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A REPRESENTATION OF DEDEKIND SUMS WITH QUASI-PERIODICITY EULER FUNCTIONS†

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Abstract. In this paper, we shall provide several properties of Dedekind sums with quasi-periodicity Euler functions. In particular, we present a representation of these Dedekind sums in terms of the Eulerian functions and the tangent functions.

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1. Introduction

Denote by

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & (x \neq \text{integer}) \\ 0 & (x = \text{integer}). \end{cases} \tag{1}$$

Here, for $x \in \mathbb{R}$, $[x]$ denotes the greatest integer not exceeding x and $\{x\}$ denotes the fractional part of real number x, thus

$$\{x\} = x - [x]. \tag{2}$$

If h, k are coprime integers, then the classical Dedekind sum $s(h, k)$ is defined by

$$s(h,k) = \sum_{\mu=0}^{k-1} \left(\left(\frac{h\mu}{k} \right) \right) \left(\left(\frac{\mu}{k} \right) \right). \tag{3}$$

This sum was introduced by Dedekind [10] in 1892. From the transformation formula of Dedekind η-functions, he deduced the following reciprocity theorem

$$12hk\{s(h,k) + s(k,h)\} = h^2 - 3hk + k^2 + 1 \tag{4}$$

(see [1, p. 62, Theorem 3.7]). There are several generalizations of the classical Dedekind sum $s(h, k)$, some of them also satisfy reciprocity formulas, see [2, 3,

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4, 5, 6, 8, 9, 14, 18, 20, 21, 22] and the references therein. The first proof of (4) which doesn't employ the theory of Dedekind η-functions is due to Rademacher [17]. And a three term version of (1.3) was discussed by Rademacher in [18].

In [16], Mikolas obtained the reciprocity theorem for

$$S_{m,n}\begin{pmatrix} a & b \\ & c \end{pmatrix} = \sum_{k=0}^{c-1} \overline{B}_m\left(\frac{ka}{c}\right)\overline{B}_n\left(\frac{kb}{c}\right), \tag{5}$$

where $m,n = 0,1,2,\ldots$, and $(a,c) = (b,c) = 1, c > 0$. He established a large number of beautiful identities involving these sums. Here $\overline{B}_n(x)$ denotes the n-th Bernoulli function defined through the following Fourier expansions:

$$\overline{B}_n(x) = B_n(x - [x]) = -\frac{n!}{(2\pi i)^n}\sum_{k=-\infty, k\neq 0}^{\infty} \frac{e^{2\pi i k x}}{k^n} \tag{6}$$

for all real x if $n \geq 1$, and for $x \neq$ integer if $n = 1$, where $B_n(x)$ is the n-th Bernoulli polynomial. Note that, for $x \neq$ integer, $\overline{B}_1(x) = ((x))$.

Let $\overline{E}_n(x)$ be the n-th quasi-periodicity Euler function defined by [8, p. 661]

$$\overline{E}_n(x) = E_n(x)\ (0 \leq x < 1), \quad \overline{E}_n(x+1) = -\overline{E}_n(x), \tag{7}$$

where $E_n(x)$ denotes the Euler polynomials (see [8, 13, 15]). Thus for $x \in \mathbb{R}$ and $r \in \mathbb{Z}$, we have

$$\overline{E}_n(x) = (-1)^{[x]}\overline{E}_n(\{x\}), \quad \overline{E}_n(x+r) = (-1)^r\overline{E}_n(x). \tag{8}$$

The n-th Euler function $\overline{E}_n(x)$ has the following Fourier expansions (comparing with (6) above)

$$\overline{E}_n(x) = \frac{4n!}{\pi^{n+1}}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi x - \frac{1}{2}\pi n)}{(2k+1)^{n+1}}, \tag{9}$$

where $0 \leq x \leq 1$ if $n \in \mathbb{N}$ and $0 < x < 1$ if $n = 0$ (see [11, Lemma 2.1] and [20, Lemma 5]).

