J. Appl. Math. & Informatics Vol. **35***(2017), No. 5 - 6, pp. 529 - 542*
https://doi.org/10.14317/jami.2017.529

UNIQUENESS OF MEROMORPHIC FUNCTIONS CONCERNING DIFFERENTIAL POLYNOMIALS WITH REGARD TO MULTIPLICITY SHARING A SMALL FUNCTION

HARINA P. WAGHAMORE* AND SANGEETHA ANAND

ABSTRACT. In this paper, using the notion of weakly weighted sharing and relaxed weighted sharing, we investigate the uniqueness problems of certain differential polynomials sharing a small function. The results obtained in this paper extend the theorem obtained by Jianren Long [9].

AMS Mathematics Subject Classification : Primary 30D35.
Key words and phrases : Uniqueness, meromorphic functions, differential polynomials, multiplicity, weighted sharing.

1. Introduction

In this paper, we use the standard notations of Nevanlinna value distribution theory (see [4, 13, 14]). Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f - a$ and $g - a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities). If we do not consider multiplicities, then f and g are said to share the value a IM (ignoring multiplicities).

Let k be a positive integer or infinity. Set $E(a, f) = \{z : f(z) - a = 0\}$, where a zero with multiplicity k is counted k times. If the zeros are counted only once, then we denote the set by $\overline{E}(a, f)$. If $E(a, f) = E(a, g)$, then we say that f and g share the value a CM; If $\overline{E}(a, f) = \overline{E}(a, g)$, then we say that f and g share the value a IM. We denote by $E_{k)}(a, f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. Also, we denote by $\overline{E}_{k)}(a, f)$ the set of distinct a-points of f with multiplicities not exceeding k.

Received February 14, 2017. Revised June 10, 2017. Accepted June 21, 2017. *Corresponding author.

In 1997, Yang and Hua [12] obtained the following uniqueness theorem.

Theorem A. Let f and g be two non-constant entire (meromorphic) functions, and let $n \geq 6$ ($n \geq 11$) be a positive integer. If $f^n f^{'}$ and $g^n g^{'}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.

In 2008, Zhang et al. [16] considered some general differential polynomials and obtained the following result.

Theorem B. Let f and g be two non-constant meromorphic functions, and $h(\not\equiv 0, \infty)$ be a small function with respect to f and g. Let $n,\ k$ and m be three positive integers with $n > 3k+m+8$ and $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$, where $a_0 \neq 0, a_1, ..., a_{m-1}, a_m \neq 0$ are complex constants. If $(f^n P(f))^{(k)}$ and $(g^n P(g))^{(k)}$ share $h(z)$ CM, then one of the following three cases hold:
(i) $f = tg$ for a constant t such that $t^d = 1$, where $d = GCD(n+m, ..., n+m-i, ..., n), a_{m-i} \neq 0$ for some $i = 0, 1, ..., m$;
(ii) f and g satisfying the algebraic function equation $R(f, g) = 0$, where $R(w_1, w_2) = w_1^n(a_m w_1^m + a_{m-1} w_1^{m-1} + ... + a_0) - w_2^n(a_m w_2^m + a_{m-1} w_2^{m-1} + ... + a_0)$;
(iii) $(f^n P(f))^{(k)} (g^n P(g))^{(k)} = h^2$.

In 2013, Bhoosnurmath and Kabbur [3] extended Theorem B and proved the following uniqueness theorem by using the concept of multiplicity.

Theorem C. Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let n and m be positive integers with $(n-m-1)s \geq max\{10, 2m+3\}$ and let $P(z)$ be defined as in Theorem B. If $f^n P(f) f^{'}$ and $g^n P(g) g^{'}$ share 1 CM, then either $f = tg$ for a constant t such that $t^d = 1$, where $d = GCD(n+m+1, ..., n+m+1-i, ..., n+1), a_{m-i} \neq 0$ for some $i = 0, 1, ..., m$ or f and g satisfy the algebraic function equation $R(f, g) = 0$, where $R(x, y) = x^{n+1}(\frac{a_m}{n+m+1} x^m + \frac{a_{m-1}}{n+m} x^{m-1} + ... + \frac{a_0}{n+1}) - y^{n+1}(\frac{a_m}{n+m+1} y^m + \frac{a_{m-1}}{n+m} y^{m-1} + ... + \frac{a_0}{n+1})$.

Recently, J. R. Long [9] generalised Theorem C by proving the following result.

