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ON THE ORDERING OF ASYMPTOTIC PAIRWISE NEGATIVELY DEPENDENT STRUCTURE OF STOCHASTIC PROCESSES†

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Abstract. In this paper, we introduced a new asymptotic pairwise negatively dependent($APND$) structure of stochastic processes. We are also important to know the degree of $APND$-ness and to compare pairs of stochastic vectors as to their $APND$-ness. So, we introduced a definitions and some basic properties of $APND$ ordering. Some preservation results of $APND$ ordering are derived. Finally, we shown some examples and applications.

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1. Introduction

Lehmann(1966) first introduced the concept of positive(negative) quadrant dependence($PQD(NQD)$) together with some other dependence concepts.
Since then, a great many papers have been written on the subject and its extensions and numerous multivariate inequalities have been obtained. For reference of available results(see Karlin and Rinott(1980), Shaked(1982b), Ebrahimi and Ghosh(1981), Sampson(1983), Barlow and Proschan(1973), Tong(1980), Joag-Deo et al(1983), Ebrahimi(1982), Bozorgnia et al(1996) and Amini et al (2004)). Concepts of these dependence have subsequently been extended to stochastic processes in different directions by many authors(see Ascher et al (1984), Baek et al(2002), Cox et al(1980), Deheuvels(1983), Ebrahimi(1994), Ebrahimi et al (1988), Pollard(1984)). Certain kinds of dependence concepts, when they are

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imposed on processes, are reflected as analogous properties of corresponding hitting times. These results are of value as they help us to understand in what ways the hitting times for dependence structures of hitting times can be inherited from the corresponding processes. Furthermore, these result sometimes can tell us how to control the reliability of a system by controlling its characteristics.

But, since these results also are qualitative form of dependence, it would seem difficult or impossible to compare different pairs of random variables as to their "degree of asymptotic pairwise negatively quadrant dependent($APND$)-ness". Fortunately, in this paper we develop a partial ordering which permits us to compare pairs of $APND$ bivariate stochastic vectors of interest as to their degree of $APND$-ness. Thus, we extended $APND$ random variables(Nili Sani et al(2017) to $APND$ stochastic processes and introduced a new $APND$ bivariate vectors of stochastic processes and importance of this concept of dependence lies in the fact that it is bigger than the negative quadrant dependence and it enjoys various properties. It is also important to know the degree of $APND$-ness and to compare pairs of $APND$ stochastic vectors as to their $APND$-ness. Furthermore, this is the first study in which a partial ordering is developed for the degree of qualitative dependence. The definitions and some basic properties of ordering are introduced in Section 2. Some preservation results of $APND$ ordering are derived in Section 3. It is shown that the $APND$ ordering is preserved under combination, mixture, transformations of stochastic vectors by increasing(decreasing) functions, and limit in distributions. Finally, in section 4, we show some examples and applications.

2. Preliminaries

In this section, we present concepts, definitions, notations and basic facts used throughout the paper. Suppose that we are given a n-dimensional $(n \geq 2)$ stochastic vector process $\{(\underline{\mathrm{X(t)}} = X_n(t))|t \in \Lambda\}$ where the index set Λ will always be a subset of $R_+ = [0, \infty)$. The state space of $\underline{\mathrm{X(t)}}$ is the cartesian product $E = E_1 \times \cdots \times E_n$, which will be a subset of n-dimensional Euclidean space R^n. For any states $a_i \in E_i$, $i = 1, 2, \cdots, n$, define the random times

$$T_i(a_i) = inf\{t|X_i(t) \leq a_i,\ 0 \leq t \leq \infty\}.$$

In other words, $T_i(a_i)$ is the hitting time that the ith component process $X_i(t)$ reaches or goes below a_i. The stochastic process $\{T_i(a),\ a \in E_i\}$ will be referred to as the hitting time process of $X_i(t)$.

Let $\beta = \beta(F, G)$ denote the class of bivariate distribution functions(df's) H on R^2 having specified marginal df's F and G, and F and G being nondegerate. Use the notation $\bar{H}(t_i, t_j) = P(T_i(a_i) > t_i,\ T_j(a_j) > t_j)$ and $P(T_i(a_i) \leq t_i, T_j(a_j) \leq t_j)$.