In this paper, we study a type of Dedekind sums analogue with (5) which is associated with the above quasi-periodic Euler functions. That is, in analogue with (5), we consider the following sums

$$T_{m,n}\begin{pmatrix} a & b \\ & c \end{pmatrix} = \sum_{k=0}^{c-1}(-1)^k\overline{E}_m\left(\frac{ka}{c}\right)\overline{E}_n\left(\frac{kb}{c}\right), \tag{10}$$

where $m,n = 0,1,2,\ldots$ with $(a,c) = (b,c) = 1, c > 0$ and $\overline{E}_n(x)$ is the n-th quasi-periodicity Euler function. We investigate their properties, and in particular we give a representation of these Dedekind sums in terms of the Eulerian functions and the tangent functions. It needs to mention that in 2016 Hu et al. [11] studied (10) in the case $n = b = 1$, and in 2017 Hu and Kim [12, Sec. 3] considered (10) under the p-adic situation where $b = 1$ and m replacing with $m - n + 1$.

2. Results

In what follows, x, y, z denote complex variables. We also denote

$$e(z) = e^{2\pi iz}.$$

Suppose $a, b, c \in \mathbb{Z}$ and $c > 0$. Put

$$T_c^{a,b}(x,y) = \sum_{k=0}^{c-1} (-1)^{k+\left[\frac{ka}{c}\right]+\left[\frac{kb}{c}\right]} e\left(\left\{\frac{ka}{c}\right\} x + \left\{\frac{kb}{c}\right\} y\right) \tag{11}$$

with $(a,c) = (b,c) = 1$. If c is odd and a, b have different parities, then this summation may extend over a complete residue system modulo c, and it is easy to see that

$$T_c^{a,b}(x,y) = T_c^{b,a}(y,x).$$

Proposition 2.1. *Let $a, b, c \in \mathbb{Z}$ and $(a,c) = (b,c) = 1$. Then we have*

$$T_c^{-a,b}(x,y) + e(x)\, T_c^{a,b}(-x,y) = 1 + e(x).$$

Proof. It is easily seen from (2) that

$$\{-u\} = \begin{cases} 0 & \text{if } u \in \mathbb{Z}, \\ 1 - \{u\} & \text{if } u \notin \mathbb{Z} \end{cases} \tag{12}$$

and

$$[u] + [-u] = \begin{cases} 0 & \text{if } u \in \mathbb{Z}, \\ -1 & \text{if } u \notin \mathbb{Z}. \end{cases} \tag{13}$$

By (11), (12) and (13) we may write

$$\begin{aligned} T_c^{-a,b}(x,y) &= \sum_{k=0}^{c-1} (-1)^{k+\left[-\frac{ka}{c}\right]+\left[\frac{kb}{c}\right]} e\left(\left\{-\frac{ka}{c}\right\} x + \left\{\frac{kb}{c}\right\} y\right) \\ &= 1 + \sum_{k=1}^{c-1} (-1)^{k+1+\left[\frac{ka}{c}\right]+\left[\frac{kb}{c}\right]} e\left(\left(1 - \left\{\frac{ka}{c}\right\}\right) x + \left\{\frac{kb}{c}\right\} y\right) \\ &= 1 - e(x) \sum_{k=1}^{c-1} (-1)^{k+\left[\frac{ka}{c}\right]+\left[\frac{kb}{c}\right]} e\left(\left\{\frac{ka}{c}\right\} (-x) + \left\{\frac{kb}{c}\right\} y\right) \\ &= 1 + e(x) - e(x)\, T_c^{a,b}(-x,y). \end{aligned}$$

This completes the proof. □

Define an auxiliary function

$$\mathfrak{F}_c^{a,b}(x,y) = [e(x) + 1]^{-1} [e(y) + 1]^{-1}\, T_c^{a,b}(x,y), \tag{14}$$

where $x, y \neq \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$. This function $\mathfrak{F}_c^{a,b}(x,y)$ has some trivial properties in analogue with its classical counterparts. For example, Proposition 2.1 implies

$$\mathfrak{F}_c^{-a,b}(x,y) + \mathfrak{F}_c^{a,b}(-x,y) = [e(y) + 1]^{-1}. \tag{15}$$

The relationship between $\mathfrak{F}_c^{a,b}$ and $T_{m,n}\binom{a\ b}{c}$ is indicated by the following proposition.

