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# Canal Surfaces in Galilean 3-Spaces

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Abstract. In this paper, we defined the admissible canal surfaces with isotropic radious vector in Galilean 3-spaces an we obtained their position vectors. Also we gave some important results by using their Gauss and mean curvatures.

## 1. Introduction

A canal surface is defined as envelope of a one-parameter set of spheres, centered at a spine curve $\gamma(s)$ with radius $r(s)$. When $r(s)$ is a constant function, the canal surface is the envelope of a moving sphere and is called a pipe surface. Canal surfaces have wide applications in CAGD, such as construction of blending surfaces, shape reconstruction, transition surfaces between pipes, robotic path planning, etc. . An envelope of a 1-parameter family of surfaces is constructed in the same way that we constructed a $1-$parameter family of curves.The family is described by a differentiable function $F(x, y, z, \lambda) = 0$, where $\lambda$ is a parameter.When $\lambda$ can be eliminated from the equations

$$F(x, y, z, \lambda) = 0$$

and

$$\frac{\partial F(x, y, z, \lambda)}{\partial \lambda} = 0,$$

we get the envelope, which is a surface described implicitly as $G(x, y, z) = 0$. For example, for a 1-parameter family of planes we get a develople surface([1], [2], [3], [5], [7] and [9]).

A general canal surface is an envelope of a 1-parameter family of surface. The envelope of a 1-parameter family $s \longrightarrow S^2(s)$ of spheres in $IR^3$ is called a *general canal surface* [3]. The curve formed by the centers of the spheres is called *center curve* of the canal surface. The radius of general canal surface is the function $r$ such that $r(s)$ is the radius of the sphere $S^2(s)$. Suppose that the center curve of

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a canal surface is a unit speed curve $\alpha : I \to IR^3$. Then the general canal surface can be parametrized by the formula

$$C(s,t) = \alpha(s) - R(s)T - Q(s)\cos(t)N + Q(s)\sin(t)B, \tag{1.1}$$

where

$$R(s) = r(s)r'(s),$$
$$Q(s) = \pm r(s)\sqrt{1 - r'(s)^2}.$$

All the tubes and the surfaces of revolution are subclass of the general canal surface.

**Theorem 1.1([3]).** *Let $M$ be a canal surface. The center curve of $M$ is a straight line if and only if $M$ is a surface of revolution for which no normal line to the surface is parallel o the axis of revolution. The following conditions are equivalent for a canal surface $M$:*

i. *$M$ is a tube parametrized by (1.1);*

ii. *the radius of $M$ is constant;*

iii. *the radius vector of each sphere in family that defines the canal surface $M$ meets the center curve orthogonally.*

## 2. Canal Surfaces in Galilean Space

The Galilean space $G_3$ is a Cayley-Klein space defined from a 3-dimensional projective space $P(R3)$ with the absolute figure that consists of an ordered triple $\{\omega, f, I\}$, where $\omega$ is the ideal (absolute) plane, $f$ the line (absolute line) in $\omega$ and $I$ the fixed elliptic involution of points off. We introduce homogeneous coordinates in $G_3$ in such a way that the absolute plane $\omega$ is given by $x_0 = 0$, the absolute line $f$ by $x_0 = x_1 = 0$ and the elliptic involution by $(0 : 0 : x_2 : x_3) \longmapsto (0 : 0 : x_3 : -x_2)$. With respect to the absolute figure, there are two types of lines in the Galilean space, isotropic lines which intersect the absolute line $f$ and non-isotropic lines which do not. A plane is called Euclidean if it contains $f$, otherwise it is called isotropic. In the given affine coordinates, isotropic vectors are of the form $(0, y, z)$, whereas Euclidean planes are of the form $x = k, k \in R$.

The scalar product in Galilean space $G_3$ is defined by

$$g(A,B) = \begin{cases} a_1b_1, & if \;\; a_1 \neq 0 \vee b_1 \neq 0, \\ a_2b_2 + a_3b_3, & if \;\; a_1 = 0 \wedge b_1 = 0, \end{cases}$$

where $A = (a_{1,}a_2, a_3)$ and $B = (b_{1,}b_2, b_3)$ .The Galilean cross product is defined by

$$A \wedge_{G_3} B = \begin{vmatrix} 0 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \quad if \;\; a_1 \neq 0 \;\; \vee b_1 \neq 0.$$

The unit Galilean sphere is defined by

$$S_{\pm}^{2}=\left\{X\in G_{3}\middle|\ g\left(X,X\right)=\mp r^{2}\right\}.$$

An admissible curve $\alpha : I \subseteq R \to G_3$ in the Galilean space $G_3$ which parameterized by the arc length $s$ defined by

$$\alpha(s) = (s, y(s), z(s)), \tag{2.1}$$

where $s$ is a Galilean invariant and the arc length on $\alpha$. The curvature $\kappa(s)$ and the torsion $\tau(s)$ are defined by

$$\kappa(s) = \sqrt{\left(y''(s)\right)^2 + \left(z''(s)\right)^2}\ , \quad \tau(x) = \frac{\det\left(\alpha'(s), \alpha''(s), \alpha'''(s)\right)}{\kappa^2(s)}\ . \tag{2.2}$$

