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SUPER VERTEX MEAN GRAPHS OF ORDER ≤ 7

A.LOURDUSAMY AND SHERRY GEORGE*

ABSTRACT. In this paper we continue to investigate the Super Vertex Mean behaviour of all graphs up to order 5 and all regular graphs up to order 7. Let $G(V,E)$ be a graph with p vertices and q edges. Let f be an injection from E to the set $\{1,2,3,\cdots,p+q\}$ that induces for each vertex v the label defined by the rule $f^v(v) = \text{Round}\left(\frac{\sum_{e\in E_v} f(e)}{d(v)}\right)$, where E_v denotes the set of edges in G that are incident at the vertex v, such that the set of all edge labels and the induced vertex labels is $\{1,2,3,\cdots,p+q\}$. Such an injective function f is called a super vertex mean labeling of a graph G and G is called a Super Vertex Mean Graph.

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1. Introduction

By a graph we mean a finite, simple and undirected one, $G(V,E)$, consisting of p elements in $V(G)$ called vertices and q elements in $E(G)$ known as edges. A graph of order p and size q is often called a (p,q) - graph [14].

A labeling of a graph G is a map that carries graph elements to integers (usually non-negative). There are varieties of labelings that are already in the literature [1], [2], [3], [5] and [11].

A super mean labeling f is an injection from V to the set $\{1,2,\cdots,p+q\}$ that induces for each edge uv the label $\left\lceil\frac{f(u)+f(v)}{2}\right\rceil$ such that the set of all vertex labels and the induced edge labels is $\{1,2,\cdots,p+q\}$. This concept was introduced by D.Ramya et al.[10]. Some results on mean labeling and super mean labeling are given in [4], [8], [9], [12] and [13].

Lourdusamy et al. [6] brought in a new extension of mean labeling, called Super vertex mean labeling of graphs and have proved many graphs, like cycles

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of any length except C_4, linear cyclic snakes etc. are Super Vertex Mean graphs in [7].

Round of a number or rounding function of a numerical value means replacing it by another value that is apporximately equal but has a shorter, simpler or more explicit representation. The round function is also called the nearest integer function and is defined such that Round(x) is the integer closest to x.

The disjoint union of m copies of a graph G is denoted by mG. The union of two graphs G_1 and G_2 is a graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The degree of a vertex v of G is the number of edges incident on it and is denoted by $d(v)$. In this paper we continue to investigate behaviour of all graphs up to order 5 and all regular graphs up to order 7.

2. Super Vertex Mean labeling

Definition 2.1. A Super Vertex Mean labeling f of a (p, q) - graph $G(V, E)$ is defined as an injection from E to the set $\{1, 2, 3, \cdots, p + q\}$ that induces for each vertex v the label defined by the rule $f^v(v) = \text{Round}\left(\frac{\sum_{e \in E_v} f(e)}{d(v)}\right)$, where E_v denotes the set of edges in G that are incident at the vertex v, such that the set of all edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, p + q\}$.

A graph that accepts super vertex mean labeling is called a Super Vertex Mean (hereafter, SVM) graph.

Remark 2.1. If $d(v) = 0$ for any vertex v of G then it is called an isolated vertex and if $d(v) = 1$ then it is called a pendant vertex. From the definition it is clear that a graph containing a vertex v whose $d(v) \leq 1$ cannot be an Super Vertex Mean (SVM) graph. For, if $deg(v) = 0$ for any vertex v of G, the above definition is not defined and if $deg(v) = 1$, the induced vertex label remains the same as the label of the edge that is incident on the vertex v. Therefore, necessarily $deg(v) \geq 2$ for all vertices v of a SVM graph. It is obvious that no tree is a SVM graph. In this paper, we discuss only those graphs with $d(v) \geq 2$ for all vertices v of G.

Remark 2.2. If $d(v) =$ r, for every vertex v of a graph G, then G is called a r - regular graph. From the above observation, we know that no zero regular or 1 - regular graph is a SVM.

Remark 2.3. A (p, q) - graph G can be r - regular graph if and only if $p.r$ is even. It is derived from the fact that 'Odd order graphs cannot be odd - regular graphs.' The number of edges of a r - regular graph is $(\frac{p.r}{2})$, i.e., $q = (\frac{p.r}{2})$.

Remark 2.4. In our previous work [7] we have proved that all cycles, C_n for any n, except C_4 are SVM graphs.

2.1. List of Regular graphs of order ≤ 7.

2.1.1. When order of a graph G is 3, there is just one 2 - regular graph. This is a cycle of length 3, known as C_3 or K_3. We have already proved in [7] that it is an SVM graph.

2.1.2. There are two regular graphs of order 4, of which C_4 is 2 - regular and K_4 is 3 - regular. We have proved in [7] that C_4 cannot be a SVM graph.

2.1.3. We have a 4 - regular graph K_5 with 10 edges and a 2 - regular graph C_5 with 5 edges of order 5, of which C_5 have been proved to be a SVM graph.

2.1.4. There are a total of 6 regular graphs of order 6. They are C_6, the disjoint union of two C_3's, both of which are 2 - regular, two non-isomorphic 3 - regular graphs with 9 edges each, a 4 - regular graph with 12 edges and the 5 - regular graph K_6 with 15 edges.

2.1.5. The number of regular graphs of order 7 is 5. They are C_7, disjoint union of C_3 and C_4, both of which are 2 - regulars, two non-isomorphic 4 - regular graphs with 14 edges and the complete graph K_7, which is 6 - regular. Now we proceed to prove that all these regular graphs are Super Vertex Mean graphs, excepting C_4. Before that we discuss the behaviour of disjoint union of graphs.

Theorem 2.2. *If G is an SVM graph, so is mG and if G_1 and G_2 are SVM graphs, so is $G_1 \cup G_2$. The converse is not true.*

Proof. For the first part of the theorem, it is enough to prove that if G_1 and G_2 are two SVM graphs, then $G_1 \cup G_2$ is also SVM.
Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be two SVM graphs with Super Vertex Mean labelings f and g respectively on them.