Proposition 2.2. *Let $a, b, c \in \mathbb{Z}$ and $(a,c) = (b,c) = 1$. Then we have*

$$\mathfrak{F}_c^{a,b}\left(\frac{x}{2\pi i}, \frac{y}{2\pi i}\right) = \frac{1}{4}\sum_{m,n=0}^{\infty} T_{m,n}\begin{pmatrix} a\ b \\ c \end{pmatrix}\frac{x^m y^n}{m!n!}.$$

Proof. In (14), replacing x by $x/2\pi i$ and y by $y/2\pi i$, from (8), (2), (10) and (11), we have

$$\begin{aligned}
\mathfrak{F}_c^{a,b}\left(\frac{x}{2\pi i}, \frac{y}{2\pi i}\right) &= \frac{1}{4}\sum_{k=0}^{c-1}(-1)^{k+\left[\frac{ka}{c}\right]+\left[\frac{kb}{c}\right]}\frac{2e^{\left\{\frac{ka}{c}\right\}x}}{e^x+1}\frac{2e^{\left\{\frac{kb}{c}\right\}x}}{e^y+1} \\
&= \frac{1}{4}\sum_{m,n=0}^{\infty}\frac{x^m y^n}{m!n!}\sum_{k=0}^{c-1}(-1)^{k+\left[\frac{ka}{c}\right]+\left[\frac{kb}{c}\right]}\overline{E}_m\left(\left\{\frac{ka}{c}\right\}\right)\overline{E}_n\left(\left\{\frac{kb}{c}\right\}\right) \\
&= \frac{1}{4}\sum_{m,n=0}^{\infty} T_{m,n}\begin{pmatrix} a\ b \\ c \end{pmatrix}\frac{x^m y^n}{m!n!}.
\end{aligned}$$

This completes the proof. □

Theorem 2.3. *Let $a, b, c \in \mathbb{Z}$ and $(a,c) = (b,c) = 1$. If c is a positive odd integer and a, b have different parities, then we have*

$$T_c^{a,b}(x,y) = \frac{1}{c}[e(x)+1][e(y)+1]\sum_{r=0}^{c-1}\left[e\left(\frac{x-br}{c}\right)+1\right]^{-1}\left[e\left(\frac{y+ar}{c}\right)+1\right]^{-1}.$$

By (14), the following corollary is an immediate consequence of the above theorem.

Corollary 2.4. *If c is a positive odd integer and a, b have different parities, then we have*

$$\mathfrak{F}_c^{a,b}(x,y) = \frac{1}{c}\sum_{r=0}^{c-1}\left[e\left(\frac{x-br}{c}\right)+1\right]^{-1}\left[e\left(\frac{y+ar}{c}\right)+1\right]^{-1},$$

where $a, b, c \in \mathbb{Z}$ and $(a,c) = (b,c) = 1$.

Note that (see [19, p. 18, Lemma 3])

$$\sum_{h=0}^{c-1}\cot\pi\left(z+\frac{h}{c}\right) = c\cot\pi\left(cz\right), \tag{16}$$

where z is not an integer. Since $\cot z = -\tan(z-\pi/2)$, from (16), we have

$$\sum_{h=0}^{c-1}\tan\pi\left(z+\frac{h}{c}\right) = c\tan\pi\left(cz+\frac{c}{2}-\frac{1}{2}\right). \tag{17}$$

It is clear from the definition that

$$[1+e(z)]^{-1} = \frac{1}{2}(1 - i\tan \pi z), \tag{18}$$

thus by Corollary 2.4, (17) and (18), we obtain the following result.

Corollary 2.5. *If c is a positive odd integer and a, b have different parities, then we have*

$$\begin{aligned}\mathfrak{F}_c^{a,b}(x,y) = \frac{1}{4}&\left[1 - i\left(\tan\pi\left(y+\frac{c}{2}-\frac{1}{2}\right)+\tan\pi\left(x-\frac{c}{2}+\frac{1}{2}\right)\right)\right]\\ &-\frac{1}{4c}\sum_{r=0}^{c-1}\tan\pi\left(\frac{x-br}{c}\right)\tan\pi\left(\frac{y+ar}{c}\right),\end{aligned}$$

where $a, b, c \in \mathbb{Z}$ and $(a,c) = (b,c) = 1$.

