac{e^{(x-a)t}}{e^t-1} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\frac{1}{[l+1]_q^k}\sum_{a=0}^{l+1}\binom{l+1}{a}(-1)^a\frac{B_{n+1}(x-a)}{n+1}\right)\frac{t^n}{n!}\end{aligned}$$

Comparing the coefficient of the result, we easily get next equation:

$$B_{n,q}^{(k)}(x) = \sum_{l=0}^{\infty}\frac{1}{[l+1]_q^k}\sum_{a=0}^{l+1}\binom{l+1}{a}(-1)^a\frac{B_{n+1}(x-a)}{n+1}.$$

□

From the binomials series and the Equation(1.3), we derive the following Theorem.

Theorem 2.5. *If $n, k \in \mathbb{Z}, n \geq 0$and $0 \leq p < 1$, then we have*

$$B_{n,q}^{(k)}(x) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{a=0}^{m+1}\binom{m+1}{a}\frac{(-1)^a(l-m-a+x)^n}{[m+1]_q^n}.$$

Proof. Let $n, k \in \mathbb{Z}, n \geq 0$ and $0 \leq p < 1$. Using the Equation(1.3), we have

$$\begin{aligned}
\frac{Li_{k,q}(1-e^{-t})}{e^t-1}e^{xt} &= \left(\sum_{m=0}^{\infty} e^{mt}\right)\left(\sum_{l=0}^{\infty}\frac{(1-e^{-t})^{l+1}}{[l+1]_q^k}\right)e^{xt} \\
&= \left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{e^{(l-m)t}}{[m+1]_q^k}\right)\left(\sum_{a=0}^{m+1}\binom{m+1}{a}(-1)^a e^{(x-a)t}\right) \\
&= \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{a=0}^{m+1}\binom{m+1}{a}\frac{(-1)^a e^{(l-m-a+x)t}}{[m+1]_q^k} \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{a=0}^{m+1}\binom{m+1}{a}\frac{(-1)^a (l-m-a+x)^n}{[m+1]_q^k}\frac{t^n}{n!}
\end{aligned}$$

Hence, we have

$$B_{n,q}^{(k)}(x) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{a=0}^{m+1}\binom{m+1}{a}\frac{(-1)^a (l-m-a+x)^n}{[m+1]_q^n}.$$

□

3. Some relation involving the Stirling numbers of the second kind

In this section, using well-known Stirling numbers, we find several identities of the q-poly-Bernoulli polynomials. Note that the Stirling numbers of the second kind is defined

$$\frac{(e^t-1)^m}{m!} = \sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!}(see[4,12,13,14]). \tag{3.1}$$

The q-analogue of polylogarithm function $Li_{k,q}$, in the Equation(1.3), is represented as below

$$\begin{aligned}
Li_{k,q}(1-e^{-t}) &= \sum_{l=1}^{\infty}\frac{(1-e^{-t})^l}{[l]_q^k} \\
&= \sum_{n=1}^{\infty}(-1)^l\frac{(e^{-t}-1)^l}{[l]_q^k} \\
&= \sum_{n=1}^{\infty}\sum_{l=1}^{n}\frac{(-1)^{l+n}}{[l]_q^k}l!S_2(n,l)\frac{t^n}{n!}
\end{aligned}$$

Therefore, we obtain

$$\frac{1}{t}Li_{k,q}(1-e^{-t}) = \sum_{n=0}^{\infty}\sum_{l=1}^{n+1}\frac{(-1)^{l+n+1}}{[l]_q^k}l!\frac{S_2(n+1,l)}{n+1}\frac{t^n}{n!}. \tag{3.2}$$

Using the Equation(3.2), we get next theorem.

Theorem 3.1. *For $n, k \in \mathbb{Z}, n \geq 0$, we have*

$$B_{n,q}^{(k)}(x) = \sum_{a=0}^{n} \binom{n}{a} \sum_{l=1}^{a+1} \frac{(-1)^{l+a+1} l! S_2(a+1,l)}{[l]_q^k (a+1)} B_{n-a}(x).$$

Proof. Let $n, k \in \mathbb{Z}, n \geq 0$. By the recomposition of q-polylogarithm function in Equation(3.2), the relation between the q-poly-Bernoulli polynomials and the ordinary Bernoulli polynomials is derived:

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,q}^{(k)}(x) \frac{t^n}{n!} &= \frac{Li_{k,q}(1-e^{-t})}{e^t - 1} e^{xt} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=1}^{n+1} \frac{(-1)^{l+n+1}}{[l]_q^k} l! \frac{S_2(n+1,l)}{n+1} \right) \frac{t^n}{n!} \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{a=0}^{n} \binom{n}{a} \sum_{l=1}^{a+1} \frac{(-1)^{l+a+1} l! S_2(a+1,l)}{[l]_q^k (a+1)} B_{n-a}(x) \frac{t^n}{n!}.
\end{aligned}$$

Thus, we have

$$B_{n,q}^{(k)}(x) = \sum_{a=0}^{n} \binom{n}{a} \sum_{l=1}^{a+1} \frac{(-1)^{l+a+1} l! S_2(a+1,l)}{[l]_q^k (a+1)} B_{n-a}(x).$$

□

By the Equation(3.2), we have the following result which is connected with q-poly-Bernoulli numbers.