Theorem D. Let f and g be two transcendental meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let n and m be two positive integers with $n - m > max\{2 + \frac{2m}{s}, \frac{(n+2)(k+4)}{ns}\}$, $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$ and let $P(z)$ be defined as in Theorem B. If $(f^n P(f))^{(k)}$ and $(g^n P(g))^{(k)}$ share $h(z)$ CM, where $h(z)(\not\equiv 0, \infty)$ is a small function of f and g, then one of the following three cases hold:
(i) $(f^n P(f))^{(k)} (g^n P(g))^{(k)} = h^2$;
(ii) $f = tg$ for a constant t such that $t^d = 1$, where $d = GCD(n+m, ..., n+m-$

$i, ..., n), a_{m-i} \neq 0$ for some $i = 0, 1, ..., m$;
(iii) f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(f,g) = f^nP(f) - g^nP(g)$.
The possibility $(f^nP(f))^{(k)}(g^nP(g))^{(k)} = h^2$ does not occur for $k = 1$.

To state our main results of this article, we need the following definitions.

Definition 1.1 ([6]). Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of f. For a positive integer k we denote by $N(r, a; f \mid\leq k)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than k. By $\overline{N}(r, a; f \mid\leq k)$ we denote the corresponding reduced counting function. Analogously, we can define $N(r, a; f \mid\geq k)$ and $\overline{N}(r, a; f \mid\geq k)$.

Definition 1.2 ([5]). Let k be a positive integer or infinity. We denote by $N_k(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \cdots + \overline{N}(r, a; f \mid\geq k).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 1.3 ([10]). We denote by $N_E(r, a; f, g)$ $(\overline{N}_E(r, a; f, g))$ the counting function (reduced counting function) of all common zeros of $f - a$ and $g - a$ with the same multiplicities and by $N_0(r, a; f, g)$ $(\overline{N}_0(r, a; f, g))$ the counting function (reduced counting function) of all common zeros of $f - a$ and $g - a$ ignoring multiplicities. If

$$\overline{N}\,(r, a; f) + \overline{N}\,(r, a; g) - 2\overline{N}_E(r, a; f, g) = S(r, f) + S(r, g),$$

then we say that f and g share the value a "CM".
If

$$\overline{N}\,(r, a; f) + \overline{N}\,(r, a; g) - 2\overline{N}_0(r, a; f, g) = S(r, f) + S(r, g),$$

then we say that f and g share the value a "IM".

Definition 1.4 ([7]). Let f and g share the value a "IM" and k be a positive integer or infinity. $\overline{N}^{E}_{k)}(r, a; f, g)$ denotes the reduced counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, and both of their multiplicities are not greater than k. $\overline{N}^{0}_{(k}(r, a; f, g)$ denotes the reduced counting function of those a-points of f which are a-points of g, and both of their multiplicities are not less than k.

Definition 1.5 ([7]). For $a \in \mathbb{C} \cup \{\infty\}$, if k is a positive integer or infinity and

$$\overline{N}(r, a; f \mid\leq k) - \overline{N}^{E}_{k)}(r, a; f, g) = S(r, f),$$
$$\overline{N}(r, a; g \mid\leq k) - \overline{N}^{E}_{k)}(r, a; f, g) = S(r, g),$$
$$\overline{N}(r, a; f \mid\geq k+1) - \overline{N}^{0}_{(k+1}(r, a; f, g) = S(r, f),$$

$$\overline{N}(r,a;g\mid\geq k+1)-\overline{N}^{0}_{(k+1}(r,a;f,g)=S(r,g),$$

or if $k=0$ and
$\overline{N}(r,a;f)-\overline{N}_0(r,a;f,g)=S(r,f),\quad \overline{N}(r,a;g)-\overline{N}_0(r,a;f,g)=S(r,g),$
then we say that f and g share the value a weakly with weight k and we write f and g share "(a,k)".

Definition 1.6 ([2])**.** We denote by $\overline{N}(r,a;f\mid=p;g\mid=q)$ the reduced counting function of common a-points of f and g with multiplicities p and q respectively.

Definition 1.7 ([2])**.** Let f, g share a "IM". Also let k be a positive integer or infinity and $a\in\mathbb{C}\cup\{\infty\}$. If for $p\neq q$,

$$\sum_{p,q\leq k}\overline{N}(r,a;f\mid=p;g\mid=q)=S(r),$$

then we say that f and g share the value a with weight k in a relaxed manner. Here we write f and g share $(a,k)^*$ to mean that f and g share a with weight k in a relaxed manner.