Let

$$E(G_1) = \{e_i : 1 \leq i \leq q_1\},$$
$$V(G_1) = \{u_i : 1 \leq i \leq p_1\},$$
$$E(G_2) = \{e_i' : 1 \leq i \leq q_2\},$$
$$V(G_2) = \{u_i' : 1 \leq i \leq p_2\}.$$

Define $h : E(G_1 \cup G_2) \rightarrow \{1, 2, 3, \cdots, p_1 + q_1 + p_2 + q_2\}$ by

$$h(e_i) = f(e_i), \text{for } 1 \leq i \leq q_1,$$

and

$$h(e_i') = p_1 + q_1 + g(e_i'), \text{for } 1 \leq i \leq q_2$$

Now we show that h is an injection.

Let

$$h(e_i) = h(e_j)$$
$$\Rightarrow f(e_i) = f(e_j)$$

Since, f is an injection, we have,

$$e_i = e_j.$$

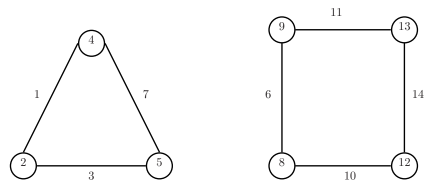

FIGURE 1. Super vertex mean labeling of $C_3 \cup C_4$.

Let

$$h(e_i') = h(e_j')$$
$$\Rightarrow p_1 + q_1 + g(e_i') = p_1 + q_1 + g(e_j')$$
$$\Rightarrow g(e_i') = g(e_j')$$

Since, g is an injection, we have,

$$e_i' = e_j'.$$

Therefore h is also an injection.

Suppose

$$h(e_i') = h^v(u_j')$$
$$\Rightarrow p_1 + q_1 + g(e_i') = p_1 + q_1 + g^v(u_j')$$
$$\Rightarrow g(e_i') = g^v(u_j')$$

which is a contradiction as g is Super Vertex Mean labeling.

So h is a SVM labeling.

To prove the second part of the theorem, we prove that although C_4 is not a SVM graph, $2C_4$ and $C_3 \cup C_4$ are SVM graphs. Also we prove the general case that $C_3 \cup C_m$ is SVM for all $m \geq 3$.

We know that C_m is SVM graph for all $m \geq 3$ and $m \neq 4$. Therefore it is enough to prove that $C_3 \cup C_4$ and $2C_4$ are SVM graphs.

Case 1: $C_3 \cup C_4$ is a SVM graph.

Let

$$E(C_3) = \{e_1, e_2, e_3\}$$

and

$$E(C_4) = \{e_1', e_2', e_3', e_4'\}$$

Define $f : E(C_3 \cup C_4) \to \{1, 2, 3, \cdots, 13, 14\}$ by

$$f(e_1) = 1, f(e_2) = 3, f(e_3) = 7$$
$$f(e_1') = 6, f(e_2') = 10, f(e_3') = 14, f(e_4') = 11$$

It is clear that f is a Super Vertex Mean labeling of $C_3 \cup C_4$. Therefore $C_3 \cup C_4$ is SVM graph, though C_4 is not.

□

Example 2.3. Super vertex mean labeling of $C_3 \cup C_4$ is shown in Figure 1.

Case 1a. General Case: $C_3 \cup C_m$ is SVM for all $m \geq 3$ including $m = 4$. All cycles, except C_4, are SVM graphs and so their union, but then $C_3 \cup C_4$ is a SVM graph. So, it is a clear fact that $C_3 \cup C_m$ is SVM for all $m \geq 3$. But we want to prove it in an alternate way without deriving from the above theorem and the fact that C_3 and C_m are SVM graphs for all $m \neq 4$.

Proof. There is nothing to prove in the case of odd m as all odd cycles are SVM graphs and their union is also SVM. Without loss of generality, we assume that m is even and $m \geq 4$.
Let $m = 2n$ for some $n \geq 2$.
Let

$$E(C_3) = \{e_1, e_2, e_3\}$$

and

$$E(C_m) = \{e'_1, e'_2, \cdots, e'_m = e'_{2n}\}$$

Define $f : E(C_3 \cup C_m) \to \{1, 2, 3, \cdots, 2m + 6 = 4n + 6\}$ by

$$f(e_1) = 1, f(e_2) = 3, f(e_3) = 7$$

$$f(e'_i) = \begin{cases} 6 & \text{if } i = 1 \\ 4i + 2 & \text{if } 2 \leq i \leq n + 1 \\ 8n - 4i + 11 & \text{if } n + 2 \leq i \leq 2n = m \end{cases}$$

Thus f is a super vertex mean labeling of $C_3 \cup C_m$ for all even $m \geq 4$, and it is a SVM graph. □

Case 2: $2C_4$ is a SVM graph.

Proof. Let C_4 and C'_4 be two cycles of length 4.
Let

$$E(C_4) = \{e_1, e_2, e_3, e_4\}$$

and

$$E(C'_4) = \{e'_1, e'_2, e'_3, e'_4\}$$

Define $f : E(C_4 \cup C'_4) \to \{1, 2, 3, \cdots, 15, 16\}$ by

$$f(e_1) = 1, f(e_2) = 3, f(e_3) = 5, f(e_4) = 10$$

$$f(e'_1) = 7, f(e'_2) = 11, f(e'_3) = 14, f(e'_4) = 16$$

Then f is a Super Vertex labeling of $2C_4$, and $2C_4$ is a SVM graph. □

Example 2.4. Super vertex mean labeling of $2C_4$ is shown in Figure 2.