Theorem 3.2. *For $n \geq 0, n, k \in \mathbb{Z}$, we have*

$$B_{n,q}^{(k)}(x) = \sum_{l=0}^{\infty} \sum_{a=l}^{n} \binom{n}{a} (x)_l S_2(a,l) B_{n-a,q}^{(k)}.$$

Proof. Let $n \geq 0, n, k \in \mathbb{Z}$. We obtain a relation between q-poly-Bernoulli numbers and q-poly-Bernoulli polynomials by the Equation(3.1):

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,q}^{(k)}(x) \frac{t^n}{n!} &= \frac{Li_{k,q}(1-e^{-t})}{e^t - 1} e^{xt} \\
&= \frac{Li_{k,q}(1-e^{-t})}{e^t - 1} \sum_{l=0}^{\infty} (x)_l \frac{(e^t-1)^l}{l!} \\
&= \sum_{n=0}^{\infty} B_{n,q}^{(k)} \frac{t^n}{n!} \sum_{l=0}^{\infty} (x)_l \sum_{a=0}^{\infty} S_2(a,l) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{\infty} \sum_{a=l}^{n} \binom{n}{a} (x)_l S_2(a,l) B_{n-a,q}^{(k)} \right) \frac{t^n}{n!}.
\end{aligned}$$

Thus, we have

$$B_{n,q}^{(k)}(x) = \sum_{l=0}^{\infty}\sum_{a=l}^{n}\binom{n}{a}(x)_l S_2(a,l) B_{n-a,q}^{(k)}.$$

where $(x)_l = x(x-1)(x-2)\cdots(x-l+1)$ is falling factorial. □

From Definition 2.1, we have the recurrence formula with the Stirling numbers of the second kind.

Theorem 3.3. *For $n \geq 1, n, k \in \mathbb{Z}$. and $0 < p < 1$, we get*

$$\begin{aligned} &B_{n,q}^{(k)}(x+1) - B_{n,q}^{(k)}(x) \\ &= \sum_{r=1}^{n}\binom{n}{r}\left(\sum_{l=0}^{r-1}\frac{(-1)^{l+1+r}}{[l+1]_q^k}(l+1)!S_2(r,l+1)\right)x^{n-r}. \end{aligned}$$

Proof. Let $n \geq 0, n, k \in \mathbb{Z}, 0 < p < 1$. Using the definition of the q-poly-Bernoulli polynomials, we have a equation as follows:

$$\begin{aligned} &\sum_{n=0}^{\infty} B_{n,q}^{(k)}(x+1)\frac{t^n}{n!} - \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x)\frac{t^n}{n!} \\ &= \frac{Li_{k,q}(1-e^{-t})}{e^t-1}e^{(x+1)t} - \frac{Li_{k,q}(1-e^{-t})}{e^t-1}e^{xt} \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n-1}\frac{(-1)^{n+l+1}}{[l+1]_q^k}(l+1)!S_2(n,l+1)\frac{t^n}{n!}\sum_{m=0}^{\infty}x^m\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\sum_{r=1}^{n}\binom{n}{r}\left(\sum_{l=0}^{r-1}\frac{(-1)^{r+l+1}}{[l+1]_q^k}(l+1)!S_2(r,l+1)\right)x^{n-r}\frac{t^n}{n!}. \end{aligned}$$

Therefore, the equation is appeared by

$$\begin{aligned} &B_{n,q}^{(k)}(x+1) - B_{n,q}^{(k)}(x) \\ &= \sum_{r=1}^{n}\binom{n}{r}\left(\sum_{l=0}^{r-1}\frac{(-1)^{r+l+1}}{[l+1]_q^k}(l+1)!S_2(r,l+1)\right)x^{n-r}. \end{aligned}$$

□

In [4], Carlitz defined the weighted Stirling numbers of the second kind as follows

$$\frac{(e^t-1)^m}{m!}e^{xt} = \sum_{n=m}^{\infty} S_2(n,m,x)\frac{t^n}{n!}. \tag{3.3}$$

By using the Equation(3.3), we get the next result.