2. Lemmas

In this section, we present some lemmas which will be needed in the sequel. We denote by H the following function:

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right), \tag{1}$$

where F and G are non-constant meromorphic functions defined in the complex plane $\mathbb{C}$.

Lemma 2.1 (see [15])**.** *Let f be a non-constant meromorphic function and p, k be positive integers, then*

$$N_p\left(r,\frac{1}{f^{(k)}}\right)\leq T(r,f^{(k)})-T(r,f)+N_{p+k}\left(r,\frac{1}{f}\right)+S(r,f), \tag{2}$$

$$N_p\left(r,\frac{1}{f^{(k)}}\right)\leq k\overline{N}(r,f)+N_{p+k}\left(r,\frac{1}{f}\right)+S(r,f). \tag{3}$$

Lemma 2.2 (see [11])**.** *Let f be a non-constant meromorphic function, let $P_n(f)=\sum\limits_{j=0}^{n}a_jf^j$ be a polynomial in f, where $a_n\neq 0,a_{n-1},...,a_1,a_0\neq 0$ are complex constants satisfying $T(r,a_j)=S(r,f)$, then*

$$T(r,P_n)=nT(r,f)+S(r,f).$$

Lemma 2.3 (see [9])**.** *Let f and g be two non-constant meromorphic functions such that $\Theta(\infty,f)+\Theta(\infty,g)>\dfrac{4}{n}$ for all integer $n\geq 3$, then $f^n(af+b)=g^n(ag+b)$ implies $f=g$, where a and b are two finite non-zero complex constants.*

Lemma 2.4. *Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer and let n, k and m be three positive integers. Let $F=(f^nP(f))^{(k)}$ and $G=(g^nP(g))^{(k)}$, where $P(z)$ be defined as in Theorem B. If there exists two non-zero constants b_1 and b_2 such that $\overline{N}(r,\frac{1}{F})=\overline{N}(r,\frac{1}{G-b_1})$ and $\overline{N}(r,\frac{1}{G})=\overline{N}(r,\frac{1}{F-b_2})$, then $n\leq\frac{3k+3}{s}+m$ when $m\leq k+1$ and $n\geq\frac{5k+5}{s}-m$ when $m>k+1$.*

Proof. By the second fundamental theorem of Nevanlinna theory, we have

$$\begin{aligned}T(r,F)&\leq\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F-b_2}\right)+S(r,F)\\&\leq\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,F).\end{aligned}\tag{4}$$

Combining (2), (3), (4) and Lemma 2.2, we get

$$\begin{aligned}(n+m)\,T(r,f)&\leq T(r,F)-\overline{N}\left(r,\frac{1}{F}\right)+N_{k+1}\left(r,\frac{1}{f^nP(f)}\right)+S(r,f)\\&\leq\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f^nP(f)}\right)+S(r,f)\\&\leq N_{k+1}\left(r,\frac{1}{f^nP(f)}\right)+N_{k+1}\left(r,\frac{1}{g^nP(g)}\right)+\overline{N}(r,f)\\&\quad+k\overline{N}(r,g)+S(r,f)+S(r,g).\end{aligned}\tag{5}$$

When $m\leq k+1$, then from (5), we have

$$\begin{aligned}(n+m)\,T(r,f)&\leq\left(\frac{k+1}{s}+m\right)T(r,f)+\left(\frac{k+1}{s}+m\right)T(r,g)+\frac{1}{s}T(r,f)\\&\quad+\frac{k}{s}T(r,g)+S(r,f)+S(r,g)\\&\leq\left(\frac{k+2}{s}+m\right)T(r,f)+\left(\frac{2k+1}{s}+m\right)T(r,g)\\&\quad+S(r,f)+S(r,g).\end{aligned}\tag{6}$$

Similarly,

$$\begin{aligned}(n+m)\,T(r,g)&\leq\left(\frac{k+2}{s}+m\right)T(r,g)+\left(\frac{2k+1}{s}+m\right)T(r,f)\\&\quad+S(r,f)+S(r,g).\end{aligned}\tag{7}$$

Combining (6) and (7), we get

$$\begin{aligned}(n+m)(T(r,f)+T(r,g))&\leq\left(\frac{3k+3}{s}+2m\right)(T(r,f)+T(r,g))\\&\quad+S(r,f)+S(r,g),\end{aligned}$$

which gives $n \leq \dfrac{3k+3}{s} + m$.
When $m > k+1$, then from (5), we have

$$\begin{aligned}(n+m)\,T(r,f) &\leq \left(\frac{2(k+1)}{s}\right)T(r,f) + \left(\frac{2(k+1)}{s}\right)T(r,g) + \frac{1}{s}T(r,f)\\ &\quad + \frac{k}{s}T(r,g) + S(r,f) + S(r,g)\\ &\leq \left(\frac{2k+3}{s}\right)T(r,f) + \left(\frac{3k+2}{s}\right)T(r,g)\\ &\quad + S(r,f) + S(r,g). \end{aligned} \tag{8}$$