Corollary 2.5. *mC_4 is a SVM graph for all even $m \geq 2$.*

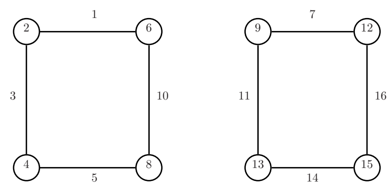

FIGURE 2. Super vertex mean labeling of $2C_4$

Proof. By the above theorem, we have proved that $2C_4$ and union of any two SVM graphs is a SVM graph.
Any even m is a multiple of 2, and therefore mC_4 is a union of $\frac{m}{2}$ times of $2C_4$. Or, mC_4, for $m \geq 4$, $m \equiv 0\ (mod\ 2)$ is equal to $m - 2C_4 \cup 2C_4$, where both of which are SVM graphs.
Thus the corollary. □

Corollary 2.6. *Disjoint union of any number of cycles of any length, except C_4 is a SVM graph.*

Proof. Since all the cycles except C_4 are SVM graphs, by the above theorem, their unions are SVM graphs.
Thus the corollary. □

Corollary 2.7. *When the disjoint union of any number of cycles of any length contains C_4, it is a SVM graph when,*

1. *There are even number of C_4 in the union, or*
2. *There exists at least one C_3 in the union.*

Proof. **1.** If there are even number of C_4 in the union, by the above corollary 1, union of these is SVM graph. All other cycles are SVM graphs. Therefore the union of both is a SVM graph by above theorem.

2. If there exists at least one C_3 in the union of cycles, then the union of this C_3 and any one C_4, if C_4 has an odd occurrence, is a SVM graph. Otherwise, C_4 occurs in even number of times, and their union is proved to be a SVM graph. □

3. Regular Graphs as SVM Graphs

Theorem 3.1. *Regular graphs of order ≤ 7 and Petersen graph are SVM graphs, C_4 being the only exception.*

3.1. Petersen graph. Given in figure 3 is an SVM labeling of $3-$ regular graph of order 10, known as Petersen Graph.

Since $f(U) \cup f^v(V) = \{1, 2, 3, \cdots, 24, 25\}$, it is a SVM labeling. While labeling Petersen graph, it is interesting to observe that the sum of all vertex labels is $\frac{2}{3}$

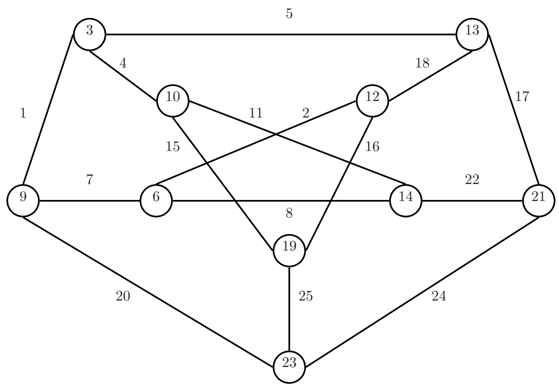

FIGURE 3. SVM labeling of Petersen Graph

times of the sum of all edge labels. i.e.,

$$\sum_{v \in V(G)} f^v(v) = \left(\frac{\sum_{e \in E(G)} f(e) \times 2}{3} \right)$$

It happens because when we calculate the induced vertex label which is rounded up average of the labels of $3-$ edges that are incident on that particular vertex, we consider the edges twice.

Being a SVM labeling, sum of all these labels is,

$$\begin{aligned} \left(\frac{(p+q)(p+q+1)}{2} \right) &= \sum_{v \in V(G)} f^v(v) + \sum_{e \in E(G)} f(e) \\ &= \left(\frac{\sum_{e \in E(G)} f(e) \times 2}{3} \right) + \sum_{e \in E(G)} f(e) \\ &= \left(\frac{\sum_{e \in E(G)} f(e) \times 5}{3} \right) \end{aligned}$$

Here for Petersen graph, the total is 325, and sum of all edge labels is 195 and that of all vertex labels is 130, perfectly in agreement with the above observation.

This need not be a necessary phenomenon for all types of SVM labeling of regular graphs. But this happens true for most of the regular graphs which we have examined. This fact is used as a hint for labeling the following graphs of order up to 7.

3.2. Regular graphs of order 3. 2 - regular graph of order 3 is the cycle C_3. We know that C_3 is a SVM graph .

3.3. Regular graphs of order 4. Regular graphs of order 4 are C_4, which is 2 - regular and K_4, that is 3 - regular. We know that C_4 is not a SVM graph. We prove that 3 - regular graph of order 4, i.e., K_4 is a SVM graph.
By above observation,

$$\begin{aligned}\left(\frac{(p+q)(p+q+1)}{2}\right) &= \sum_{v\in V(G)} f^v(v) + \sum_{e\in E(G)} f(e) \\ &= \left(\frac{\sum_{e\in E(G)} f(e)\times 2}{3}\right) + \sum_{e\in E(G)} f(e) \\ &= \left(\frac{\sum_{e\in E(G)} f(e)\times 5}{3}\right)\end{aligned}$$

In this case of K_4, we have,

$$\sum_{e\in E(G)} f(e) = \frac{3}{5}\times\left(\frac{(p+q)(p+q+1)}{2}\right)$$

ie.,

$$\begin{aligned}\sum_{e\in E(G)} f(e) &= \frac{3}{5}\times\left(\frac{10\times 11}{2}\right) \\ &= 33\end{aligned}$$

and,

$$\begin{aligned}\sum_{v\in V(G)} f^v(v) &= \frac{2}{5}\times\left(\frac{10\times 11}{2}\right) \\ &= 22\end{aligned}$$

So we can select the set $\{4,5,6,7\}$, as the vertex label set, the sum of whose elements is equal to 22. Consequently the edge label set is $\{1,2,3,8,9,10\}$, sum of whose elements is 33. We have partitioned the positive integers up to $p+q$, (here it is 10) in the above manner by the following logic. These numbers, 1 to 10, have to be distributed into two mutually disjoint sets in such a way that except any 4 numbers that are reserved as induced vertex labels, have to be clubbed in 4 sets of 3 elements (K_4 is a 3 - regular graph) and have to appear exactly twice without two numbers of one set coming together in some other set. it is because two vertices are connected by a single edge. And in a complete graph like K_4, each vertex is connected by an edge to every other vertex of the graph.
It is impossible to include the numbers $1,2,9$ and 10 in vertex label set. While calculating the average we cannot obtain one of these numbers as the rounded up average of any 3 numbers up to 10, without including the same number. Using the same number both as vertex label and edge label is ruled out by the

