Siyuan Yujian in the Joseon Mathematics

朝鮮 算學의 四元玉鑑

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As is well known, the most important development in the history of Chinese mathematics is materialized in Song–Yuan era through tianyuanshu up to siyuanshu for constructing equations and zengcheng kaifangfa for solving them. There are only two authors in the period, Li Ye and Zhu Shijie who left works dealing with them. They were almost forgotten until the late 18th century in China but Zhu's Suanxue Qimeng(1299) had been a main reference for the Joseon mathematics. Commentary by Luo Shilin on Zhu's Siyuan Yujian(1303) was brought into Joseon in the mid–19th century which induced a great attention to Joseon mathematicians with a thorough understanding of Zhu's tianyuanshu. We discuss the history that Joseon mathematicians succeeded to obtain the mathematical structures of Siyuan Yujian based on the Zhu's tianyuanshu.

Keywords: Li Ye, Zhu Shijie, tianyuanshu, Siyuan Yujian, polynomial equations, systems of equations, Nam Byeong-gil, Lee Sang-hyeog, Jo Hui-sun, Sanhag Jeong-eui, Igsan, Sanhag Seub-yu.

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1   Introduction

Since Jiuzhang Suanshu (九章筭術) adopts the format of solving practical problems to reveal mathematical structures, it has been retained throughout the history of Chinese mathematics. In the fourth chapter Shaoguang (少廣), the extractions of square and cubic roots are not that practical problems compared with the problems of areas and volumes and eventually lead to the theory of equations, in particular the theory of solving equations. In this case, they don't have to construct equations. Linear equations are not introduced for they solve them simply by divisions and

hence they indicate them by numerators(實) and denominators (法). But using matrices, systems of linear equations are perfectly represented. Wang Xiaotong (王孝通, fl. 7th C.) dealt with general equations of higher degrees in his Jigu Suanjing (緝古筭經), which treats volumes of various constructions and granaries, and right triangles. The latter is the first case for solving right triangles, i.e., gougushu (勾股術) by equations. Wang just indicated the coefficients of equations. Since then, there have been not much development of Chinese mathematics until the 11th century. Jia Xian (賈憲, fl. 11th C.) introduced zengchengfa (增乘法) and then the method of solving equations, called zengcheng kaifangfa. Liu Yi (劉益, fl. 11th C.) also discussed the construction of equations and their solutions. But their works were lost and parts of them were quoted by Yang Hui (楊輝) in his Xiangjie Jiuzhang Suanfa (詳解九章算法, 1261) for Jia's contributions and Yang Hui Suanfa (楊輝算法, 1274–1275) for Liu's ones respectively. All equations in Yang's works are of the form $p(x) = a_0$, where $p(x)$ does not contain a constant term, or $x|p(x)$. Qin Jiushao (秦九韶, 1202–1261) also dealt with theory of equations in his Shushu Jiuzhang (數書九章, 1247) where his main concern is how to solve equations. Although his equations are of the form as that in Yang Hui Suanfa, he applied zengcheng kaifangfa to $p(x) - a_0 = 0$ which involves a much more natural procedure than that of Jia Xian and Liu Yi [7].

We now discuss the constructions of equations in Song–Yuan era. As works by Jia and Liu, mathematical works written before the 13th century were all lost. Although prefaces of Siyuan Yujian (四元玉鑑) mentioned books dealing with tianyuanshu (天元術), eryuanshu (二元術) and sanyuanshu (三元術), they are not transmitted to the present. There are only two authors, Li Ye (李冶, 1192–1279) and Zhu Shijie (朱世傑) who left works dealing with constructions of equations using tianyuanshu and up to siyuanshu (四元術).

Li Ye wrote Ceyuan Haijing (測圓海鏡, 1248, published in 1282) and Yigu Yanduan (益古演段, 1259). Li introduced tianyuanshu which represents rational polynomials, namely those with negative powers along with their operations in Ceyuan Haijing and constructed equations for the radius or diameter of the inscribed circle to a given right triangle under the given conditions of dimensions of its sub triangles. In Yigu Yanduan, Li also constructed quadratic equations and linear equations by tianyunashu for problems of Yiguji (益古集) of an anonymous author which deal with circles, rectangles, squares, and a trapezoid under given conditions like areas and their relative positions. Although Li did make a great contribution, if not greatest, to tianyuanshu in the above two books, they lack slightly for attracting the attentions of the later mathematicians because Ceyuan Haijing treats solely problems with the same solutions under so many geometrical observations and Yigu Yand-

uan is concerned with quadratic equations arising from plane figures.

Zhu Shijie also published two books, Suanxue Qimeng (算學啓蒙, 1299) and Si-yuan Yujian (四元玉鑑, 1303). Zhu introduced tianyuanshu at the last problem of the fourth chapter Fangcheng Zhengfumen (方程正負門) in the last book of Suan-xue Qimeng. He first solved a problem of a right triangle by the method in Jiu-zhang Suanshu and then by tianyuanshu in the problem. Contrasting two prob-lems he showed that the algebraic approach, namely tianyuanshu to gougushu is much more useful. Indeed, the traditional method uses geometrical consequences from the Pythagorean theorem but tianyuanshu method solely from the theorem. He then devoted to the theory of equations in the last chapter Kaifang Shisuomen (開方釋鎖門) with first 7 problems for solving equations and 27 problems for con-structing equations of quadratic up to the fifth degree by tianyuanshu. The latter deals with geometrical problems so that readers could easily construct equations in Jigu Suanjing by tianyuanshu. He introduced the terminology kaifangshi (開方式) for polynomial equations $p(x) = 0$ of any degree which clearly differentiates equa-tion $p(x) = 0$ with the polynomial $p(x)$.