Definition 2.1. The stochastic vector process $\{(X_i(t), X_j(t))|t \in \Lambda\}$ is asymptotic negatively dependent type 1 if for all $t_i,\ t_j \in \Lambda$, where $q(n) = o(n^{-\omega}), \omega >$

0,

$P(T_i(a_i) > t_i,\ T_j(a_j) > t_j) \leq (1 + q(|i-j|))P(T_i(a_i) > t_i)P(T_j(a_j) > t_j).$

Definition 2.2. The stochastic vector processis $\{(X_i(t), X_j(t))|t \in \Lambda\}$ asymptotic negatively dependent type 2 if for all $t_i,\ t_j \in \Lambda$, where $q(n) = o(n^{-\omega}), \omega > 0$,

$P(T_i(a_i) \leq t_i,\ T_j(a_j) \leq t_j) \leq (1 + q(|i-j|))P(T_i(a_i) \leq t_i)P(T_j(a_j) \leq t_j).$

Similarly, the stochastic vector process $\{(X_i(t), X_j(t))|t \in \Lambda\}$ is asymptotic positively dependent type 1 and asymptotic positively dependent type 2 if Definition 2.1 and Definition 2.2 hold with the inequalities sign reserved. The stochastic processes $X_1(t), X_2(t). \cdots$ are said to be asymptotic pairwise negatively dependent($APND$) if stochastic vector process $\{(X_i(t), X_j(t))|t \in \Lambda\}$is asymptotic negatively dependent type 1 and type 2 for every $i \neq j,\ i,j \geq 1$. If for $i, j, q|i-j| = 0$, then the random variables(r.v.'s) and are negatively dependent(Lehmann (1966)). A special subclass of $APND$ r.v.'s is the pairwise negatively dependent r.v.'s studied by Nili Sani et al. (2005).

Let β^+ denote the subclass of β where H is $APND$. Suppose H_1 and H_2 both belong to β^+.

Definition 2.3. The bivariate distribution H_2 is said to be more asymptotic pairwise negatively dependent than H_1 if

$$\bar{H}_2(t_i, t_j) \leq \bar{H}_2(t_i, t_j).\ for\ all\ t_i, t_j \in \Lambda, i \neq j. \tag{1}$$

We write $H_2 >^{APND} H_1$.

Remark 2.1. Note that an equivalent form of (1) is $H_2(t_i, t_j) \leq H_2(t_i, t_j)$. for all $t_i, t_j \in \Lambda, i \neq j$.

Definition 2.4. A stochastic vector process $\underline{\mathrm{Y(t)}}$ is stochastically increasing (decreasing) $(SI(SD))$ in the stochastic vector process $\underline{\mathrm{X(t)}}$ if $E(f(\underline{\mathrm{S(a)}}) \mid \underline{\mathrm{T(a)}} = \underline{\mathrm{t}}, \underline{\mathrm{t}} \in \Lambda)$ is increasing(decreasing) in $\underline{\mathrm{t}}$ for all real valued of increasing(decreasing) function f.

The next Lemma shows that ordering of stochastic vector process is preserved under convolution.

Lemma 2.5. *Suppose that $\underline{X(t)} = (X_i(t), X_j(t))$ and $\underline{Y(t)} = (Y_i(t), Y_j(t))$ are stochastic vector process with increasing sample paths and have distributions H_1 and H_2 respectively, where H_1 and H_2 belongs to β^+ such that $H_1 >^{APND} H_2$ and coefficients $q_1,\ q_j$ for $i \neq j,\ i,j \geq 1$, and $\underline{Z} = (Z_i,\ Z_j)$ with an arbitrary $APND$ distribution H and independent and increasing sample paths of both $\underline{X(t)}$ and $\underline{X(t)}$. Then $\underline{X(t)} + \underline{Z} >^{APND} \underline{Y(t)} + \underline{Z}$.*

Proof. Firstly, to show that $\underline{\mathrm{X(t)}} + \underline{\mathrm{Z}}$ are $APND$, the proof will be given for $i = 1, 2$.
Let $W_i(a_i) = inf\{s|X_i(s) + Z_i \leq a_i\},\ i = 1, 2$. Then,