Proof of the Theorem 2.3. Let c be a positive odd integer. It is easy to see that

$$\begin{aligned}&\sum_{j=0}^{c-1}(-1)^j e\left(\frac{jx}{c}\right) = [e(x)+1]\left[e\left(\frac{x}{c}\right)+1\right]^{-1},\\ &\sum_{j=0}^{c-1}(-1)^j e\left(\frac{jx}{c}\right)e\left(\frac{j}{c}\right) = [e(x)+1]\left[e\left(\frac{x+1}{c}\right)+1\right]^{-1},\\ &\qquad\vdots\\ &\sum_{j=0}^{c-1}(-1)^j e\left(\frac{jx}{c}\right)e\left(\frac{j(c-1)}{c}\right) = [e(x)+1]\left[e\left(\frac{x+c-1}{c}\right)+1\right]^{-1}.\end{aligned} \tag{19}$$

For fixed $h = 0, 1, \ldots, c-1$, multiplying (19) by $e\left(-\frac{hj}{c}\right)$ for each $j = 0, 1, \ldots, c-1$, we have

$$\begin{aligned}&\sum_{j=0}^{c-1}(-1)^j e\left(\frac{jx}{c}\right)e\left(-\frac{0h}{c}\right) = [e(x)+1]\left[e\left(\frac{x}{c}\right)+1\right]^{-1}e\left(-\frac{0h}{c}\right),\\ &\sum_{j=0}^{c-1}(-1)^j e\left(\frac{jx}{c}\right)e\left(\frac{j}{c}\right)e\left(-\frac{1h}{c}\right)\\ &\qquad = [e(x)+1]\left[e\left(\frac{x+1}{c}\right)+1\right]^{-1}e\left(-\frac{1h}{c}\right),\\ &\qquad\vdots\\ &\sum_{j=0}^{c-1}(-1)^j e\left(\frac{jx}{c}\right)e\left(\frac{j(c-1)}{c}\right)e\left(-\frac{(c-1)h}{c}\right)\\ &\qquad = [e(x)+1]\left[e\left(\frac{x+c-1}{c}\right)+1\right]^{-1}e\left(-\frac{(c-1)h}{c}\right).\end{aligned} \tag{20}$$

Summing both sides of (20), we get

$$\begin{aligned}
&\sum_{j=0}^{c-1} e\left(-\frac{jh}{c}\right) + (-1)^1 e\left(\frac{x}{c}\right)\sum_{j=0}^{c-1} e\left(\frac{j(1-h)}{c}\right) \\
&+ (-1)^2 e\left(\frac{2x}{c}\right)\sum_{j=0}^{c-1} e\left(\frac{j(2-h)}{c}\right) \\
&+ \cdots + (-1)^{h-1} e\left(\frac{(h-1)x}{c}\right)\sum_{j=0}^{c-1} e\left(\frac{j(h-1-h)}{c}\right) \\
&+ (-1)^{h} e\left(\frac{hx}{c}\right)\sum_{j=0}^{c-1} 1 + (-1)^{h+1} e\left(\frac{(h+1)x}{c}\right)\sum_{j=0}^{c-1} e\left(\frac{j(h+1-h)}{c}\right) \\
&+ \cdots + (-1)^{c-1} e\left(\frac{(c-1)x}{c}\right)\sum_{j=0}^{c-1} e\left(\frac{j(c-1-h)}{c}\right) \\
&= [e(x)+1]\sum_{j=0}^{c-1}\left[e\left(\frac{x+j}{c}\right)+1\right]^{-1} e\left(-\frac{hj}{c}\right).
\end{aligned} \tag{21}$$

Since

$$\sum_{r \pmod c} e\left(\frac{r(m-\alpha)}{c}\right) = \begin{cases} c & \text{if } m \equiv \alpha \pmod c, \\ 0 & \text{if } m \not\equiv \alpha \pmod c, \end{cases}$$

the left-hand side of (21) gives

$$(-1)^h c e\left(\frac{hx}{c}\right).$$

Thus

$$e\left(\frac{hx}{c}\right) = (-1)^h \frac{1}{c}[e(x)+1]\sum_{j=0}^{c-1}\left[e\left(\frac{x+j}{c}\right)+1\right]^{-1} e\left(-\frac{hj}{c}\right), \tag{22}$$

where $h = 0, 1, \ldots, c-1$ and c is a positive odd integer.