Zhu Shijie published Siyuan Yujian just 4 years later after his Suanxue Qimeng. He deals with 288 problems to construct equations by tianyuanshu through siyuan-shu but he includes quite short explanations for one problem corresponding to the four methods, tianyuanshu up to siyuanshu in the introductory section. Jialing Sicao (假令四艸) and for the remaining 284 problems, Zhu indicated the choice of yuan and then the final equation for the chosen yuan. Since there are only 36 problems for eryuanshu, 13 ones for sanyuanshu and 7 ones for siyuanshu, the main body of the book treats tianyuanshu so that mathematicians in Joseon (朝鮮) and Japan would have no difficulty to decipher those 232 tianyuanshu problems for they have studied already it in Suanxue Qimeng. Mei Juecheng (梅瑴成, 1681–1763) claimed that tianyuanshu in Siyuan Yujian is the same with jiegenfang (借根方) of Shuli Jing-yun (數理精蘊, 1723) in his Chishui Yizhen (赤水遺珍, 1759). Thus, later commenta-tors like Li Rui (李銳, 1769–1817) and Luo Shilin (羅士琳, 1784–1853) have had some difficulty to understand fully tianyuanshu. Zhang Dunren (張敦仁, 1754–1834) com-pleted Jigu Suanjing Xicao (緝古算經細草, 1813) in 1801 solely based on tianyuanshu of Ceyuan Haijing to construct equations in Jigu Suanjing and Li Rui also completed Gougu Suanshu Xicao (句股算術細草) in 1807 to solve gougushu by tianyuanshu in-dependent of jiegenfang. He might be influenced by Zhang's Xicao and also dis-closed a much more advantage of tianyuanshu over the geometrical approach ini-tiated by Xu Guangqi (徐光啓, 1562–1633) in his Gouguyi (句股義).

Luo Shilin published his commentary, Siyuan Yujin Xicao (四元玉鑑細艸, 1835) [20] which became a main reference for the study of Siyuan Yujian in China. After

Xicao, he got a copy of Suanxue Qimeng republished in 1660 by Kim Si-jin (1618–1667) and published a corrected version of Suanxue Qimeng in 1839 and then wrote appendices to his Xicao. In the mid-19th century, Luo's Xicao was brought into Joseon presumably by Nam Byeong-gil (南秉吉, 1820–1869) and gave rise to great attractions of Joseon mathematicians and Siyuan Yujian became the last important topic of the traditional mathematics in Joseon as in China.

In this paper, we discuss first mathematical structures of Siyuan Yujian and the progress of initially accepting and then developing new structures of Siyuan Yujian in Joseon which took place in such a short period, about two decades. It is a very exceptional case in the history of Joseon mathematics.

For the Korean books and Chinese books included in [13] and [1, 12], respectively, they will not be numbered as an individual reference.

2 Mathematical Structures of Siyuan Yujian

We assume that readers are familiar with tianyuanshu. Indeed, it is a sequence of coefficients of a polynomial which are represented by calculating rods with a dual role, namely representation of a polynomial and a tool for their algebraic operations, sums, products, and divisions by a power of tianyuan.

We recall that for a *monic* polynomial equation $p(x) = 0$ with a polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ ($a_n = 1$), it generates a system of n equations

$x_1 + x_2 + \cdots + x_n = a_{n-1}$

$x_1 x_2 + x_1 x_3 + \cdots + x_{n-1} x_n = a_{n-2}$

$\cdots$

$x_1 x_2 \cdots x_{n-1} x_n = a_0$

and vice versa, because $p(x)$ has a linear factorization $\prod_{k=1}^{n}(x + x_k)$.

Liu Yi introduced general quadratic equations by the problem of a rectangle with a given area ab and sum or difference $|a - b|$ of its two sides but the equations are obtained by a geometrical approach as indicated in Tianmu Bilei Chengchu Jiefa (田畝比類乘除捷法, 1275) of Yang Hui Suanfa.

Chinese mathematics deals with word problems so that the equations should be derived by a system of equations as Liu Yi did. Before tianyuanshu was introduced, they should utilize all the related information to obtain a suitable polynomial equations from the given system of equations.

As mentioned above, the last two problems of Fangcheng Zhengfumen in Suanxue Qimeng are to find three sides of a right triangle with given gouxianhe (勾弦和, $a + c$) α and guxianhe (股弦和, $b + c$) β where a, b, c denote *throughout this paper* the three sides of a right triangle. This means to solve the system of equations:

$a + c = \alpha$; $b + c = \beta$; $a^2 + b^2 = c^2$.

Zhu took c as a tianyuan x and a, b are represented by $\alpha - x$, $\beta - x$, respectively. By the third equation, one has an equation $(\alpha - x)^2 + (\beta - x)^2 = x^2$ which gives rise to the equation $(\alpha - x)^2 + (\beta - x)^2 - x^2 = 0$ for xian.