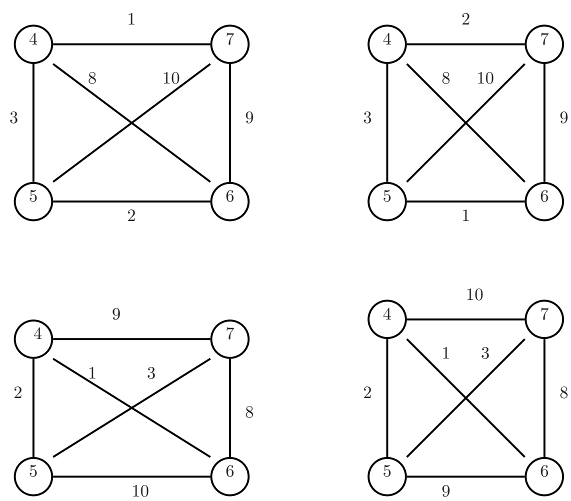

FIGURE 4. Super vertex mean labeling of K_4 in 4 different ways

definition of SVM labeling.
Therefore,

$$\{1, 2, 9, 10\} \subseteq f(E)$$

The sum of these numbers is 22. If we take two more numbers in the edge label set (K_4 has 6 edges), so that the sum equal to 33, we are done. By careful way of inspection, we have found that the only possibility is to include the numbers 3 and 8 in to the above set.
So,

$$f(E) = \{1, 2, 3, 8, 9, 10\}$$

and,

$$f^v(V) = \{4, 5, 6, 7\}$$

Using these sets, we can label K_4 in 4 different ways as shown below in figure 4.

3.4. Regular graphs of order 5. r - regular graphs, $3 \leq r \leq p-1$ of order 5 are C_5, which is a 2 - regular graph, and K_5 which is 4 - regular graph. Being an odd order, there cannot be any odd regular graphs. In our previous works [7] we have proved that C_5, a cycle of length 5 is SVM graph. Now we proceed to prove that K_5, the complete graph of order 5 is SVM graph.

We observe that any complete graph K_n for some $n \geq 3$ is a $n-1$ regular graph. Therefore it has $\frac{n\times(n-1)}{2}$ edges.
Therefore,

$$p + q = n + \frac{n \times (n-1)}{2}$$

Equivalently, for any r - regular graph,

$$p+q = n + \frac{n\times r}{2} = \frac{n\times(r+2)}{2}$$

When $r = n-1$, we get

$$p+q = \frac{n\times(n+1)}{2}$$

For K_5,

$$p+q = \frac{5\times 6}{2} = 15$$

As in the case of K_4, here

$$\sum_{v\in V(G)} f^v(v) = \frac{\sum_{e\in E(G)} f(e)\times 2}{4}$$

may be true. It is because every edge is counted twice while finding the induced vertex label which is the rounded up average of labels of 4 edges incident on that particular vertex. Therefore,

$$\begin{aligned}\frac{(p+q)\times(p+q+1)}{2} &= \sum_{v\in V(G)} f^v(v) + \sum_{e\in E(G)} f(e)\\ &= \sum_{e\in E(G)} f(e) + \frac{1}{2}\times\sum_{e\in E(G)} f(e)\\ &= \frac{3}{2}\times\sum_{e\in E(G)} f(e)\end{aligned}$$

$$\Rightarrow \sum_{e\in E(G)} f(e) = \frac{2}{3}\times\frac{(p+q)\times(p+q+1)}{2}$$

$$Now,\ \frac{(p+q)\times(p+q+1)}{2} = \frac{15\times 16}{2} = 120$$

$$\sum_{e\in E(G)} f(e) = 80$$

$$\sum_{v\in V(G)} f^v(v) = 40$$

From the set $\{1,2,3,\cdots,14,15\}$, the subset $\{1,2,14,15\}$ has to be a subset of $f(E)$ in SVM labeling. If 3 becomes a vertex label, then 5 and 6 cannot become vertex labels because when 3 and 6 or 3 and 5 become vertex labels, then among $10,11,12$ and 13, only three numbers could be chosen as induced vertex labels. For example,

$$3 = Round\left(\frac{1+2+3+4}{4}\right)$$

$$6 = Round\left(\frac{1+7+8+9}{4}\right)$$

If we select 10 and 13 as the next two vertex labels then only 11 can be the fifth one,
i.e.,

$$13 = Round\left(\frac{9+12+14+15}{4}\right)$$

$$13 = Round\left(\frac{11+12+14+15}{4}\right)$$

Then

$$13 = Round\left(\frac{9+12+14+15}{4}\right)$$

The remaining numbers that could be used to get rounded up average of 10 and 11 are $2, 4, 5, 7, 8, 12, 14$ and 15, and they can be classified into 3 sets which appeared elsewhere. So we cannot have any option to have rounded up average of 10 and 11 without repeating any numbers which have already appeared in pair.
If we select 11 and 13 as vertex labels where,

$$13 = Round\left(\frac{9+12+14+15}{4}\right)$$

or,

$$13 = Round\left(\frac{10+12+14+15}{4}\right)$$

then 12 cannot be the fifth vertex label. 10 is already ruled out to be the vertex label with 13 as another vertex label.
Therefore, 13 has to be an edge label and $10, 11$ and 12 can be vertex labels along with 3.
Here too, 12 has only three options left,

$$12 = Round\left(\frac{4+13+14+15}{4}\right)$$

$$12 = Round\left(\frac{5 \ or \ 6+13+14+15}{4}\right)$$

$$12 = Round\left(\frac{7+13+14+15}{4}\right)$$

This implies 10 and 11 are obtained as averages by making use of any one of the numbers among $13, 14$ and 15.
For example, 11 cannot be made a vertex label without repeating any one of the above numbers.
Therefore when 3 becomes a vertex label, the only next vertex label can be 7 or any number greater than 7. By continuing our inspection in a similar way we get the possible sets which can be vertex label set as follows;
1. $\{3, 7, 8, 10, 12\}$

2. $\{3, 7, 9, 10, 11\}$
3. $\{4, 6, 7, 10, 13\}$
4. $\{4, 6, 7, 11, 12\}$
5. $\{4, 6, 8, 9, 13\}$
6. $\{4, 6, 8, 10, 12\}$
7. $\{4, 6, 9, 10, 11\}$
8. $\{6, 7, 8, 9, 10\}$
9. $\{6, 7, 8, 9, 11\}$

It is interesting to note that except the 9th set, all the others follow the rule,

$$\sum_{v \in V(G)} f^v(v) = \frac{1}{2} \times \sum_{e \in E(G)} f(e) = 40$$

Therefore for r - regular graphs, the condition

$$\sum_{v \in V(G)} f^v(v) = \frac{2}{r} \times \sum_{e \in E(G)} f(e)$$

is not a necessary condition, but only a hint to SVM labeling. Given in figure 5 are the pictorial representations of different SVM labelings of K_5.