Similarly,

$$\begin{aligned}(n+m)\,T(r,g) &\leq \left(\frac{2k+3}{s}\right)T(r,g) + \left(\frac{3k+2}{s}\right)T(r,f)\\ &\quad + S(r,f) + S(r,g). \end{aligned} \tag{9}$$

Combining (8) and (9), we get

$$(n+m)(T(r,f)+T(r,g)) \leq \left(\frac{5k+5}{s}\right)(T(r,f)+T(r,g)) + S(r,f) + S(r,g),$$

which gives $n \leq \dfrac{5k+5}{s} - m$. This proves the lemma. □

Lemma 2.5 (see [9]). *Let f and g be two transcendental meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let $P(z)$ be defined as in Theorem B and n, m and k be three positive integers and $\alpha(z)(\not\equiv 0,\ \infty)$ be a small function of f and g, then $(f^nP(f))^{(k)}(g^nP(g))^{(k)} \not\equiv \alpha^2$ holds for $k=1$ and $(n+m-2)p > 2m(1+\frac{1}{s})$, where p is the number of distinct roots of $P(z)=0$.*

Lemma 2.6 (see [2]). *Let F and G be non-constant meromorphic functions that share "(1,2)" and $H \not\equiv 0$, then*

$$\begin{aligned}T(r,F) &\leq N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G) - \sum_{p=3}^{\infty}\overline{N}_{(p}\left(r,\frac{G}{G'}\right)\\ &\quad + S(r,F) + S(r,G),\end{aligned}$$

and the same inequality hold for $T(r,G)$.

Lemma 2.7 (see [2]). *Let F and G be non-constant meromorphic functions that share $(1,2)^*$ and $H \not\equiv 0$, then*

$$\begin{aligned}T(r,F) &\leq N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G)\\ &\quad + \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) - m\left(r,\frac{1}{G-1}\right) + S(r,F) + S(r,G),\end{aligned}$$

and the same inequality hold for $T(r,G)$.

Lemma 2.8 (see [1])**.** *Let* F *and* G *be non-constant meromorphic functions. If* $\overline{E}_{4)}(1,F)=\overline{E}_{4)}(1,G)$, $\overline{E}_{2)}(1,F)=\overline{E}_{2)}(1,G)$ *and* $H\not\equiv 0$, *then*

$$T(r,F)+T(r,G)\le 2\left\{N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right\} + S(r,F)+S(r,G).$$

Now the following question is inevitable, which is the motivation of the paper: Is it possible to relax the nature of sharing the small function in Theorem D? Considering this question, we prove the following results.

3. Main results

Theorem 3.1. *Let* f *and* g *be two transcendental meromorphic functions, whose zeros and poles are of multiplicities atleast* s, *where* s *is a positive integer and* $\alpha(z)(\not\equiv 0)$ *be a small function of* f *and* g. *Let* $P(z)$ *be defined as in Theorem B and* $\Theta(\infty,f)+\Theta(\infty,g)>\frac{4}{n}$. *Let* n, m *and* k *be three positive integers satisfying* $n-m>max\left\{2+\frac{2m}{s},\frac{3k+8}{s}\right\}$ *when* $m\le k+1$ *and* $n+m> max\left\{2+\frac{2m}{s},\frac{5k+12}{s}\right\}$ *when* $m>k+1$. *If* $(f^nP(f))^{(k)}$ *and* $(g^nP(g))^{(k)}$ *share "*$(\alpha(z),2)$*", then one of the following three cases hold:*

(i) $(f^nP(f))^{(k)}(g^nP(g))^{(k)}=\alpha^2$ *for* $k\neq 1$;

(ii) $f=tg$ *for a constant* t *such that* $t^d=1$, *where* $d=GCD(n+m,...,n+m-i,...,n), a_{m-i}\neq 0$ *for some* $i=0,1,...,m$;

(iii) f *and* g *satisfy the algebraic equation* $R(f,g)=0$, *where* $R(f,g)=f^nP(f)-g^nP(g)$.