We must point out that x in the above polynomials is not an *indeterminate* of modern polynomials but specifically *xian* in the given problem (see also [7]) and hence two sides indicate numbers. We also recall that Li Ye used *tongshu* (同數) to indicate the two sides being equal in his Ceyuan Haijing. In Yigu Yanduan, the given areas calculated with tianyuan are called xuji (虛積) in the first two problems and then ruiji (如積) or ruijishu (如積數) in the remaining problems

Zhu Shijie extended the terminology ruiji (如積) to arbitrary cases and then called the above processes ruiji qiuzhi (如積求之) in Siyuan Yujian. Thus tianyuanshu is a method to construct an equation from a system of equations by choosing a single unknown, tianyuan and then obtaining an equation of the form $p(x) = q(x)$, where $p(x), q(x)$ are *numbers* represented by polynomials. The polynomial forms $p(x), q(x)$ are obtained by algebraic operations of polynomials. We also point out that the equality sign = in Shuli Jingyun was not used for identities between polynomials but for equations as follows:

凡數有相等者用此號 ＝如二立方與十六相等 則此列之 二立方＝十六.

Because of these dual roles of $p(x), q(x)$, we take these $p(x), q(x)$ as polynomials for the convenience of readers in this paper.

We take the first problem, called Yiqi Hunyuan (一氣混元) of Siyuan Yujian as follows.

今有黃方乘直積得二十四步 只云股弦和九步 問勾幾何

The problem asks to find a from a system of equations:
$ab(a + b - c) = 24$; $b + c = 9$; $a^2 + b^2 = c^2$.

Zhu didn't explain anything but he included only the equation, $x^5 - 9x^4 - 81x^3 + 728x^2 = 162 \times 24$ as ruiji from the first equation in the system. By Luo Shilin's commentary, one has to have an identity $a^2 = c^2 - b^2 = (c + b)(c - b)$ to use the second and third equations of the system.

Furthermore, as the above two examples show, the equations in the given system of equations are always of the form $p = q$, where p, q may be *polynomials, rational expressions* or *irrational expressions* as those in a system of linear equations.

As tianyuan is denoted by x, the next three yuan's, diyuan (地元), renyuan (人元) and wuyuan (物元) will be denoted by y, z, u in this paper.

Yi Zhihan (易之瀚) completed Buzeng Zhuli (補增諸例) consisting of Kaifang Geli (開方各例) and Tianyuan Siyuan Geli (天元四元各例), which was included in Luo's Siyuan Yujian Xicao. In the sixth item of Tianyuan Siyuan Geli, Yi solved the above problem by eryuanshu as follows:

Choose a and b as tianyuan (x) and diyuan (y). Then $c = 9 - y$ from the second equation of the system and hence the first equation implies $x^2y + 2xy^2 - 9xy = 24$ as Ruiji so that one has the equation $x^2y + 2xy^2 - 9xy - 24 = 0$ which is called jinshi (今式), and similarly the equation $18y - x^2 - 81 = 0$, called yunshi (云式) from the third equation of the system. Using these two equations $p(x, y) = 0, q(x, y) = 0$, one has the same equation as above. In this case, one does not need to refer to the above identity derived from the Pythagorean theorem and factorization.

Incidentally, we choose b as tianyuan x and then one has the equation
$4x^5 - 36x^4 + 81x^3 + 48x^2 - 216x + 32 = 0$ for b with a solution 4 as in the above Suanxue Qimeng problem. This shows that a suitable choice of tianyuan is very important in tianyuanshu.

Inductively, choosing three (four, resp.) unknowns as tianyuan, diyuan, renyuan (wuyuan, resp.), one has a system of equations of the type $p(x, y, z) = 0$ ($p(x, y, z, u) = 0$, resp.), which give rise to an equation.

To represent these polynomials, Zhu Shijie used the Cartesian coordinate system (see [17, 20] for the detail).

We should mention the premise for Siyuan Yujian that every equation in Chinese mathematics must have *positive* solution because it deals with practical problems. In other words, tianyuan through siyuan are always assumed to be *positive*.

We now discuss the operating devices for eliminations to have an equation from a system of equations of the *form $p = 0$*.

Using the above premise and *properties of the zero* 0 in the field $\mathbb{Q}$ of rational numbers, Zhu Shijie recognized the following *propositions* where A, B denote polynomials in Siyuan Yujian, or those in tianyuanshu through siyuanshu:

1) For equations $A = 0$ and $B = 0$ and any rationals α and β, $\alpha A + \beta B = 0$.

In particular, we have $A + B = 0$ and $A - B = 0$

2) For an equation $A = 0$ and any *integer k*, $x^k A = 0$.

The same holds for $y^k A$, $z^k A$, $u^k A$.

3) For an equation $A = 0$ and any polynomial $p(x, y, z, u)$, $pA = 0$.

4) If $A + B = 0$, then $A^2 - B^2 = 0$.

5) If $y^k A_1(x) + B_1(x) = 0$, $y^k A_2(x) + B_2(x) = 0$, then $A_1(x)B_2(x) - A_2(x)B_1(x) = 0$.

The same holds for replacing y^k by z^k.

The proof for them is trivial and the first three propositions imply that the elimination process for a system of linear equations can be applied to the elimination for that of polynomial equations. In the representation of polynomial equations in Siyuan Yujian, its column which is also a polynomial, can be used as an element of the matrix of a system of linear equations so that by cross multiplications by columns of two equations $A = 0, B = 0$, one can eliminate a column.