Theorem 3.4. *For $n \geq 1, k \in \mathbb{Z}, 0 < p < 1$, we have*

$$B_{n,q}^{(k)}(x) = \sum_{m=0}^{n}\frac{(-1)^{m+n}m!}{[m+1]_q^k}S_2(n,m,x).$$

Proof. Let $n \geq 1, k \in \mathbb{Z}, 0 < p < 1$.

$$\begin{aligned}\frac{Li_{k,q}(1-e^{-t})}{e^t-1}e^{xt} &= \sum_{m=0}^{\infty}\frac{(1-e^{-t})^m}{[m+1]_q^k}e^{xt}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^{m+n}m!}{[m+1]_q^k}\sum_{n=m}^{\infty}S_2(n,m,x)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{(-1)^{m+n}m!}{[m+1]_q^k}S_2(n,m,x)\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, the formula is expressed as below:

$$B_{n,q}^{(k)}(x) = \sum_{m=0}^{n}\frac{(-1)^{m+n}m!}{[m+1]_q^k}S_2(n,m,x)$$

□

4. Symmetric identities for the q-analogue of the poly-Bernoulli polynomials

In this section, we investigate several symmetric identities for the q-poly-Bernoulli polynomials by given special functions.

Theorem 4.1. *For $n \geq 0, n, k \in \mathbb{Z}$, $a, b > 0 (a \neq b)$, the following identity is obtained:*

$$\begin{aligned}&\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^m B_{m,q}^{(k)}(ax)B_{n-m,q}^{(k)}(bx)\\ &\qquad = \sum_{m=0}^{n}\binom{n}{m}a^m b^{n-m}B_{n-m,q}^{(k)}(ax)B_{m,q}^{(k)}(bx).\end{aligned}$$

Proof. Let $n, k \in \mathbb{Z}, n \geq 0$ and $a, b > 0 (a \neq b)$. If we start with the function,

$$F(t) = \frac{Li_{k,q}(1-e^{-at})Li_{k,q}(1-e^{-bt})}{(e^{at}-1)(e^{bt}-1)}e^{2abxt}, \tag{4.1}$$

then the Equation(4.1) is written by

$$\begin{aligned}F(t) &= \sum_{n=0}^{\infty}B_{n,q}^{(k)}(bx)\frac{(at)^n}{n!}\sum_{m=0}^{\infty}B_{m,q}^{(k)}(ax)\frac{(bt)^m}{m!}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^m B_{m,q}^{(k)}(ax)B_{n-m,q}^{(k)}(bx)\frac{t^n}{n!}.\end{aligned} \tag{4.2}$$

In similarly, we can see that

$$F(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}a^m b^{n-m}B_{n-m,q}^{(k)}(ax)B_{m,q}^{(k)}(bx)\frac{t^n}{n!}. \tag{4.3}$$

From Equation(4.2) and (4.3), it is clear to Theorem 4.1. □

By substituting $b = 1$, we easily get next corollary.

Corollary 4.2. *For $a > 0$ anf $n \geq 0, n \in \mathbb{Z}$, we have*

$$\sum_{m=0}^{n} \binom{n}{m} a^{n-m} B_{m,q}^{(k)}(ax) B_{n-m,q}^{(k)}(x) = \sum_{m=0}^{n} \binom{n}{m} a^{m} B_{n-m,q}^{(k)}(ax) B_{m,q}^{(k)}(x).$$

Note that $S_m(a) = \sum_{k=0}^{a} k^m$ is a power sum polynomials(cf [15]). We consider the following exponential generating function with indeterminate t,

$$\frac{e^{(a+1)t} - 1}{e^t - 1} = \sum_{m=0}^{\infty} S_m(a) \frac{t^m}{m!}. \tag{4.4}$$

Using the generating function, we get a symmetric relation of the q-poly-Bernoulli polynomials.

Theorem 4.3. *For $n \in \mathbb{Z}$, $n \geq 0$, $a, b > 0$ and $a \neq b$, we have*

$$\begin{aligned}\sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^{m-1} B_m(ax) S_{n-m}(b-1) \\ = \sum_{m=0}^{n} \binom{n}{m} a^{m-1} b^{n-m} B_m(bx) S_{n-m}(a-1)\end{aligned}$$

Proof. Let $n \in \mathbb{Z}$, $n \geq 0$, $a, b > 0$ and $a \neq b$.