Proof. Let $F=\frac{(f^nP(f))^{(k)}}{\alpha(z)}$ and $G=\frac{(g^nP(g))^{(k)}}{\alpha(z)}$. Then F and G are transcendental meromorphic functions that share "$(1,2)$" except the zeros and poles of $\alpha(z)$.
Suppose that $H\not\equiv 0$.
Using (2) and Lemma 2.2, we get

$$\begin{aligned}N_2\left(r,\frac{1}{F}\right)&\le N_2\left(r,\frac{1}{(f^nP(f))^{(k)}}\right)+S(r,f)\\&\le T(r,F)-(n+m)T(r,f)+N_{k+2}\left(r,\frac{1}{f^nP(f)}\right)+S(r,f).\qquad(10)\end{aligned}$$

Using (3), we deduce that

$$N_2\left(r,\frac{1}{F}\right)\le k\overline{N}(r,(f^nP(f))^{(k)})+N_{k+2}\left(r,\frac{1}{f^nP(f)}\right)+S(r,f)$$

$$\leq k\overline{N}(r,f) + N_{k+2}\left(r, \frac{1}{f^nP(f)}\right) + S(r,f). \tag{11}$$

From (10), we have

$$(n+m)T(r,f) \leq T(r,F) + N_{k+2}\left(r, \frac{1}{f^nP(f)}\right) - N_2\left(r, \frac{1}{F}\right) + S(r,f). \tag{12}$$

By using (12) and Lemma 2.6, we get

$$\begin{aligned}(n+m)T(r,f) \leq{} & N_2(r,F) + N_2(r,G) + N_2\left(r, \frac{1}{G}\right) + N_{k+2}\left(r, \frac{1}{f^nP(f)}\right)\\ & + S(r,f) + S(r,g).\end{aligned} \tag{13}$$

We suppose that $m \leq k+1$. Then from (13), we get

$$\begin{aligned}(n+m)T(r,f) \leq{} & \left(\frac{k+4}{s} + m\right)T(r,f) + \left(\frac{2k+4}{s} + m\right)T(r,g)\\ & + S(r,f) + S(r,g).\end{aligned} \tag{14}$$

Similarly,

$$\begin{aligned}(n+m)T(r,g) \leq{} & \left(\frac{k+4}{s} + m\right)T(r,g) + \left(\frac{2k+4}{s} + m\right)T(r,f)\\ & + S(r,f) + S(r,g).\end{aligned} \tag{15}$$

From (14) and (15) together, we get

$$\left(n - m - \frac{3k+8}{s}\right)(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g),$$

a contradiction to our assumption that $n - m > max\left\{2 + \frac{2m}{s}, \frac{3k+8}{s}\right\}$.
Next we assume that $m > k+1$. Then from (13), we get

$$\begin{aligned}(n+m)T(r,f) \leq{} & \left(\frac{2k+6}{s}\right)T(r,f) + \left(\frac{3k+6}{s}\right)T(r,g)\\ & + S(r,f) + S(r,g).\end{aligned} \tag{16}$$

Similarly,

$$\begin{aligned}(n+m)T(r,g) \leq{} & \left(\frac{2k+6}{s}\right)T(r,g) + \left(\frac{3k+6}{s}\right)T(r,f)\\ & + S(r,f) + S(r,g).\end{aligned} \tag{17}$$

From (16) and (17) together, we get

$$\left(n + m - \frac{5k+12}{s}\right)(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g),$$

a contradiction to our assumption that $n+m > max\left\{2+\frac{2m}{s}, \frac{5k+12}{s}\right\}$.
Therefore, we must have $H=0$. Then

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating both sides twice, we get

$$\frac{1}{F-1} = \frac{a}{G-1} + b, \tag{18}$$

where $a(\neq 0)$ and b are constants. From (18), it is clear that F and G share 1 CM and hence they share "(1, 2)". Therefore, $n-m > max\left\{2+\frac{2m}{s}, \frac{3k+8}{s}\right\}$ when $m \leq k+1$ and $n+m > max\left\{2+\frac{2m}{s}, \frac{5k+12}{s}\right\}$ when $m > k+1$. We now discuss the following cases separately.