3.5. Regular graphs of order 6. Regular graphs having no isolated or pendant vertex of order 6 are the cycle, C_6 and $2C_3$, which are 2 - regulars, $K_{3,3}$ and another graph with 9 edges, both of them are 3 - regulars, the octahedral graph with 12 edges, which is 4 - regular and the complete graph K_6. In total there are 6 non-isomorphic r - regular graphs of order 6, where $2 \leq r \leq 5$.
We have already proved that C_6 and $2C_3$ are SVM graphs. We show now that $K_{3,3}$ is a SVM graph.
For $K_{3,3}$, $p = 6$ and $q = 9$.
Therefore,

$$f(E) \cup f^v(V) = \{1, 2, 3, \cdots, 14, 15\}$$

While inspecting the possibility of SVM labeling of $K_{3,3}$ we have to keep the following in mind:

(1) Partition the above set into two sets, keeping the hint for labeling r-regular graphs, i.e., $\sum_{v \in V(G)} f^v(v) = \frac{2}{r} \times \sum_{e \in E(G)} f(e)$
(2) Clearly $f^v(V)$ contains 6 elements and $f(E)$ has 9 elements.
(3) Now the set $f(E)$ is distributed into six sets of 3 elements each in such a way that,
- The rounded up average of each set is one of the numbers in the set $f^v(V)$. These numbers are not repeated.
- These six sets form two partitions, each partition having 3 sets and no number in one set of one partition is repeated in another set of the same partition.
- All the three numbers in one set of one partition are distributed equally in each set of the second partition.

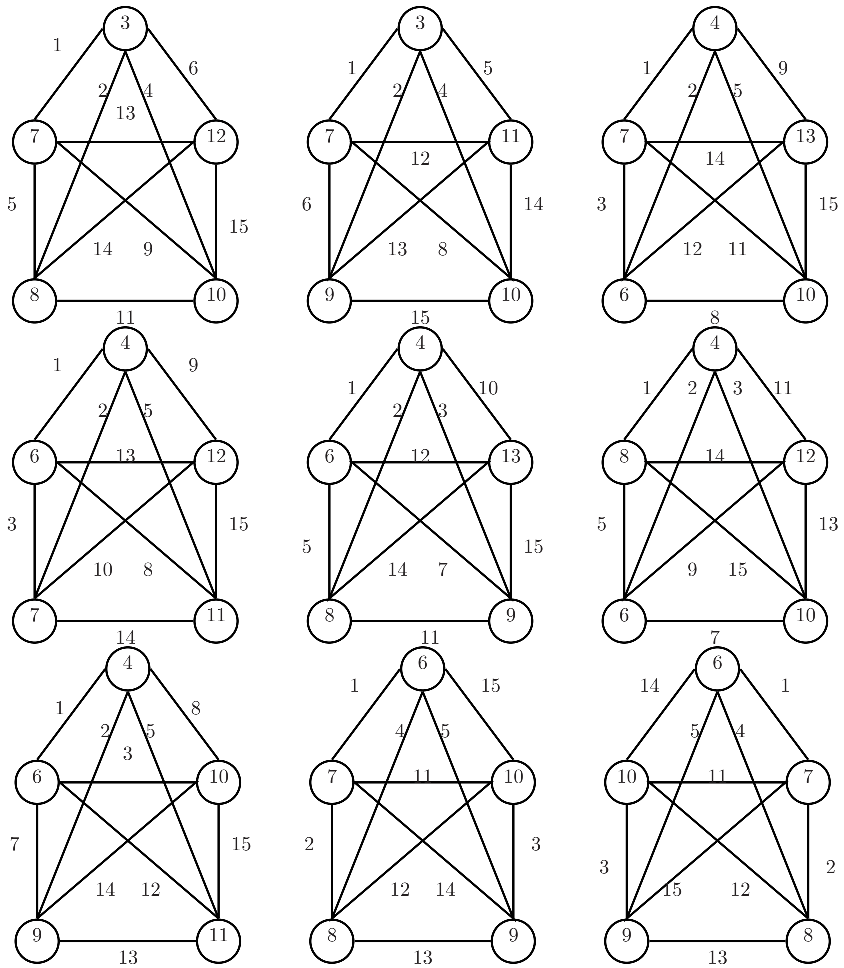

FIGURE 5. Pictorial representations of different SVM labelings of K_5

Following above directions we form six subsets of $f(E)$ as given below; $\{1, 2, 6\}, \{4, 7, 12\}, \{11, 14, 15\}$ and $\{1, 4, 11\}, \{6, 7, 14\}, \{2, 12, 15\}$ whose rounded up averages are $3, 8, 13, 5, 9$ and 10 respectively. Note that the first three sets and the last three sets are having the same elements, the only difference being that two elements of any set do not appear together in any other set. The first three sets and the last three sets in themselves form two different partitions of the set $f(E)$.

The above labeling and another SVM labeling of $K_{3,3}$ are shown pictorially in figure 6.