We consider the generating function:

$$F(t) = \frac{Li_{k,q}(1 - e^{-at}) Li_{k,q}(1 - e^{-bt})(e^{abt} - 1)(e^{abxt})t}{(e^{at} - 1)^2 (e^{bt} - 1)^2}. \tag{4.5}$$

The Equation(4.5) follows as below

$$\begin{aligned} F(t) &= \frac{Li_{k,q}(1 - e^{-at}) Li_{k,q}(1 - e^{-bt})(e^{abt} - 1)(e^{abxt})t}{(e^{at} - 1)^2 (e^{bt} - 1)^2} \\ &= \sum_{n=0}^{\infty} B_{n,q}^{(k)} \frac{(at)^n}{n!} \sum_{n=0}^{\infty} B_{n,q}^{(k)} \frac{(bt)^n}{n!} \sum_{m=0}^{\infty} S_m(b-1) \frac{(at)^m}{m!} b^{-1} \sum_{n=0}^{\infty} B_n(ax) \frac{(bt)^n}{n!} \\ &= \sum_{n=0}^{\infty} B_{n,q}^{(k)} \frac{(at)^n}{n!} \sum_{n=0}^{\infty} B_{n,q}^{(k)} \frac{(bt)^n}{n!} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^{m-1} B_m(ax) S_{n-m}(b-1) \frac{t^n}{n!}. \end{aligned}$$

In similar method, the Equation(4.5) is written by

$$\begin{aligned} F(t) &= \sum_{n=0}^{\infty} B_{n,q}^{(k)} \frac{(at)^n}{n!} \sum_{n=0}^{\infty} B_{n,q}^{(k)} \frac{(bt)^n}{n!} \sum_{m=0}^{\infty} S_m(a-1) \frac{(bt)^m}{m!} a^{-1} \sum_{n=0}^{\infty} B_n(bx) \frac{(at)^n}{n!} \\ &= \sum_{n=0}^{\infty} B_{n,q}^{(k)} \frac{(at)^n}{n!} \sum_{n=0}^{\infty} B_{n,q}^{(k)} \frac{(bt)^n}{n!} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} a^{m-1} b^{n-m} B_m(bx) S_{n-m}(a-1) \frac{t^n}{n!}. \end{aligned}$$

Now, comparing the coefficient of t^n, then it gives the symmetric identity,

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^{m-1}B_m(ax)S_{n-m}(b-1)=\sum_{m=0}^{n}\binom{n}{m}a^{m-1}b^{n-m}B_m(bx)S_{n-m}(a-1).$$

□

Theorem 4.4. *For $n\in\mathbb{Z}$, $n\geq 0$, $a,b>0$ $(a\neq b)$, we have*

$$\begin{aligned}&Li_{k,q}(1-e^{-bt})\sum_{m=0}^{n}\binom{n}{m}a^{m}b^{n-m}B_{m,q}^{(k)}(bx)S_{n-m}(a-1)\\&\qquad=Li_{k,q}(1-e^{-at})\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^{m}B_{m,q}^{(k)}(ax)S_{n-m}(b-1)\end{aligned}$$

Proof. Let $n\in\mathbb{Z}$, $n\geq 0$, $a,b>0$ and $a\neq b$.

We consider a function that is given below:

$$F(t)=\frac{Li_{k,q}(1-e^{-at})Li_{k,q}(1-e^{-bt})(e^{abt}-1)(e^{abxt})}{(e^{at}-1)(e^{bt}-1)}. \tag{4.6}$$

The Equation(4.6) follows:

$$\begin{aligned}F(t)&=\frac{Li_{k,q}(1-e^{-at})Li_{k,q}(1-e^{-bt})(e^{abt}-1)(e^{abxt})}{(e^{at}-1)(e^{bt}-1)}\\&=Li_{k,q}(1-e^{-bt})\sum_{n=0}^{\infty}B_{n,q}^{(k)}(bx)\frac{(at)^n}{n!}\sum_{m=0}^{\infty}S_n(a-1)\frac{(bt)^n}{n!}\\&=\sum_{n=0}^{\infty}Li_{k,q}(1-e^{-bt})\sum_{m=0}^{n}\binom{n}{m}a^{m}b^{n-m}B_{m,q}^{(k)}(bx)S_{n-m}(a-1)\frac{t^n}{n!}.\end{aligned}$$

In similar method, we have

$$\begin{aligned}F(t)&=Li_{k,q}(1-e^{-at})\sum_{n=0}^{\infty}B_{n,q}^{(k)}(ax)\frac{(bt)^n}{n!}\sum_{n=0}^{\infty}S_n(b-1)\frac{(at)^n}{n!}\\&=\sum_{n=0}^{\infty}Li_{k,q}(1-e^{-at})\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^{m}B_{m,q}^{(k)}(ax)S_{n-m}(b-1)\frac{t^n}{n!}.\end{aligned}$$

Now, comparing the coefficient of t^n, then it gives the symmetric identity,

$$\begin{aligned}&Li_{k,q}(1-e^{-bt})\sum_{m=0}^{n}\binom{n}{m}a^{m}b^{n-m}B_{m,q}^{(k)}(bx)S_{n-m}(a-1)\\&\qquad=Li_{k,q}(1-e^{-at})\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^{m}B_{m,q}^{(k)}(ax)S_{n-m}(b-1).\end{aligned}$$

□

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