Case 1. Let $b \neq 0$, and $a = b$. Then from (18), we get

$$\frac{1}{F-1} = \frac{bG}{G-1}. \tag{19}$$

If $b = -1$, then from (19), we obtain $FG = 1$.
Then $(f^nP(f))^{(k)}(g^nP(g))^{(k)} = \alpha^2$.
This is a contradiction when $k = 1$ by Lemma 2.5.
If $b \neq -1$, from (19), we have $\frac{1}{F} = \frac{bG}{(1+b)G-1}$, hence $\overline{N}\left(r, \frac{1}{G-1/(1+b)}\right) = \overline{N}(r, \frac{1}{F})$.
Using (2), (3), Lemma 2.2 and the second fundamental theorem of Nevanlinna, we deduce that

$$\begin{aligned} T(r,G) &\leq \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G-1/(1+b)}\right) + \overline{N}(r,G) + S(r,G) \\ &\leq \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}(r,G) + S(r,G). \end{aligned}$$

Hence,

$$\begin{aligned} (n+m)T(r,g) &\leq \overline{N}(r, \frac{1}{F}) + \overline{N}(r,G) + N_{k+1}\left(r, \frac{1}{g^nP(g)}\right) + S(r,g) \\ &\leq k\overline{N}(r,f) + N_{k+1}\left(r, \frac{1}{f^nP(f)}\right) + \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g^nP(g)}\right) \\ &\quad + S(r,g). \end{aligned} \tag{20}$$

If $m \leq k+1$, then from (20), we get

$$\begin{aligned} (n+m)T(r,g) \leq &\left(\frac{2k+1}{s} + m\right)T(r,f) + \left(\frac{k+2}{s} + m\right)T(r,g) \\ &+ S(r,f) + S(r,g). \end{aligned} \tag{21}$$

Similarly,

$$(n+m)T(r,f) \leq \left(\frac{2k+1}{s}+m\right)T(r,g)+\left(\frac{k+2}{s}+m\right)T(r,f)$$
$$+S(r,f)+S(r,g). \tag{22}$$

From (21) and (22) together, we get

$$\left(n-m-\frac{3k+3}{s}\right)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g),$$

a contradiction since $n-m > \dfrac{3k+8}{s}$.

Similarly, if $m > k+1$, then from (20), we get

$$\left(n+m-\frac{5k+5}{s}\right)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g),$$

a contradiction since $n+m > \dfrac{5k+12}{s}$.

Case 2. Let $b \neq 0$ and $a \neq b$. Then from (18), we have $F = \dfrac{(b+1)G-(b-a+1)}{bG+(a-b)}$ and hence

$$\overline{N}\left(r, \frac{1}{G-\frac{b-a+1}{b+1}}\right) = \overline{N}\left(r, \frac{1}{F}\right).$$

Proceeding in a manner similar to case 1, we get a contradiction.

Case 3. Let $b=0$ and $a \neq 0$. Then from (18), we have $F = \dfrac{G+a-1}{a}$ and $G = aF-(a-1)$.

If $a \neq 1$, it follows that

$$\overline{N}\left(r, \frac{1}{F-\frac{a-1}{a}}\right) = \overline{N}\left(r, \frac{1}{G}\right) \text{ and } \overline{N}\left(r, \frac{1}{G-(1-a)}\right) = \overline{N}(r, \frac{1}{F}).$$

By applying Lemma 2.4, we arrive at a contradiction. Therefore $a=1$ and hence $F=G$.

Hence, $(f^nP(f))^{(k)} = (g^nP(g))^{(k)}$.

By integration, we get

$(f^nP(f))^{(k-1)} = (g^nP(g))^{(k-1)} + c_{k-1}$, where c_{k-1} is a constant. If $c_{k-1} \neq 0$, by Lemma 2.4, it follows that

$n-m \leq \dfrac{3k}{s} < \dfrac{3k+3}{s}$ when $m \leq k+1$

and $n+m \leq \dfrac{5k}{s} < \dfrac{5k+5}{s}$ when $m > k+1$, a contradiction to the hypothesis.

Hence, $c_{k-1} = 0$.
Repeating the same process $k-1$ times, we get

$$f^n P(f) = g^n P(g). \tag{23}$$

If $m = 1$ in (23), then we get $f = g$ by using Lemma 2.3.
Suppose that $m \geq 2$ and $b = \frac{f}{g}$.
If b is a constant, then substituting $f = bh$ in (23), we get

$$a_m g^{n+m}(b^{n+m} - 1) + a_{m-1} g^{n+m-1}(b^{n+m-1} - 1) + ... + a_0 g^n (b^n - 1) = 0,$$

which implies $b^d = 1$, where $d = GCD(n+m, ..., n+m-i, ..., n)$. Hence, $f = tg$ for a constant t such that $t^d = 1$, $d = GCD(n+m, ..., n+m-i, ..., n)$, $i = 0, 1, ..., m$.
If b is not constant, then from (23), we find that f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(f,g) = f^n P(f) - g^n P(g)$.
This completes the proof of Theorem 3.1. □

Remark 3.1. When $m = 0$, $s = 1$ and $k = 1$ in Theorem 3.1, we get Theorem A.