Proposition 4) is a very powerful and ingenious tool. An equation $p = 0$ can be divided into two parts A, B so that $p = A + B$, the process called tifen (剔分) and then one has another equation $A^2 - B^2 = 0$ from $p = 0$. This process provides a device to raise the exponents of two terms in $p = 0$ independently.

Proposition 5) means that its assumptions imply the final equation of a single unknown and hence the final goal for elimination becomes to obtain the assumption part.

Aside from these, Zhu used a process of interchanging the roles of siyuan, called yiwei (易位) to apply Proposition 5) and the final equation should be positioned in tianyuanshu for the traditional vertical writing.

In Siyuan Yujian, Zhu treated much complicated problems of areas of plane figures and volumes of solid figures than those in his Suanxue Qimeng. He also constructed equations relating to commercial problems and linear equations.

Creating problems on gougushu and finite series, Zhu made a great contribution to the theory of equations through them. It is rather difficult to understand the fourth power x^4 and up by volumes but by practical gougushu and sums of finite series, one can easily have higher powers so that they give rise to equations of much higher degrees. The first four problems in Jialing Sicao are all gougushu problems and problems in sanyuanshu and siyuanshu are chosen in gougushu except two problems. Zhu is also the first mathematician in the history who connected equations with finite series. Without tianyuanshu Zhu constructed equations related to finite series in Suanxue Qimeng. He extended the theory of finite series far more in Siyuan Yujian and solved problems by tianyuanshu, even though he neither included definitions nor arranged them systematically. They were dealt in sections, Ruyi Hunhe (如意混和), Jiaocao Xingduan (茭艸形段), Jianji Jiaocan (箭積交參), Ruxiang Zhaoshu (如像招數) and Guaduo Diecang (果垛疊藏), where the terminologies for finite series are not unified.

Zhu retained his strong tendency in Siyuan Yujian for the educational value of the book as in Suanxue Qimeng. Indeed, he included 12 problems to introduce eryuanshu in Liangyi Hezhe (兩儀合轍) which ask wuhe (五和), wujiao (五較) and the squares of three sides of a right triangle under the given conditions of its area ($= 30$) and gouguhe (勾股和 $a + b = 17$). We note that these problems can be solved by a simple tianyuanshu or gufa (古法) to have three sides. But in each case, he chose the sought number as tianyuan and a suitable diyuan which induces the system of equations in the above Proposition 5) except three problems where he has equations simply by elimination processes. Zhu also included the 6 problems for sanyuanshu in Sancai Biantong (三才變通) which ask various combinations of two sides and diagonal of a rectangle under the same conditions on them. Choosing two

sides as tianyuan and diyuan and the sought solution as renyuan, he has the same jinshi and yunshi for the problems but along the chosen renyuan, he has different sanyuanshi (三元式), say $p(x, y, z) = 0$ and then applies elimination processes.

We note that the second condition, $\dfrac{\dfrac{ab^2}{b} + ab - a^2}{a} = c$, in these problems implies $c = 2b - a$ and hence by Pythagorean theorem, $a^2 + b^2 = (2b - a)^2$, one has $3b = 4a$ which implies $a : b : c = 3 : 4 : 5$. By the first condition, one has $a^2b - a^2 - 4ab + b^2 + 2b - a = 0$ and hence $a = 3$ or $a = \dfrac{5}{12}$ and consequently the values of b, c from those of a. Indeed, Luo did have yunshi $3y = 4x$. Since Zhu proposed the problem to find $2a + 4b + 7c + ab$, chose tianyuan a, diyuan b and renyuan the sought number, and indicated the equation for renyuan, Luo was obliged to have the equation for renyuan along the elimination processes and a single solution.

There are numerous cases like the above ones. Indeed, among 6 problems for siyuanshu in Sixiang Zhaoyuan (四象朝元), the last problem is really a problem for siyuanshu. The second problem can be solved simply by tianyuanshu.

For solving equations, Zhu Shijie included only four problems among 288 problems, whose soultions can be obtained by the linear interpolation, called simply bujin mingfen (不盡 命分) by Zhu (problem 11, 13 in Sanlu Jiuyuan (三率究圓), problem 11 in Huowen Getuan (或問歌彖) and problem 19 in Suotao Tunrong (鎖套吞容)). Thus he might have set up problems with presupposed answers because his main concern to write the book is tianyuanshu up to siyuanshu.

When one has a repeating decimal as a solution, it is impossible to have a solution by zengcheng kaifangfa for it involves infinite steps. A repeating decimal can be represented by a fraction. For the extraction of a square root of a fraction $\dfrac{b}{a}$, i.e., solving equation $ax^2 - b = 0$, one can solve it by $a^2x^2 - ab = 0$, in other words, $\dfrac{\sqrt{ab}}{a}$ which is called lianzhishu (連枝術) in the section, piaotian tuiji (漂田推積) of Shushu Jiuzhang. Zhu Shijie extended this to equations of any degree as follows:

For an equation $p(x) = 0$, we assume that the equation for the fractional part of its *positive* solution in the process of zengcheng kaifangfa, is

$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = o$.

Then, multiplying a_n^{n-1} to the equation, one has a monic equation

$y^n + a_{n-1}y^{n-1} + a_na_{n-2}y^{n-2} + \cdots + a_n^{n-2}a_1y + a_n^{n-1}a_0 = 0$ where $y = a_nx$.