Letting $\frac{h}{c} = \left\{\frac{ak}{c}\right\}$ with $(a,c)=1$ in (22), we have

$$e\left(\left\{\frac{ak}{c}\right\}x\right) = (-1)^{ak+\left[\frac{ak}{c}\right]}\frac{1}{c}[e(x)+1]\sum_{j=0}^{c-1}\left[e\left(\frac{x+j}{c}\right)+1\right]^{-1} e\left(-j\frac{ak}{c}\right), \tag{23}$$

where we have used the equality $-\frac{hj}{c} = -j\left\{\frac{ak}{c}\right\} = -j\frac{ak}{c} + j\left[\frac{ak}{c}\right]$ and the fact that $(-1)^h = (-1)^{c\left\{\frac{ak}{c}\right\}} = (-1)^{c\left(\frac{ak}{c}-\left[\frac{ak}{c}\right]\right)} = (-1)^{ak+\left[\frac{ak}{c}\right]}$ in the case c is a positive odd integer. Similarly, putting $\frac{h}{c} = \left\{\frac{bk}{c}\right\}$ with $(b,c)=1$ and replacing

x by y in (22), we get

$$e\left(\left\{\frac{bk}{c}\right\}y\right)=(-1)^{bk+\left[\frac{bk}{c}\right]}\frac{1}{c}\left[e\left(y\right)+1\right]\sum_{j=0}^{c-1}\left[e\left(\frac{y+j}{c}\right)+1\right]^{-1}e\left(-j\frac{bk}{c}\right). \tag{24}$$

Then, from (23) and (24), we have

$$\begin{aligned}T_c^{a,b}(x,y)&=\sum_{k=0}^{c-1}(-1)^{k+\left[\frac{ka}{c}\right]+\left[\frac{kb}{c}\right]}e\left(\left\{\frac{ka}{c}\right\}x+\left\{\frac{kb}{c}\right\}y\right)\\&=\frac{1}{c^2}\left[e\left(x\right)+1\right]\left[e\left(y\right)+1\right]\\&\times\sum_{p,q\pmod c}\left[e\left(\frac{x+p}{c}\right)+1\right]^{-1}\left[e\left(\frac{y+q}{c}\right)+1\right]^{-1}\\&\times\sum_{k=0}^{c-1}(-1)^{k(a+b+1)}e\left(-\frac{k(ap+bq)}{c}\right).\end{aligned}$$

Suppose a and b have different parities. If we consider the complete residue systems (mod c): $p=-br, q=a\rho$ $(r,\rho=0,1,\dots,c-1)$ and take into account that

$$\sum_{k=0}^{c-1}(-1)^{k(a+b+1)}e\left(-\frac{k(ap+bq)}{c}\right)=\sum_{k=0}^{c-1}e\left(-k\frac{ab(\rho-r)}{c}\right)$$

vanishes except for $\rho=r$ when it has value c, then we have

$$T_c^{a,b}(x,y)=\frac{1}{c}[e(x)+1][e(y)+1]\sum_{r=0}^{c-1}\left[e\left(\frac{x-br}{c}\right)+1\right]^{-1}\left[e\left(\frac{y+ar}{c}\right)+1\right]^{-1}.$$

This completes the proof. □

Theorem 2.6. *Let $a,b,c\in\mathbb{Z}$ and $(a,c)=(b,c)=1$. If c is a positive odd integer and a,b have different parities, then we have*

$$T_{m,n}\begin{pmatrix}a\ b\\c\end{pmatrix}=\frac{1}{c^{m+n+1}}\left[E_m(0)E_n(0)+4\sum_{r=1}^{c-1}\frac{H_m(-\eta^{br})H_n(-\eta^{-ar})}{(1+\eta^{-br})(1+\eta^{ar})}\right],$$

where $m,n=0,1,2,\dots$ and the Eulerian numbers $H_n(\eta^k)$ defined for a root of unity $\eta^k=e\left(\frac{k}{c}\right), c>1, c\nmid k$ is given by the following generating function

$$\frac{1-\eta^k}{e^z-\eta^k}=\sum_{n=0}^{\infty}H_n(\eta^k)\frac{z^n}{n!},\quad |z|<2\pi\left\{\frac{k}{c}\right\}. \tag{25}$$

Remark 2.1. In [7, (6.5)], Carlitz proved a similar result using different methods.