Remark 3.2. When $s = 1$ in Theorem 3.1, we get Theorem B.

Theorem 3.2. *Let f and g be two transcendental meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer and $\alpha(z)(\not\equiv 0)$ be a small function of f and g. Let $P(z)$ be defined as in Theorem B and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$. Let n, m and k be three positive integers satisfying $n - 2m > max\left\{2 + \frac{2m}{s}, \frac{5k+10}{s}\right\}$ when $m \leq k+1$ and $n + m > max\left\{2 + \frac{2m}{s}, \frac{8k+15}{s}\right\}$ when $m > k+1$. If $(f^n P(f))^{(k)}$ and $(g^n P(g))^{(k)}$ share $(\alpha(z), 2)^*$, then conclusions of Theorem 3.1 hold.*

Proof. Let F and G be defined as in Theorem 3.1. Then F and G are transcendental meromorphic functions that share $(1,2)^*$ except the zeros and poles of $\alpha(z)$.
We suppose that $H \not\equiv 0$.
Using (3) and Lemma 2.7 in (10), we get

$$(n+m)T(r,f) \leq N_2\left(r, \frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + \overline{N}(r, \frac{1}{F}) + \overline{N}(r,F)$$
$$+ N_{k+2}\left(r, \frac{1}{f^n P(f)}\right) + S(r,f) + S(r,g). \tag{24}$$

Suppose that $m \leq k+1$, then from (24), we get

$$(n+m)T(r,f) \leq N_{k+2}\left(r, \frac{1}{g^n P(g)}\right) + k\overline{N}(r,g) + 2\overline{N}(r,f) + 2\overline{N}(r,g)$$

$$
\begin{aligned}
&+ N_{k+1}\left(r, \frac{1}{f^nP(f)}\right) + k\overline{N}(r,f) + \overline{N}(r,f)\\
&+ N_{k+2}\left(r, \frac{1}{f^nP(f)}\right) + S(r,f) + S(r,g)\\
&\leq \left(\frac{3k+6}{s} + 2m\right)T(r,f) + \left(\frac{2k+4}{s} + m\right)T(r,g)\\
&+ S(r,f) + S(r,g).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(n+m)T(r,g) \leq & \left(\frac{3k+6}{s} + 2m\right)T(r,g) + \left(\frac{2k+4}{s} + m\right)T(r,f)\\
& + S(r,f) + S(r,g).
\end{aligned}
$$

Hence,

$$
\left(n - 2m - \frac{5k+10}{s}\right)(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g),
$$

a contradiction to our assumption that $n - 2m > max\left\{2 + \frac{2m}{s}, \frac{5k+10}{s}\right\}$. Similarly, if $m > k+1$, then from (24), we get

$$
\left(n + m - \frac{8k+15}{s}\right)(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g),
$$

a contradiction to the fact that $n + m > max\left\{2 + \frac{2m}{s}, \frac{8k+15}{s}\right\}$. Thus, $H \equiv 0$ and rest of the theorem follows from the proof of Theorem 3.1. This completes the proof of Theorem 3.2. □

Theorem 3.3. *Let f and g be two transcendental meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer and $\alpha(z)(\not\equiv 0)$ be a small function of f and g. Let $P(z)$ be defined as in Theorem B and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$. Let n, m and k be three positive integers satisfying $n - m > max\left\{2 + \frac{2m}{s}, \frac{3k+8}{s}\right\}$ when $m \leq k+1$ and $n + m > max\left\{2 + \frac{2m}{s}, \frac{5k+12}{s}\right\}$ when $m > k+1$. If $\overline{E}_{4)}(\alpha(z), (f^nP(f))^{(k)}) = \overline{E}_{4)}(\alpha(z), (g^nP(g))^{(k)})$ and $\overline{E}_{2)}(\alpha(z), (f^nP(f))^{(k)}) = \overline{E}_{2)}(\alpha(z), (g^nP(g))^{(k)})$, then the conclusions of Theorem 3.1 hold.*

Proof. Let F and G be defined as in Theorem 3.1. Then F and G are transcendental meromorphic functions such that $\overline{E}_{4)}(1, F) = \overline{E}_{4)}(1, G)$ and $\overline{E}_{2)}(1, F) = \overline{E}_{2)}(1, G)$ except for the zeros and poles of $\alpha(z)$. Let $H \not\equiv 0$.