Suppose that b is a solution for the above, then the fractional part of the solution of $p(x) = 0$ is clearly $\dfrac{b}{a_n}$. The above process is called lianti tongtishu (連枝 同體術) or later zhifenfa (之分法) or zhifenshu. The first one is somewhat a bad choice for it refers to the case of $a^2x^2 - b^2 = 0$ in Shushu Jiuzhang [16]. Further, if the equation for the fractional part is a quadratic equation, say $ax^2 + bx + c = 0$, then the resulting eqation for $y = ax$ is simply $y^2 + by + ac = 0$. Zhu devoted whole 13 problems in the

section Hefen Suoyin (和分索隱) with 11 quadratic, a cubic and a quartic equations. He treats 5 more problems with zhifenfa in the other sections so that he presumably is very much satisfied with his invention, zhifenfa (also see [19]).

When Ceyuan Haijing and Yigu Yanduan were included in Siku Quanshu (四庫全書), anonymous commentators inserted commentaries on both books. Contrary to the commentator of Ceyuan Haijing, the commentator of Yigu Yanduan explained tianyuanshu by jiegenfang in Shuli Jingyun without fully understanding tianyuanshu. Indeed, all the equations in Shuli Jingyun are of the form $p(x) = q(x)$, where two sides of equations are denoted by bi (彼) and ci (此) and he changed the original equations of the form $p(x) = 0$ into the form of $q(x) = a_0$ with $x|q(x)$. Li Rui added another detailed commentary on the two books based on Shuli Jingyun [1]. They all missed the fact that tianyuanshu up to siyuanshu are firmly established on the structure of the zero as explained in the above. Moreover, they don't even take 0 as a number. Thus the commentary of Li Rui had impeded for the succeeding mathematicians to understand the structure of tianyuanshu although Li himself returned to the original tianyuanshu in his Gougu Suanshu Xicao. Luo Shilin is one of those mathematicians and tries to rectify Li's misunderstanding. With Yi Zhihan, Luo observed that an equation $p(x) = 0$ has a positive solution iff there are positive and negative terms together in $p(x)$. Further let $p_1(x)$ and $p_2(x)$ be the sums of positive and negative terms of $p(x)$, then the *equations* $p(x) = 0$ and $p_1(x) = -p_2(x)$ are equivalent or they have the same solution sets. Luo calls this fact zhengfu xiangdang (正負相當) in his appendix, Tianyuan Shili (天元釋例) to Siyuan Yujian Xicao. By this, he explains the equation $p(x) = 0$ by an equation with two sides, i.e., bi and ci. Clearly this observation is meaningless and useless, because every equation in tianyuanshu is obtained from an equation $s(x) = t(x)$ by subtracting one of two sides, or xiangxiao (相消). The form of zhengfu xiangdang is simply one of those with fixed positive and negative parts. Further, they always solve an equation $p(x) = 0$ for it is natural for zengcheng kaifangfa. Further, the type of equations $p(x, y, z, u) = 0$ is essential for elimination processes as we have discussed in the above.

3   Siyuan Yujian in Joseon Mathematics

The fourth king Sejong (世宗, 1397–1450, r. 1418–1450) imported Suanxue Qimeng and Yang Hui Suanfa with various astronomical books to improve Joseon calendrical system before 1430 when he himself studied Suanxue Qimeng [21]. But all the mathematical works of the Joseon dynasty before the 17th century were lost. Since Kim Si-jin republished Suanxue Qimeng in 1660, Joseon mathematics was successfully revived along the study of Suanxue Qimeng. As Zhao Yuanzhen (趙元鎮)

indicated in his preface of Suanxue Qimeng, the most important contribution of the book is tianyuanshu. Begining with Sanhak Wonbon (筭學原本, 1701) of Park Yul (朴繘, 1621–1668), the study of tianyuanshu in Joseon was renewed [14].

The real contribution to tianyuanshu was achieved by a Jung-in (中人) mathematician Hong Jeong-ha (洪正夏, 1684–1727) who served as an officer of Hojo (戶曹). He completed the greatest mathematics book in the Joseon dynasty, Gu-il Jib (九一集) before 1713 and added an appendix in 1724 and it was republished in 1868 with the preface by Nam Byeong-gil and the postscript by Lee Sang-hyeog [2]. Hong Jeong-ha applied tianyuanshu to solve problems about the areas and volumes, finite series, gougushu and then solved the equations constructed by tianyuanshu by zengcheng kaifangfa [10]. For the zengcheng kaifangfa, he discovered its mathematical structure by synthetical multiplications [9].

Hong Jeong-ha's contemporary mathematician, Yu Su-seok (劉壽錫) also completed a book, Gugo Sulyo (勾股術要) which included altogether 210 problems on gougushu. Even though it included problems, answers, and then indicated equations as Zhu did in Siyuan Yujian, he presumably applied tianyuanshu to construct equations as Hong Jeong-ha did for his theory of right triangles [8].

After these three works, Joseon mathematicians had not paid their attentions to tianyuanshu until the mid–19th century, because they did not recognize the superiority of theory of equations based on tianyuanshu and zengcheng kaifangfa over that on jiegenfang and the method of solving equations in Shuli Jingyun [5].

Joseon mathematics in the first half of 19th century has developed under the strong influence of Shuli Jingyun.

Jeong Yag-yong (丁若鏞, 1762–1836) wrote a book, Gugo Wonlyu (勾股源流) during the period 1801–1818 dealing with algebraic and order structures of Pythagorean polynomials [11].