$P(W_1(a_1) > t_1,\ W_2(a_2) > t_2)$

$$\begin{aligned}
&= P[(inf\{s|X_1(s)+Z_1 \le a_1\}) > t_1, (inf\{s|X_2(s)+Z_2 \le a_2\}) > t_2] \\
&= \int\int P(T_1(a_1-z_1) > t_1,\ T_2(a_2-z_2) > t_2) dH_{Z_1,z_2}(z_1,z_2) \\
&\le (1+q_1)\int\int P(T_1(a_1-z_1) > t_1)P(T_2(a_2-z_2) > t_2) dH_{z_1}(z_1)dH_{z_2}(z_2) \\
&= (1+q_1)P(W_1(a_1) > t_1)P(W_2(a_2) > t_2)
\end{aligned}$$

The inequality follows from $APND$ and independent assumptions.
So, $\underline{\mathrm{X(t)}}+\underline{\mathrm{Z}}$ are $APND$, similarly we can show that $\underline{\mathrm{Y(t)}}+\underline{\mathrm{Z}}$ are $APND$.

Secondly, we have to show that $\underline{\mathrm{X(t)}}+\underline{\mathrm{Z}} >^{APND} \underline{\mathrm{Y(t)}}+\underline{\mathrm{Z}}$.
Let $W_i(a_i) = inf\{s|X_i(s)+Z_i \le a_i\}$ and $V_i(a_i) = inf\{s|Y_i(s)+Z_i \le a_i\},\ i = 1,2$. Then,

$$\begin{aligned}
&P(W_1(a_1) > t_1,\ W_2(a_2) > t_2) \\
&= P[(inf\{s|X_1(s)+Z_1 \le a_1\}) > t_1,\ (inf\{s|X_2(s)+Z_2 \le a_2\}) > t_2] \\
&= \int\int P(T_1(a_1-z_1) > t_1,\ T_2(a_2-z_2) > t_2) dH_{Z_1,Z_2}(z_1,z_2) \\
&\le \int\int P(S_1(a_1-z_1) > t_1,\ S_2(a_2-z_2) > t_2) dH_{Z_1,Z_2}(z_1,z_2) \\
&= P(V_1(a_1) > t_1,\ V_2(a_2) > t_2)
\end{aligned}$$

The inequality follows from that assumption that $\underline{\mathrm{X(t)}} >^{APND} \underline{\mathrm{Y(t)}}$. □

3. Closure properties of $APND$ ordering

In this section we show preservation of the $APND$ ordering under combination, mixture, transformation of stochastic vector processes by increasing (decreasing) function, limits in distributions.
The following theorem is very important in recognizing $APND$ ordering in compound distribution which arise naturally in stochastic vector process.

Theorem 3.1. *Suppose (X_i, Y_i) and (U_i, V_i) are such that $(X_i, Y_i) >^{APND}$ (U_i, V_i) for $i = 1, 2, ...$ and let $N(t)$ be the Poisson process with number of shocks received by time t. If $\{(X_i, Y_i)|i = 1, 2, ...\}$ and $\{(U_i, V_i)|i = 1, 2, ...\}$ are independent bivariate processes and independent of $N(t)$ respectively, then $(\sum_{i=1}^{N(t)} X_i, \sum_{i=1}^{N(t)} Y_i) >^{APND} (\sum_{i=1}^{N(t)} U_i, \sum_{i=1}^{N(t)} V_i)$.*

Proof. First, using Lemma 2,1, we can know that for coefficients $q_i, q_j,\ i \ne j,\ i,j \ge 1$, $(\sum_{i=1}^{N(t)} X_i, \sum_{i=1}^{N(t)} Y_i)$ and $(\sum_{i=1}^{N(t)} U_i, \sum_{i=1}^{N(t)} V_i)$ are $APND$ respectively. Next, the proof will be given for $i = 1, 2$.

