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Euler Characteristics of Log Calabi–Yau Threefolds

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Abstract. For any even integer n, we show that there exists a log Calabi–Yau threefold (Y, D) such that the Euler characteristic of Y is n. Furthermore Y is smooth and D is smooth anticanonical section of Y that is a $K3$ surface.

A *log Calabi–Yau pair* (Y, D) consists of a proper variety Y and an effective $\mathbb{Q}$-divisor D such that (Y, D) is log canonical and $K_Y + D$ is Q-linearly equivalent to zero. Recently much interest has been given to log Calabi–Yau pairs ([5, 7, 3, 2]). Some types of boundedness properties for klt log Calabi–Yau pairs were proved in [5, 3]. For a Fano variety Y, (Y, D) is a log Calabi–Yau pair, where D is an anticanonical section of Y. For a Calabi–Yau variety Y, $(Y, 0)$ is also a log Calabi–Yau pair. It is well-known that there are only finitely many deformation types for Fano threefolds. The Euler Characteristics of elliptic Calabi–Yau threefolds are bounded ([6, 4]) and many expect that it is true for the general Calabi–Yau threefolds. So it would be a natural question to ask whether the Euler Characteristics for log Calabi–Yau threefolds is also bounded. Suppose that Y is smooth and D is a smooth anticanonical section of Y that is a $K3$ surface. Then the pair (Y, D) is a log Calabi–Yau threefold. The authors in [2] gave an unbounded family of such log Calabi–Yau threefolds with the Euler characteristics $e(Y) = -48n - 46$, where $n \geq N_0$ for some integer N_0. In this note, we show the following theorem, which is an improvement of the result in [2].

Theorem 1.1. *For any even integer n, there exists a log Calabi–Yau threefold (Y, D) such that the Euler characteristic $e(Y)$ of Y is n. Moreover, Y is smooth and D is smooth anticanonical section of Y that is a $K3$ surface.*

Its proof is short and quite elementary. Note that the Euler characteristic $e(Y)$ of Y in Theorem 1.1 is given by

$$2(1 + h^{1,1}(Y) - h^{1,2}(Y)),$$

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which is *even*. So Theorem 1.1 basically asserts that the Euler characteristic of such a log Calabi–Yau threefold Y can be any possible number.

proof of Theorem 1.1. Let (X, D) be a log Calabi–Yau threefold such that X is smooth and that D is a smooth anticanonical section of X which is a $K3$ surface. Furthermore assume that there is a smooth curve c on D. We let $X_0 = X$, $D_0 = D$ and $c_0 = c$. Let $X_1 \to X_0$ be the blow-up along c_0 and D_1 be the proper transform of D_0. Then D_1 is isomorphic to D_0 and it is an anticanonical section of X_1. So (X_1, D_1) is also a log Calabi–Yau threefold. Let c_1 be the intersection of D_1 with the exceptional divisor. Then c_1 is a smooth curve on D_1 and it is isomorphic to c_0. Inductively we repeat this procedure to construct log Calabi–Yau threefolds: Let $X_{i+1} \to X_i$ be the blow-up along c_i, D_{i+1} be the proper transform of D_i and c_{i+1} be the intersection of D_{i+1} with the exceptional divisor. Then (X_{i+1}, D_{i+1}) is also a log Calabi–Yau threefold. Note that Euler characteristic $e(X_{i+1})$ of X_{i+1} is

$$e(X_{i+1}) = e(X_i) + 2 - 2g(c_i)$$

and $g(c_{i+1}) = g(c_i)$, where $g(c_i)$ is the genus of the curve c_i. Hence we have

$$e(X_i) = e(X) + i(2 - 2g(c)).$$

Let $X = \mathbb{P}^3$ and D be a smooth quartic that is a Kummer surface (e.g., the Fermat quartic). Then D has two canonical elliptic fibrations with fibers Γ_1, Γ_2 respectively such that $\Gamma_1 \cdot \Gamma_2 = 1$ (see [1] for example). Note that the linear system $|\Gamma_1 + \Gamma_2|$ is base-point-free. So we can choose a smooth divisor c from the linear system. Note that $g(c) = 2$. Apply the above procedure to get the log Calabi–Yau threefolds (X_i, D_i)'s. We have

$$e(X_i) = e(X) + i(2 - 2g(c)) = e(\mathbb{P}^3) - 2i = 4 - 2i, \tag{1.1}$$

where $i \geq 0$.

We note that D also contains a smooth rational curve c'. Apply the above procedure again to c' to get log Calabi–Yau threefolds (X_i', D_i')'s. Then we have

$$e(X_i') = e(X) + i(2 - 2g(c')) = 4 + 2i, \tag{1.2}$$

where $i \geq 0$. By (1.1) and (1.2), any even integer n is the Euler characteristic of one of X_i's or X_i''s. □

References

[1] P. W. Barth, K. Hulek, A. M. C. Peters, Van de Ven, Antonius Compact complex surfaces, Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 4. Springer-Verlag, Berlin, 2004.

[2] G. Bini, F. F. Favale, *An Unbounded Family of log Calabi–Yau Pairs*, arXiv:1608.08804

[3] G. Di. Cerbo, R. Svaldi, *Log birational boundedness of Calabi–Yau pairs*, arXiv:1608.02997

[4] M. Gross, *A finiteness theorem for elliptic Calabi–Yau threefolds*, Duke Math. J., **74**(1994), no. 2, 271299.

[5] C. Hacon, C. Xu, *Boundedness of log Calabi–Yau pairs of Fano type*, Math. Res. Lett., **22**(2015), no. 6, 1699–1716.

[6] B. Hunt, *A bound on the Euler number for certain Calabi–Yau 3 -folds*, J. Reine Angew. Math., **411**(1990), 137170.

[7] J. Kollár, C. Xu, *The dual complex of log Calabi–Yau pairs*, arXiv:1503.08320 to appear in Invent. Math. (2016)