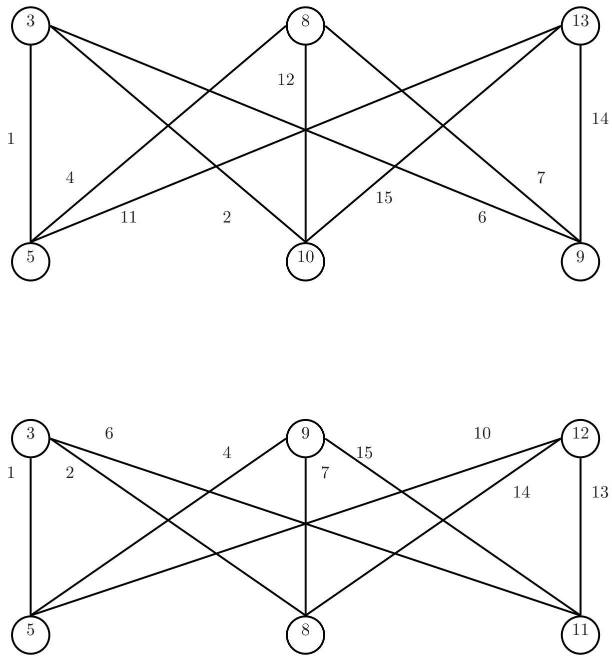

FIGURE 6. SVM labelings of $K_{3,3}$

3.5.1. Another 3 - regular graph of order 6. There is another 3 - regular graph that is non-isomorphic to $K_{3,3}$ with 9 edges and 6 vertices. Therefore we cannot use the same method that we used in the case of the previous graph. The SVM labelings of this graph is shown below in figure 7.

3.5.2. 4 - regular graph of order 6 (Octahedral graph). There is a 4 - regular graph of order 6 having 12 edges.
Therefore

$$p + q = 18$$

$$\frac{(p+q) \times (p+q+1)}{2} = 171$$

$$\sum_{v \in V(G)} f^v(v) = \frac{2}{r} \times \sum_{e \in E(G)} f(e)$$

$$= \frac{2}{4} \times \sum_{e \in E(G)} f(e)$$

$$171 = \frac{3}{2} \times \sum_{e \in E(G)} f(e)$$

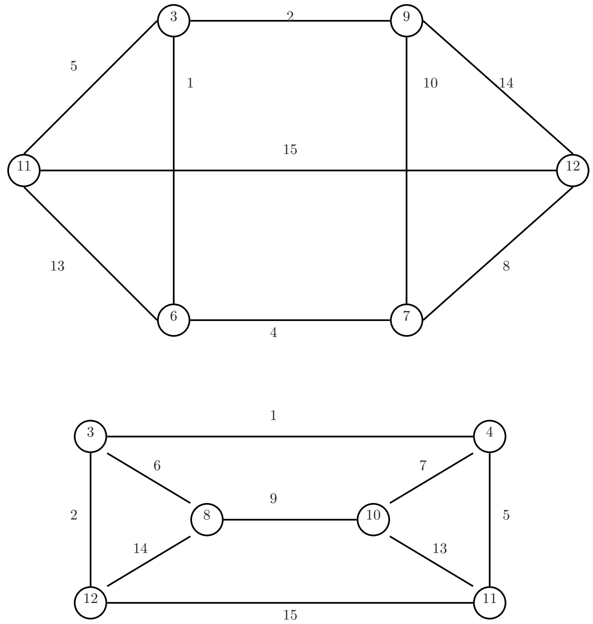

Figure 7

$$\sum_{v \in V(G)} f^v(v) = \frac{171}{3} = 57$$

Using this hint we can partition the numbers up to 18 into two sets as given below;

$$f(E) = \{1, 2, 4, 5, 6, 7, 12, 13, 14, 15, 18\}$$
$$f^v(V) = \{3, 8, 9, 10, 11, 16\}$$

where $f(E)$ contains q elements and $f^v(V)$ has p elements. The elements of $f(E)$ are repeated exactly once to find the rounded up average of four numbers of $f(E)$, in order to obtain the elements in $f^v(V)$. Care should be taken so as not to place two numbers together while finding a second rounded up average. Thus we find that this 4 - regular graph of order 6 too is a SVM graph with the following SVM labeling in figure 8.

3.5.3. The Complete graph K_6. Now we have the task of labeling K_6, the complete graph of order 6. Being a 5- regular graph the hint that we could use is that,

$$\sum_{v \in V(G)} f^v(v) = \frac{2}{5} \times \sum_{e \in E(G)} f(e)$$

The total sum of all the numbers up to $p + q$, i.e., up to 21 is 231, where $p = 6$ and $q = 15$.

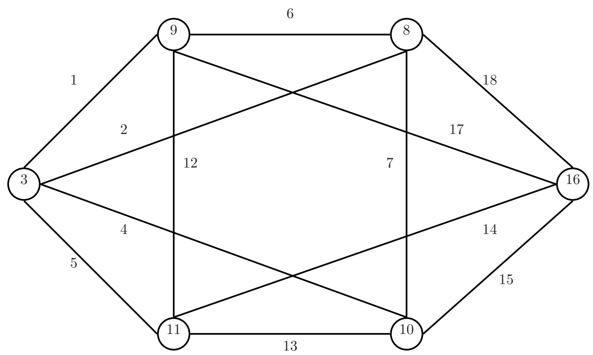

FIGURE 8. A 4 - regular graph of order 6 is a SVM graph

By the definition of SVM labeling, we have,

$$231 = \sum_{v \in V(G)} f^v(v) + \sum_{e \in E(G)} f(e)$$

This implies that,

$$\frac{7}{5} \times \sum_{e \in E(G)} f(e) = 231$$

$$\sum_{e \in E(G)} f(e) = \frac{231 \times 5}{7} = 165$$

and,

$$\sum_{v \in V(G)} f^v(v) = \frac{231 \times 2}{7} = 66$$

So we partition the numbers up to 21 into two sets,

$$f(E) = \{1, 2, 3, 5, 6, 7, 8, 9, 14, 15, 17, 18, 19, 20, 21\}$$

$$f^v(V) = \{4, 10, 11, 12, 13, 16\}$$

having q and p elements respectively and the respective sum of its members being 165 and 66.
The other aspects are kept in mind as in previous cases of labeling regular and complete graphs.
In a complete graph's SVM labeling, the $(n-1)$ elements of $f(E)$ that are taken to calculate the rounded up average, in order to get one of the elements of $f^v(V)$, are used a total of $(n-1)$ instances. But they are used one at a time, and without repeating. Whereas in a r - regular graph's labeling only r - elements are used only in any r - instances, one at a time and without repeating. Thus we obtain a SVM labeling of K_6 and it is shown in figure 9.