$$\begin{aligned}
&P(T_1(a_1) > t_1,\ T_2(a_2) > t_2) \\
&= P(\sum_{i=1}^{N(t)} X_i \le a_1,\ t_1 \le t < \infty,\ \sum_{i=1}^{N(t)} Y_i \le a_2, t_2 \le t < \infty)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} P(N(t_1)=k_1,\ N(t_2)=k_2)P(\sum_{i=1}^{k_1} X_i \le a_1,\ \sum_{i=1}^{k_2} Y_i \le a_2) \\
&\le \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} P(N(t_1)=k_1,\ N(t_2)=k_2)P(\sum_{i=1}^{k_1} U_i \le a_1,\ \sum_{i=1}^{k_2} V_i \le a_2) \\
&= P(\sum_{i=1}^{N(t_1)} U_i \le a_1,\ \sum_{i=1}^{N(t_2)} V_i \le a_2) \\
&= P(S_1(a_1) > t_1,\ S_2(a_2) > t_2)
\end{aligned}$$

□

Theorem 3.2. *Let (a) stochastic vector process with increasing sample paths $(X_i(t),\ X_j(t))|\lambda$ be a conditionally APND and coefficients $q_1,\ q_2$ for $i \neq j$, $i,j \geq 1$, and (b) $X_i(t)$ be $SI(SD)$ in λ and $X_j(t)$ be $SD(SI)$ in λ. Then $(X_i(t), X_j(t))$ is APND.*

Proof. Let f_1 and f_2 concordant functions. Then for fixed $t_i,\ t_j \geq 0$, $a_i \in E_i$, $a_j \in E_j$, $i \neq j$, $i,j \geq 1$,

$$\begin{aligned}
P(T_i(a_i) > t_i,\ T_j(a_j) > t_j) &= E(I_{f(T_i(a_i))} I_{g(T_i(a_i))}) \\
&\le (1+q_1)E_\lambda(E(I_{f(X_i)|\lambda}))(E(I_{g(X_i)|\lambda})),\ \textit{by (a), (b)} \\
&= (1+q_1)E(I_{f(T_i(a_i))})E(I_{g(T_i(a_i))}) \\
&= (1+q_1)P(T_i(a_i) > t_i)P(T_j(a_j) > t_j).
\end{aligned}$$

Thus, we obtain that stochastic vector process $(X_i(t),\ X_j(t))$ are $APND$. □

The next theorem deals with the preservation of the $APND$ ordering under mixture. For $i \neq j,\ i,j \geq 1$, we may define the following class that $\beta_\lambda^+ = \{H_\lambda | H(t_i,\infty) = F(t_i|\lambda),\ H(\infty,t_j) = G(t_j|\lambda),\ H_\lambda|\lambda$ is $APND,\ X_i(t)$ is $SD(SI)$ in λ and $Y_j(t)$ is $SI(SD)$ in $\lambda\}$.
Consider $(\beta_\lambda^+,\ >^{APND})$. The following theorem shows that if two elements of β_λ^+ are ordered according to $>^{APND}$, then after mixing λ, the resulting element in β^+ preserve the same order.

Theorem 3.3. *Suppose that stochastic vector processes with increasing sample paths $(X_i(t),\ X_j(t))|\lambda$ and $(Y_i(t),\ Y_j(t))|\lambda$ belong to β_λ^+ and coefficients $q_i,\ q_j$ for $i \neq j,\ i,j \geq 1$, and let $(X_i(t),\ X_j(t)|\lambda >^{APND} (Y_i(t),\ Y_j(t))|\lambda$ for all λ. Then $((X_i(t),\ X_j(t)),\ (Y_i(t),\ Y_j(t))$ belong to β^+ and $(X_i(t),\ X_j(t)) >^{APND} (Y_i(t),\ Y_j(t))$.*

Proof. From theorem 3.2, $(X_i(t),\ X_j(t))$ and $(Y_i(t),\ Y_j(t))$ are $APND$ respectively and for fixed $t_i, t_j \geq 0$, $a_i \in E_i,\ a_j \in E_j,\ i \neq j,\ i,j \geq 1$, define the following hitting times $T_i(a_i) = inf\{t : X_i(t) \leq a_i\}$ and $S_i(a_i) = inf\{t : Y_i(t) \leq a_i\}$. Then,

$$P(T_i(a_i) > t_i,\ T_j(a_j) > t_j) = E_\lambda(E(I_{f(T_i(a_i))} I_{g(T_i(a_i))}|\lambda))$$

$$\begin{aligned} &\leq E_\lambda(E(I_{f(S_i(a_i))}I_{g(S_i(a_i))}|\lambda)) \\ &= E(I_{f(S_i(a_i))}I_{g(S_i(a_i))}) \\ &= P(S_i(a_i) > t_i,\ S_j(a_j) > t_j). \end{aligned}$$

□

Next, we show that $APND$ ordering is preserved under transformation of increasing(decreasing) functions.