Lee Sang-hyeog published Chageunbang Mong-gu (借根方蒙求, 1854), where the author took problems mostly in Shuli Jingyun and then constructed equations, 78 linear equations (線類), 35 quadratic equations (面類) and 17 equations (體類) - 5 quadratic, 10 cubic and 2 quartic equations - by jiegenfang. Among 130 problems, only 6 problems were taken in three Book 34–36 of Shuli Jingyun which deal with constructions of equations by Jiegenfang. Lee Sang-hyeog shows in Chageunbang Mong-gu that one can solve most of problems in Shuli Jingyun by equations [4]. One year after Chageunbang Mong-gu, Lee also published Sansul Gwan-gyeon (算術管見, 1855) where he discarded jiegenfang but constructed equations by tianyuanshu.

Nam Byeong-gil completed Jibgo Yeondan (緝古演段, 1854?–1855?) and Mu-i-hae (無異解, 1855). Jibgo Yeondan is a book with the commentary to construct equa-

tions by Jiegenfang and Mu-i-hae is also one with the commentary for 4 problems in Yigu Yanduan and 3 problems in Ceyuan Haijing by jiegenfang to rectify Li Rui's commentaries on those problems based on Shuli Jingyun as mentioned above [5]. In the preface of Jibgo Yeondan, Nam mentioned Lee Sang-hyeog's assistance and hence two scholars' collaboration on their study of mathematics was already begun. Moreover, Nam mentioned zhengfu xiangdang in his preface of Mu-i-hae which implies that he had already studied Siyuan Yujian Xicao of Luo Shilin in 1855. Further, we should point out that the books Ceyuan Haijing and Yigu Yanduan in Siku Quanshu added by commentaries of Li Rui were included in Zhibuzuzhai Congshu (知不足齋叢書, 1798) and hence Nam also had Zhibuzuzhai Congshu in 1855. In all, we safely assume that Luo's Xicao was brought into Joseon before 1855. Indeed, Nam is the first mathematician in the history of Joseon mathematics who collected all the classics of mathematics including Jiuzhang Suanshu and books of most of Qing mathematicians before the 19th century, which were shared by Lee Sang-hyeog for their collaborations.

We now investigate the history of Siyuan Yujin in Joseon mathematics.

The first step to study a new book is to transcribe it in Joseon because it requires a great fortune and processes to import books from China. We have one, titled Sawon Og-gam (四元玉鑑) [15] but it is a transcription of a part of Luo's Xicao. It includes Jialing Xicao, Liangyi Hezhe, and then parts of Zuoyou Fengyuan (左右逢元), Sancai Biantong, Sixiang Zhaoyuan, and only problems of Huowen Getuan dealing with eryuanshu, sanyuanshu. These inclusions imply that the transcriber pays his attentions to those except tianyuanshu with which he is familiar. He then chose two problems of Shanggong Xiuzhu (商攻修築) and the problem 20 of Guoduo Diecang (果垜疊藏) and collection of terminologies of finite series and formulas of their sums. By these, the transciber understands fully the connection between finite series and volumes of solids obtained by Shen Kuo (沈括, 1031–1095). He also transcribed Siyuan Shili (四元釋例), Luo's appendix to his Xicao and then a part of Tianyuan Siyuan Geli (天元四元各例), Yi Zhihan's appendix to Luo's Xicao which discusses equations with *multiple* solutions. Finally, he added Ruxiang Zhaoshu. In all, the transcriber picked items in Siyuan Yujin which were not treated in the preceding history of Joseon mathematics. With his later works, the transcriber should be Lee Sang-hyeog [6].

Nam Byeong-gil also left a commentary to Luo's Xicao, titled Og-gam Secho Sanghae (玉鑑細艸詳解) [18]. He did not deal with tianyuanshu in the commentary but dealt with 3 problems in Jialing Xicao, 5 problems in Zuoyou Fengyuan, 4 problems in Sancai Biantong and 3 problems in Sixiang Zhaoyuan. These selections also indicate that Nam devoted the commentary to eryuanshu to siyuanshu as well. Nam

Byeong-gil is drawn so much to Luo's zhengfu xiangdang and hence every equation in the book is indicated by zhengfu xiangdang as he did for the jinshi, $-x^3 - 2xy - 2x^2y + 2y^2 + xy^2 = 0$ of Liangyi Huayuan (兩儀化元) where gu (股) is tianyuan (x) and gouxian (勾弦和) diyuan (y):

股立方一負 勾弦和乘股二負 勾弦和乘股冪二負與

勾弦和冪二正 股乘勾弦和冪一正等 正負相當,

which says that $-x^3 - 2xy - 2x^2y = 2y^2 + xy^2$. It is clearly a wrong sentence and also confuses readers that it might be an identity.

When he applies proposition 5) for elimination processes, he has $B_2(yA_1 + B_1) - B_1(yA_2 + B_2) = 0$ and hence $y(A_1B_2 - A_2B_1) = 0$. Dividing it by y, he has the final equation. He also mentions that one can have it by hucheng jifen (互乘齊分) which is clearly the method meant by Zhu Shijie but we don't know why he took this detour. But he did not just transcribe Luo's Xicao. In Liangyi Huayuan, Luo did not get yunshi from the observations for jinshi so that he also took a detour. Nam tried to redress this and eventually succeeded to have yunshi and also had eqautions of the form $-p(x) = 0$ where Luo had $p(x) = 0$ which are given by Zhu Shijie. He also added zengcheng kaifangfa processes of solving equations indicated in Luo's appendix Kaifang Shili (開方釋例) to Xicao to his problems.