Proof of the Theorem 2.6. If replacing $\frac{x}{2\pi i}$ by x and $\frac{y}{2\pi i}$ by y in Corollary 2.4, then from usual simplifications, we find that

$$\mathfrak{F}_c^{a,b}\left(\frac{x}{2\pi i},\frac{y}{2\pi i}\right)=\frac{1}{c}\sum_{r=0}^{c-1}\frac{1}{\eta^{-br}e^{\frac{x}{c}}+1}\frac{1}{\eta^{ar}e^{\frac{y}{c}}+1} \tag{26}$$

By expanding the right-hand side of (26), we obtain

$$\begin{aligned}\mathfrak{F}_c^{a,b}\left(\frac{x}{2\pi i},\frac{y}{2\pi i}\right)&=\frac{1}{c}\frac{1}{e^{\frac{x}{c}}+1}\frac{1}{e^{\frac{y}{c}}+1}+\frac{1}{c}\sum_{r=1}^{c-1}\frac{\eta^{br}}{e^{\frac{x}{c}}+\eta^{br}}\frac{\eta^{-ar}}{e^{\frac{y}{c}}+\eta^{-ar}}\\&=\frac{1}{4c}\frac{2}{e^{\frac{x}{c}}+1}\frac{2}{e^{\frac{y}{c}}+1}\\&\quad+\frac{1}{c}\sum_{r=1}^{c-1}\frac{1}{1+\eta^{ar}}\frac{1}{1+\eta^{-br}}\frac{1+\eta^{br}}{e^{\frac{x}{c}}+\eta^{br}}\frac{1+\eta^{-ar}}{e^{\frac{y}{c}}+\eta^{-ar}}\\&=\sum_{m,n=0}^{\infty}\frac{x^ny^n}{m!n!}\frac{1}{c^{m+n+1}}\left[\frac{E_m(0)E_n(0)}{4}\right.\\&\quad\left.+\sum_{r=1}^{c-1}\frac{H_m(-\eta^{br})H_n(-\eta^{-ar})}{(1+\eta^{-br})(1+\eta^{ar})}\right].\end{aligned} \tag{27}$$

Finally comparing the above equality with Proposition 2.2, we have

$$\frac{1}{4}T_{m,n}\begin{pmatrix}a\ b\\c\end{pmatrix}=\frac{1}{c^{m+n+1}}\left[\frac{E_m(0)E_n(0)}{4}+\sum_{r=1}^{c-1}\frac{H_m(-\eta^{br})H_n(-\eta^{-ar})}{(1+\eta^{-br})(1+\eta^{ar})}\right], \tag{28}$$

where $m,n=0,1,\ldots$. □

Remark 2.2. If m and n have different parities with $m,n>0$, then $E_m(0)E_n(0)=0$, and Theorem 2.6 reduces to the following relation:

$$T_{m,n}\begin{pmatrix}a\ b\\c\end{pmatrix}=\frac{4}{c^{m+n+1}}\sum_{r=1}^{c-1}\frac{H_m(-\eta^{br})H_n(-\eta^{-ar})}{(1+\eta^{-br})(1+\eta^{ar})}.$$

In particular, for $m=n=1$ Theorem 2.6 becomes

$$T_{1,1}\begin{pmatrix}a\ b\\c\end{pmatrix}=\frac{1}{4c^3}+\frac{1}{c^3}\sum_{r=1}^{c-1}\frac{\eta^{(a-b)r}}{(1+\eta^{ar})^2(1+\eta^{-br})^2}.$$

A similar result for Bernoulli numbers has been discussed by Mikolas [16, (3.8)].

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