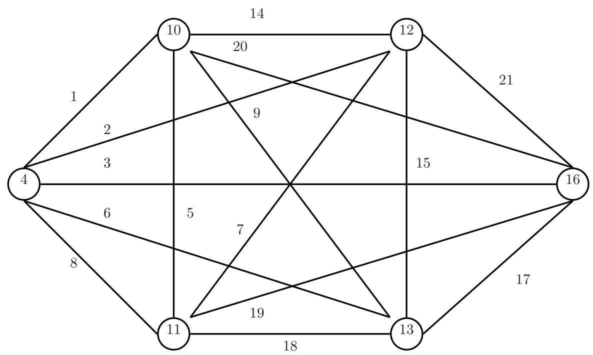

FIGURE 9. SVM labeling of K_6

3.6. Regular graphs of order 7. There are 5 regular graphs of order 7 that do not have any isolated and pendant vertex. They are $C_7, C_3 \cup C_4$, which are 2 - regulars, two non-isomorphic 4 - regular graphs and K_7, the complete graph which is 6 - regular.

We have already proved that C_7 and $C_3 \cup C_4$ are SVM graphs.

Let us investigate the SVM behaviour of the rest of graphs of order 7. We start with 4 - regular graphs of order 7. As in previous cases we can use the following hint that;

$$\sum_{v \in V(G)} f^v(v) = \frac{2 \times \sum_{e \in E(G)} f(e)}{4}$$

and, since $p + q = 7 + 14 = 21$, we have

$$\frac{(p+q)(p+q+1)}{2} = 231$$

$$\frac{3}{2} \times \sum_{e \in E(G)} f(e) = 231$$

$$\sum_{v \in V(G)} f^v(v) + \sum_{e \in E(G)} f(e) = 231$$

$$\sum_{e \in E(G)} f(e) = \frac{2 \times 231}{3} = 154$$

$$\sum_{v \in V(G)} f^v(v) = \frac{231}{3} = 77$$

So we partition the numbers up to 21 into two possible sets, having q and p elements respectively;

$$f(E) = \{1, 2, 4, 5, 6, 7, 8, 9, 13, 15, 16, 17, 18, 20, 21\}$$

$$f^v(V) = \{3, 8, 10, 11, 12, 12, 19\}$$

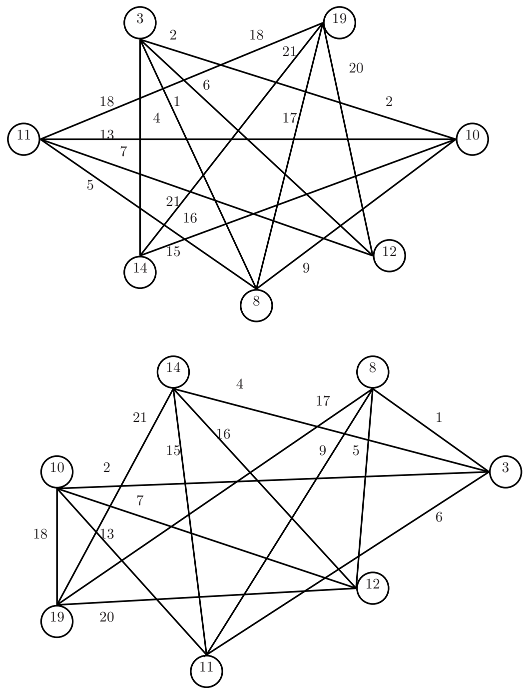

FIGURE 10. Two different labelings of two non-isomorphic 4 - regular graphs of order 7

These two partitions give rise to two different SVM labelings for the following two non-isomorphic 4 -regular graphs of order 7 as shown in figure 10.

3.6.1. The complete graph K_7. Now we proceed to prove that K_7 is a SVM graph. K_7 being a 6 - regular graph of order 7 and each vertx is connected to every other vertex by a unique edge, we have to partition the numbers up to $\frac{n\times(n+1)}{2}$, since for a complete graph, $p+q=\frac{n\times(n+1)}{2}$.
i.e.,

$$\frac{7\times 8}{2}=28$$

The hint that we could use as in previous cases is that

$$\sum_{v\in V(G)} f^v(v)=\frac{2\times\sum_{e\in E(G)} f(e)}{6}$$

and, since $p+q=28$, we have

$$\frac{(p+q)(p+q+1)}{2}=406$$

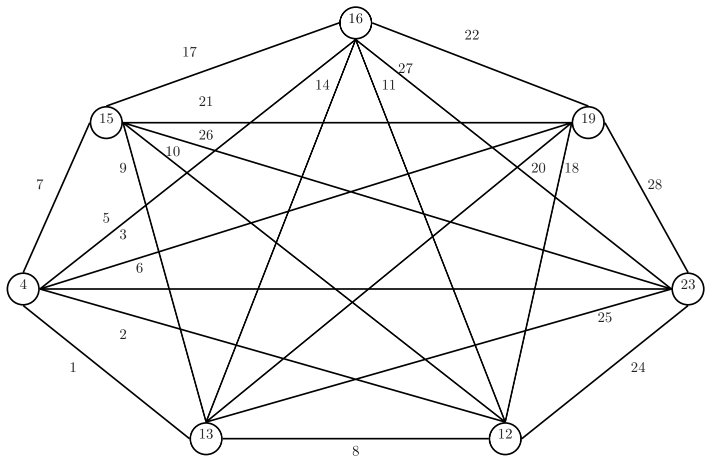

FIGURE 11. SVM labeling of K_7

$$\frac{4}{3} \times \sum_{e \in E(G)} f(e) = 406$$

$$\sum_{v \in V(G)} f^v(v) + \sum_{e \in E(G)} f(e) = 406$$

$$\sum_{e \in E(G)} f(e) = \frac{406 \times 3}{4} = 101.5$$

$$\sum_{v \in V(G)} f^v(v) = \frac{406}{4} = 304.5$$

For our convenience, we take this as

$$\sum_{e \in E(G)} f(e) = 102$$

$$\sum_{v \in V(G)} f^v(v) = 304$$

Based on this, we obtain the following partitions of the numbers upto 28

$$f(E) = \{1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 14, 17, 18, 20, 21, 22, 24, 25, 26, 27, 28\}$$

$$f^v(V) = \{4, 12, 13, 15, 16, 19, 23\}$$

having q and p elements respectively.
Careful distribution of these numbers as various edge as well as vertex labels, keeping the facts mentioned in earlier cases, we obtain the SVM labeling of K_7 as shown in figure 11.