The next study of Siuan Yujian appears in Sanhak Jeong-eui (算學正義, 1867). Although Lee Sang-hyeog is quoted as a proofreader, Lee is a coauthor by Nam's preface. The book takes topics in Shuli Jingyun and treats them with traditional Chinese contributions, in particular solving equations, tianyuanshu and eryanshu up to siyuanshu in Siyuan Yujian and dayanshu (大衍術).

Constructing equations by tianyuanshu, they took Lee's Chageunbang Mong-gu as a reference. They constructed 20 linear, 27 quadratic, 14 cubic and 1 quartic equations. Most of them are originated from those in Chageunbang Mong-gu but their numeral parts are changed. There are additional problems dealing with finite series introduced in Shuli Jingyun. Surprisingly, they didn't use calculating rods to represent polynomials and equations except the first problem. Further, equation $-8,064 + 1,220x - 76x^2 + x^3 = 0$ obtained by xiangxiao in Problem 49 as follows:

相消得式 八千六十四太負 一千四百四十元正 七十六平方負 一立方正.

As the comment shows in the last problem, they compared tianyuanshu with jiegenfang, just 太, 元, 正, 負 in tianyuanshu replace 眞數, 根, 多, 少 in jiegenfang. This shows that they didn't fully appreciate the advantage of tianyuanshu over jiegenfang. Anyhow, all the equations obtained in the section are of the form $p(x) = 0$.

For eryuanshu to siyuanshu, they introduced a terminology, dawon, or duoyuan

(多元) in Chinese and dealt with 14 problems, indeed, 10 for eryuanshu, 3 for sanyuan-shu and 1 for siyuanshu. They took them from Siyuan Yujian and Suanxue Qimeng as follows.

#1 :	#14 in Zuoyou Fengyuan	#2 :	#16 in Zuoyou Fengyuan
#3 :	#9 in Huowen Getuan	#4 :	#11 in Zuoyou Fengyuan
#5 :	#16 in Kaifang Shisuomen	#6 :	Liangyi Huayuan
#7 :	$ab = 2,065, (b-a)(b+a) = 2,256$	#8 :	#8 in Zuoyou Fengyuan
#9 :	#4 in Zuoyou Fengyuan	#10 :	#18 in Zuoyou Fengyuan
#11 :	#12 in Huowen Getuan	#12 :	#1 in Sancai Biantong
#13 :	Sancai Yunyuan (三才運元)	#14 :	Sixiang Huiyuan (四象會元)

We should point out that #2 and #8 are slightly different problems with those indicated and the equations in #7 are given conditions and it is a new one. #5 is clearly a tianyuanshu problem in Suanxue Qimeng but authors took it eryuanshu problem. They discarded zhengfu xiangdang in the section but cleared it by the fact that zhengfu xiangdang holds only on the solution of the original equation in the section daejong pyeongbangbeob (帶縱平方法) dealing with solutions of quadratic equations. Every misunderstanding in Sawon Og-gam was corrected.

Problem 2 in Dawon is given with slightly altered conditions of the original one, but the first condition of them implies the value of a and hence they are simply problems of tianyuanshu. There are numerous cases like this in Siyuan Yujian. Problem 8 in Dawon is solved along that in Zuoyou Fengyuan for tianyuan x. The authors have the jinshi $11y + 11xy + 2x - 219 = 0$ and solve it by the factorization $11y(1 + x) = 219 - 2x$ and substituion $x = 4$ in it. There are numerous cases in Siyuan Yujian where they are solved by yiwei but not by a simple substitution.

Lee Sang-hyeog published his last work, Ig San (翼算) in 1868. It consists of two books, Jeongburon (正負論) and Toetaseol (堆垜說) based on his study of Luo's Si-yuan Yujian Xicao. Jeongburon concerns with theory of equations based on zhengfu xiangdang. It became a partial theory for it deals only with equations with positive solutions. We note that Luo Shilin said that a linear equation $ax = b$ has a positive (negative, resp.) solution if a, b have different (same, resp.) signs as his statement 視上下兩層正負異名可開正數 同名可開負數 says. Lee's Toetaseol is really one of the most important contributions in the history of Joseon mathematics. Lee dealt in Toetaseol with finite series from the works of Shen Kuo, Guo Shuojing (郭守敬, 1231–1316) and Zhu Shijie and completely systematized them by their mathematical structures [3]. He then added 12 problems along with those in Ruxiang Zhaoshu and Guoduo Diecang among which two problems 9 and 10 are exactly quotes of problem 4 and 5 in Ruxiang Zhaoshu, and remaining ones are all new problems with

detailed solutions based on his theory. Even the last one is a problem of eryuan-shu. We just quote the problem and a sketch of its solution:

設如招工興造 每日工銀一錢 共用銀一百四十六兩六錢 木工每日減招一人

土工每日加招一人 土工之初日木工之末日各招一人而訖役

只云土工都數比木工役日自乘數不足八人 問木工土工各役幾日

答 木工十二日 土工十六日

Using the numbers n, m of sums of sanjiaoduo (三角垜) $\sum_{k=1}^{n} \frac{k(k+1)}{2}$ and sijiao-duo (四角垜) $\sum_{k=1}^{m} k^2$ as tianyuan and diyuan respectively, Lee obtains jinshi $(-8796 + 2x + 3x^2 + x^3) + (y + 3y^2 + 2y^3) = 0$ and yunshi $(-16 - x - x^2) + 2y^2 = 0$ and hence the equation of degree 6

$307,782,688 - 247,282x - 317,509x^2 - 70,252x^3 + 3x^4 + 30x^5 + 2x^6 = 0$ with a solution 16 which is the working days of earthen workers. Using yunshi and sub-stitution, he has the working days of carpenters but also adds the method of yiwei to have it.