Then by using (3), (10) and Lemma 2.8, we get

$$\begin{aligned}(n+m)(T(r,f)+T(r,g)) \leq\; & 2N_2(r,F)+2N_2(r,G)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)\\ & +N_{k+2}\left(r,\frac{1}{f^nP(f)}\right)+N_{k+2}\left(r,\frac{1}{g^nP(g)}\right)+S(r,F)\\ & +S(r,G). \qquad (25)\end{aligned}$$

Suppose that $m \leq k+1$, then from (25), we get

$$\begin{aligned}(n+m)(T(r,f)+T(r,g)) \leq & \left(\frac{3k+8}{s}+2m\right)(T(r,f)\\ & +T(r,g))+S(r,f)+S(r,g). \qquad (26)\end{aligned}$$

Hence,

$$\left(n-m-\frac{3k+8}{s}\right)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g),$$

a contradiction to our assumption that $n-m > max\left\{2+\frac{2m}{s}, \frac{3k+8}{s}\right\}$.
Similarly, if $m > k+1$, then from (25), we get

$$\left(n+m-\frac{5k+12}{s}\right)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g),$$

a contradiction to the fact that $n+m > max\left\{2+\frac{2m}{s}, \frac{5k+12}{s}\right\}$.
Thus, $H \equiv 0$ and rest of the theorem follows from the proof of Theorem 3.1. This completes the proof of Theorem 3.3. □

References

1. A. Banerjee, *On uniqueness of meromorphic functions when two differential monomials share one value*, Bull. Korean Math. Soc. **44** (2007), no. 4, 607–622.
2. A. Banerjee and S. Mukherjee, *Uniqueness of meromorphic functions concerning differential monomials sharing the same value*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **50** (2007), no. 3, 191-206.
3. S. S. Bhoosnurmath and S. R. Kabbur, *Uniqueness and value sharing of meromorphic functions with regard to multiplicity*, Tamkang J. Math. **44** (2013), no. 1, 11-22.
4. W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.
5. I. Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Variables Theory Appl. **46** (2001), no. 3, 241-253.
6. I. Lahiri, *Value distribution of certain differential polynomials*, Int. J. Math. Math. Sci. **28** (2001), no. 2, 83-91.
7. S. Lin and W. Lin, *Uniqueness of meromorphic functions concerning weakly weighted-sharing*, Kodai Math. J. **29** (2006), no. 2, 269-280.
8. X. Lin and W. Lin, *Uniqueness of entire functions sharing one value*, Acta Math. Sci. Ser. B Engl. Ed. **31** (2011), no. 3, 1062-1076.

9. J. Long, *Nonlinear differential polynomials of meromorphic functions with regard to multiplicity sharing a small function*, J. Comput. Anal. Appl. **22** (2017), no. 7, 1220-10230.
10. P. Sahoo and H. Karmakar, *Results on uniqueness of entire functions whose certain difference polynomials share a small function*, Anal. Math. **41** (2015) no. 4, 257-272.
11. C. C. Yang, *On deficiencies of differential polynomials. II*, Math. Z. **125** (1972), 107-112.
12. C.-C. Yang and X. Hua, *Uniqueness and value-sharing of meromorphic functions*, Ann. Acad. Sci. Fenn. Math. **22** (1997), no. 2, 395-406.
13. C.-C. Yang and H.-X. Yi, *Uniqueness theory of meromorphic functions*, Mathematics and its Applications, 557, Kluwer Acad. Publ., Dordrecht, 2003.
14. L. Yang, *Value distribution theory*, translated and revised from the 1982 Chinese original, Springer, Berlin, 1993.
15. J.-L. Zhang and L.-Z. Yang, *Some results related to a conjecture of R. Brück*, JIPAM. J. Inequal. Pure Appl. Math. **8** (2007), no. 1, Article 18, 11 pp.
16. X.-Y. Zhang, J.-F. Chen and W.-C. Lin, *Entire or meromorphic functions sharing one value*, Comput. Math. Appl. **56** (2008), no. 7, 1876-1883.

Harina P. Waghamore
Associate Professor, Department of Mathematics, Jnanabharathi Campus, Bangalore University, Bangalore-56.
e-mail:harinapw@gmail.com

Sangeetha Anand
Research Scholar, Department of Mathematics, Jnanabharathi Campus, Bangalore University, Bangalore-56.
e-mail:sangeetha.ads13@gmail.com