Theorem 3.4. *Let* $\{(X_i(t),\ Y_i(t))^{H_i}|i = 1, 2, ..., n\}$ *be* n *independent pairs from a bivariate distribution* $H_j,\ j = 1, 2$ *and increasing sample paths with coefficients* $q_1,\ q_2, ..., q_n$. *Suppose* H_1 *and* H_2 *such that* $H_1 >^{APND} H_2$.
If f_1 *and* f_2 *are concordant functions, then* $P_{H_1}(f_1(X_i(t)),\ f_2(Y_i(t)),\ i = 1, 2, ..., n)$ $>^{APND} P_{H_2}(f_1(X_i(t)),\ f_2(Y_i(t)),\ i = 1, 2, ..., n)$.

Proof. Let f_1 and f_2 concordant functions. To show that$(f_1(X_i(t)),\ f_2(Y_i(t)),$ $i = 1, 2, ..., n)$ are $APND$ for general n, the proof will be given for $n = 2$.
We introduce the random variables $V_1 = X_2(t_1),\ V_2 = Y_2(t_2),\ U_1 = sup_{0\leq s\leq t_1}$ $f_1(X_1(s),\ X_2(s))$ and $U_2 = sup_{0\leq s\leq t_2} f_2(Y_1(s),\ Y_2(s))$ for fixed $t_1,\ t_2 \geq 0$ and simplicity, $t_1,\ t_2$ have been suppressed in V_i and U_i, for $i = 1, 2$. Define the following any hitting times of $Z_1(s) = f_1(X_1(s),\ X_2(s)),\ Z_2(s) = f_2(Y_1(s),\ Y_2(s))$ by $W_1(a_1) = inf\{s : Z_1(s) \leq a_1\}$ and $W_2(a_2) = inf\{s : Z_2(s) \leq a_2\}$. Note the facts that $U_1 = sup_{0\leq s\leq t_1} f_1(X_1(s), V_1)$ and $U_2 = sup_{0\leq s\leq t_2} f_2(Y_1(s), V_2)$ and that V_1 and V_2 are $APND$ by assumptions. Thus,

$$\begin{aligned} & P(W_1(a_1) > t_1,\ W_2(a_2) > t_2) \\ =\ & EP(U_1 < a_1,\ U_2 < a_2|V_1, V_2) \\ \leq\ & (1 + q_1)E(P(U_1 < a_1|V_1)P(U_2 < a_2|V_2)) \\ \leq\ & (1 + q_1)EP(U_1 < a_1|V_1)EP(U_2 < a_2|V_2) \\ =\ & (1 + q_1)P(W_1(a_1) > t_1)P(W_2(a_2) > t_2). \end{aligned}$$

So, $(f_1(X_i(t)),\ f_2(Y_i(t)),\ i = 1, 2, ..., n)$ are $APND$ and note that $(X_i(t),\ Y_i(t))$ $\sim H,\ (X'_i(t), Y'_i(t)) \sim H',\ H >^{APND} H'$ and two ordered elements belong to β^+, then the corresponding elements in $\beta^+_{f_1,f_2}$ maintain the same order, and so $(f_1(X_i(t)),\ f_2(Y_i(t)))_H >^{APND} (f_1(X_i(t)),\ f_2(Y_i(t)))_{H'}$. Thus we can obtain that $P_{H_1}(f_1(X_i(t)),\ f_2(Y_i(t)),\ i = 1, 2, ..., n) >^{APND} P_{H_2}(f_1(X_i(t)),\ f_2(Y_i(t),$ $i = 1, 2, ..., n)$
In the following theorem we show that $APND$ ordering is preserved under limits in distributions. □

Theorem 3.5. *Suppose that* $H_n >^{APND} H'_n$ *such that* $H_n \to H,\ H'_n \to H'$ *weakly as* $n \to \infty$ *for every* n. *Then* $H >^{APND} H'$