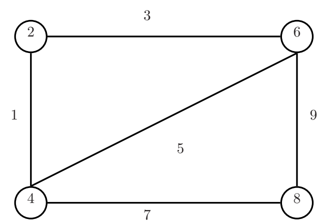

FIGURE 12

4. Super Vertex Mean Graphs of order ≤ 5

Theorem 4.1. *All the graphs of order ≤ 5 having no isolated or pendant vertex are Super Vertex Mean graphs, having C_4 the only exception.*

We have so far proved that all the complete and regular graphs, (except C_4) of order up to 7, are SVM graphs and graphs containing any isolated or pendant vertex are not SVM graphs. In this section we examine all other graphs of order ≤ 5 and do not fall into the above category of graphs. There are 3 graphs with $d(v) \geq 2$ of order 4, out of which a graph with 5 edges fulfill our requirement and so we examine its SVM behaviour and find that it is a SVM graph. Its labeling is given in figure 12.

Of the order 5, there are altogether 10 non-isomorphic graphs with $d(v) \geq 2$. Among those, 8 graphs need to be investigated of their SVM nature. We have found that they are all SVM graphs as shown in figure 13.

5. Conclusion

We conclude by stating that all the r - regular graphs of order ≤ 7 and all graphs having no isolated or pendant vertex and order ≤ 5, excluding C_4 are SVM graphs.

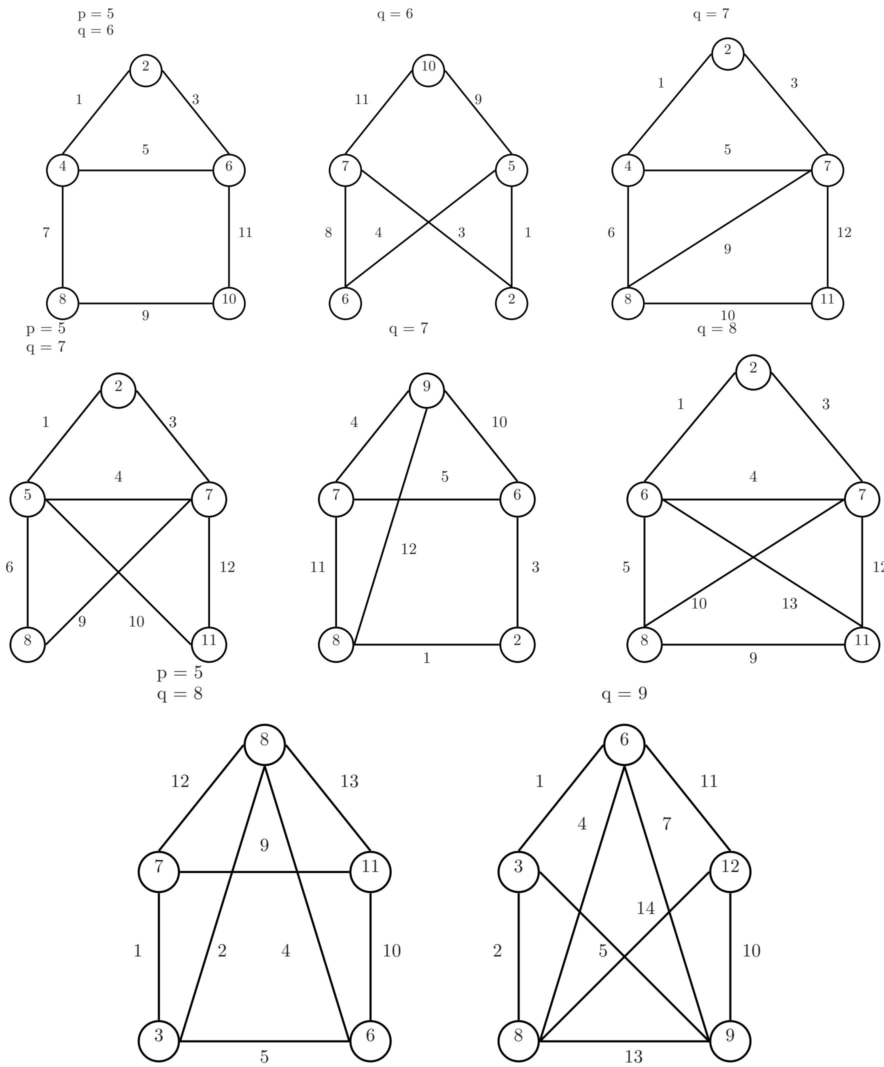

FIGURE 13

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A.Lourdusamy received M.Sc. from St.Joseph's College, Trichy, India (affiliated to Bharathidasan University, Trichy) and Ph.D at Manonmaniam Sundaranar University, Tirunelveli in India. His Ph.D. was in Graph Theory. At present he is an Associate Professor and IQAC Coordinator of St.Xavier's College, Palayamkottai. Since 1986 he has served many colleges in India as assistant professor. He has published 76 publications in National/International Journals so far. There is also a paper in the international journal in his name; Dong-Lin Hao, Jian-Hua Yin and Ze-TuGao, *Lourdusamy's conjecture for* ${K_n^{-k_1}} \times {K_m^{-k_2}}$, Utilitas Mathematica **101**(2016), 69-77.

Department of Mathematics, St.Xavier's College(Autonomous), Manonmaniam Sundaranar University, Tirunelveli, India.
e-mail: lourdusamy15@gmail.com

Sherry George received M.Sc. from St.Xavier's College, Palayamkottai, (affiliated to Manonmaniam Sundaranar University, Tirunelveli) in India and is pursuing Ph.D. at Manonmaniam Sundaranar University under the guidance of Dr.A.Lourdusamy.

Department of Mathematics, St.Xavier's College(Autonomous), Manonmaniam Sundaranar University, Tirunelveli, India.
e-mail: 1212sherry@gmail.com