One year later after Ig San, a military officer Jo Hui-sun (趙羲純, 1824–1890) com-pleted Sanhag Seubyu (籌學拾遺, 1869) with the preface of Nam Byeong-gil who praised very highly Jo's results. It contains 7 sections. He treats eryuanshu to siyuan-shu in the final section, Saji Sanryag (四之籌略). The section contains 14 problems but the problem 12 is ill posed. There are 3 eryuanshu, 8 sanyuanshu and 2 siyuan-shu problems in due order, all of which are devised by the author himself.

In Siyuan Yujin, the sought answers were taken as one of yuans, but there are many problems in Sanhag Seubyu which were solved by not choosing them as yuans. We presume that he tried to find a short cut to solve them. Further, he indicated a term or a column to be eliminated and hence readers could easily understand the processes. We point out that Luo Shilin mentioned just equations for eliminations. Jo devised problems with given conditions for right triangles and then extra condi-tions of rectangles or a right triangle constructed by solutions of the first problem.

The first 3 problems deal with geometrical problems, inscribed three squares of a given circle (品字問題) and inscribed and circumscribed circle of a given regular pen-tagon. They were dealt in Sansul Gwan-gyeon (算術管見, 1855) of Lee Sang-hyeog, based on the theory of Shuli Jingyun and tianyuanshu. Jo Hui-sun solved them by different geometirical properties with those in Lee's book and eryuanshu.

The remaining problems are concerned with right triangles. We quote problem 4 as follows:

設取股三之二與(勾股)較相加與弦等 積加弦較和之半開方得勾

求倍積和三弦較和爲實開平方所得之數

Choosing jiao $b - a$ as tianyuan and gu b as diyuan, one has jinshi $-15x + 7y = 0$

and yunshi $18x - 18x^2 + 6y + 27xy - 9y^2 = 0$ and hence $x = 21$. Thus, by jinshi, one has $y = 45 = (b)$ and hence $a = 24$, consequently $c = 51$. In all, the sought number $\sqrt{ab + 3(c + b - a)}$ can be easily found. But as Luo Shilin did in his Xicao in numerous problems, Jo took the sought number as renyuan and then solves the problem. But he added that the problem is indeed a problem of eryuanshu.

此緣只立天地兩元而可得勾股弦 故以人元爲所求數不則必須

We just quote two more problem 11 and 14 to show that Jo extended the range of Siyuan Yujian.

設四股二勾一弦併爲實開方得較 又勾弦和爲首率弦較較爲末率
所得中率仍爲弦較較有半 求一積-卽勾股相乘積-一弦二勾爲實之開方數

We recall that proportional means (中率) are used widely in Shuli Jingyun. Problem 12 is ill posed because Jo added another conditions $a+b = c$ and $2c-3(b-a) = b$ to those of problem 11 which is inconsistent with conditions of problem. Even the first condition is also wrong for any triangle.

設勾股半積–相乘折半-減勾股較及勾弦較爲實開方得勾
又倍積-相乘倍之-加五股開方得股
若以勾作弦較較 弦和較爲勾則求其弦幾何.

The problem involves two right triangles with conditions
$\sqrt{\frac{ab}{2} - \{(b-a) + (c-a)\}} = a$; $\sqrt{2ab + 5b} = b$ and $c' - (b' - a') = a$; $(a + b) - c = a'$
and asks c'.

4 Conclusions

There have been two occasions in the history of Joseon mathematics which show unprecedented indigenous developments in a comparatively short period. Their main subjects were related with theory of equations, in particular, tianyuanshu and its generalizations, eryuanshu to siyuanshu. The first one has occurred during about five decades after the republication of Suanxue Qimeng in 1660 which initiated tianyuanshu a major subject for Joseon mathematicians. Among them the greatest mathematician, Hong Jeong-ha in the dynasty did reveal the fundamental structures of tianyuanshu in his Gu-il Jib and applied them to much more extended topics than those in Suanxue Qimeng which had been lost in China until the 19th century.

Unlike the first occasion, we have the next one without any preparatory stage which was evolved simply by an importation of Siyuan Yujian Xicao of Luo Shilin in the mid–19th century. In the meantime, Joseon mathematicians have neglected

tianyuanshu because of Jiegenfang in Shuli Jingyun. We have shown that the typical processes of a study of Chinese mathematics, namely first transcriptions of the original book and then their own contributions, were proceeded in the study of Siyuan Yujian for two decades. Indeed, works by Nam Byeong-gil, Lee Sang-hyeog and Jo Hui-sun completed in each year from 1857 to 1859 precisely reveal these processes, where they also discarded unnecessary claims. Further, they recognized the mathematical structures of 0 for the whole theory unlike Chinese mathematicians in the 18th century.

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