Proof. For fixed $t_i, t_j \geq 0,\ a_i \in E_i,\ a_j \in E_j,\ i \neq j,\ i, j \geq 1$, define the following hitting times $T_i(a_i) = inf\{t : X_i(t) \leq a_i\}$ and $S_i(a_i) = inf\{t : Y_i(t) \leq a_i\}$, then

$$P_H(T_i(a_i) > t_i,\ T_j(a_j) > t_j)$$

$$\begin{aligned} &= \lim_{n\to\infty} P_{H_n}[inf\{t_i|X_{in}(t_i) \le a_i\} > t_i,\ inf\{t_j|X_{jn}(t_j) \le a_j\} > t_j] \\ &\le \lim_{n\to\infty} P_{H_n}[inf\{t_i|X_{in}(t_i) \le a_i\} > t_i,\ inf\{t_j|X_{jn}(t_j) \le a_j\} > t_j] \\ &= P_{H^{\cdot}}(S_i(a_i) > t_i,\ S_j(a_j) > t_j) \end{aligned}$$

□

4. Examples and Applications

In this section we list some examples and applications.

Example 4.1. Suppose that $(X_{ni}(t),\ X_{nj}(t))^{H_n}$ and $(Y_{ni}(t),\ Y_{ni}(t))^{H'_n}$ are sequence of bivariate vector process, for $n \ge 1$, $i \ne j$, $i, j \ge 1$, and let $H_n >^{APND} H'_n$. Let H_n and H'_n converge weakly to another vector process $(X_i(t),\ X_j(t))^H$ and $(Y_i(t),\ Y_j(t))^{H'}$ respectively (with respect to any Skorohod(1956) topology as $n \to \infty$, and if $(X_{ni}(t),\ X_{nj}(t))^{H_n}$, $Y_{ni}(t),\ Y_{nj}(t))^{H'_n}$, $(X_i(t),\ X_j(t))^H$ and $(Y_i(t),\ Y_j(t))^{H'}$ have sample paths that are right-continuous on R_+ with finite left limits at all t, then, by Theorem 3.5, we have the limiting distribution $H >^{APND} H'$.

Example 4.2. Suppose that stochastic vector process $(X_1(t),\ X_2(t))$ be $APND$ with increasing sample paths and coefficients q_1, and let $Z(t)$ be independent and have increasing sample paths of $(X_1(t),\ X_2(t))$. If we define $X(t) = X_1(t) + \lambda_1 Z(t)$ and $Y(t) = X_2(t) + \lambda_2 Z(t)$, where $\lambda_1 \ge 0$, $\lambda_2 \le 0$, then $X_1(t) + \lambda_1 Z(t)$ is SI in $Z(t)$ and $X_2(t) + \lambda_2 Z(t)$ is SD in $Z(t)$. Therefore, since $(X(t), Y(t))$ given $Z(t)$ is $APND$, by Theorem 3.2, we can obtain that stochastic vector process $(X(t), Y(t))$ is $APND$.

Application 4.3. Consider a system with two components which is subjected to shocks. Let $N(t)$ be the number of shocks received by time t and let $\sum_{i=1}^{N(t)} X_i$, $\sum_{i=1}^{N(t)} Y_i$,$\sum_{i=0}^{N_1(t)} U_i$ and $\sum_{i=0}^{N_2(t)} V_i$ be total damages to components 1, 2, 3 and 4 by time t, respectively. If X_i, Y_i, U_i and V_i are damages to components 1, 2, 3 and 4 by shock i, respectively, then we can obtain Theorem 3.1.

Application 4.4. Consider a following bivariate vector process comes from the fact that a Brownian motion has continuous paths. Let $\{X_n, Y_n)|n \ge 1\}$ and $\{(V_n, W_n)|n \ge 1\}$ be a bivariate process such that $(X_i, Y_i) >^{APND} (V_i, W_i)$ for $i = 1, 2, ...$ Suppose that $\{(X_i, Y_i)|i \ge 1\}$ and $\{(U_i, V_i)|i \ge 1\}$ are independent bivariate processes respectively. Then, from the result given by Pitt (1982) about multivariate normal distribution, we can know that $\{(X_n, Y_n)|n \ge 1\}$ and $\{(V_n, W_n)|n \ge 1\}$ are $APND$ respectively and we can obtain that $\{(X_n, Y_n)|n \ge 1\} >^{APND} \{(V_n, W_n)|n \ge 1\}$.

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