The orthonormal frame in the sense of Galilean space $G_3$ is defined by

$$\begin{aligned} T(s) &= \alpha'(s) = (1, y'(s), z'(s)), \\ N(s) &= \frac{1}{\kappa(s)}\left(0, y''(s), z''(s)\right), \\ B(s) &= \frac{1}{\kappa(s)}\left(0, -z''(s), y''(s)\right). \end{aligned} \tag{2.3}$$

The vectors $T, N$ and $B$ in (2.3) are called the vectors of the tangent, principal normal and the binormal line of $\alpha$, respectively. They satisfy the following Frenet equations

$$T' = \kappa N\ ,\ N' = \tau B\ ,\ B' = -\tau N. \tag{2.4}$$

A $C^r-$surface $M$, $r \geq 1$, immersed in the Galilean space, $\mathrm{x} : U \to M$, $U \subset R^2$,

$$\mathrm{x}(u,v) = (x(u,v), y(u,v), z(u,v)),$$

has the following first fundamental form

$$I = (g_1 du + g_2 dv)^2 + \epsilon(h_{11} du^2 + 2h_{12} du dv + h_{22} dv^2),$$

where the symbols $g_i = x_i$ and $h_{ij} = g\left(\widetilde{\mathrm{x}}_i, \widetilde{\mathrm{x}}_j\right)$ stand for derivatives of the first coordinate function $x(u,v)$ with respect to $u$,$v$ and for the Euclidean scalar product of the projections $\widetilde{\mathrm{x}}_k$ of vectors $\mathrm{x}_k$ onto the $yz-$plane, respectively. Furthermore,

$$\epsilon = \begin{cases} 0, & \text{if direction } du : dv \text{ is non-isotropic,} \\ 1, & \text{if direction } du : dv \text{ is isotropic.} \end{cases}$$

In every point of a surface there exists a unique isotropic direction defined by $g_1 du + g_2 dv = 0$. In that direction, the arc length is measured by

$$\begin{aligned} ds^2 &= h_{11} du^2 + 2h_{12} du dv + h_{22} dv^2 \\ &= \frac{1}{\left(g_1\right)^2}\left\{h_{11}\left(g_2\right)^2 - 2h_{12} g_1 g_2 + h_{22}\left(g_1\right)^2\right\} dv^2 \\ &= \frac{W^2}{\left(g_1\right)^2} dv^2, \quad g_1 \neq 0, \end{aligned}$$

where

$$h_{11}=\frac{x_2^2}{W^2},\quad h_{12}=-\frac{x_1x_2}{W^2},\quad h_{12}=\frac{x_1^2}{W^2},$$

$$x_1=\frac{\partial x}{\partial u},\quad x_2=\frac{\partial x}{\partial v},\quad W^2=(x_2\mathrm{x}_1-x_1\mathrm{x}_2)^2.$$

A surface is called *admissible* if it has no Euclidean tangent planes. Therefore, for an admissible surface $g_1\neq 0$ or $g_2\neq 0$ holds. An admissible surface can always locally be expressed as $z=f(x,y)$.

The Gaussian $K$ and mean curvature $H$ are $C^{r-2}$-functions, $r\geq 2$, defined by

$$K=\frac{L_{11}L_{22}-L_{12}^2}{W^2},\quad H=\frac{(g_2)^2L_{11}-2g_1g_2L_{12}+(g1)^2L_{22}}{2W^2},$$

where

$$L_{ij}=g\left(\frac{x_1\mathrm{x}_{ij}-x_{ij}\mathrm{x}_1}{x_1},\eta\right),\quad x_1=g_1\neq 0.$$

The unit normal vector $\eta$ given by an isotropic vector is defined by

$$\eta=\frac{\mathrm{x}_1\wedge_{G_3}\mathrm{x}_2}{W}=\frac{1}{W}(0,-x_2z_1+x_1z_2,x_2y_1-x_1y_2)$$

([4], [6], [8]).

In Galilean geometry, there are two types sphere depending radious vector whether it is a isotropic or non-isotropic. Spheres with non-isotropic radious vector are Euclidean circles in yoz-plane and spheres with isotropic radious vector are parallel planes such as $x=\pm r$. We denote the Euclidean circles by $S_{\pm}^1(s)$.

**Definition 2.1.** The envelope of a 1-parameter family $s\to S_{\pm}^1(s)$ of the circles in $G_3$ is called a *canal surface* in Galilean 3-space. The curve formed by the centers of the Euclidean circles is called *center curve* of the canal surface. The radius of the canal surface is the function $r$ such that $r(s)$ is the radius of the Euclidean circles $S_{\pm}^1(s)$.

Let $\gamma(s)$ be an admissible curve as centered curve and canal surface is a patch that parametrizes the envelope of Euclidean circles which can be defined as

$$C(s,t)=\gamma(s)+\psi(s,t)T(s)+\varphi(s,t)N(s)+\omega(s,t)B(s)\tag{2.5}$$

with the regularity conditions $C_s\neq 0$, $C_t\neq 0$ and $C_s\times C_t\neq 0$, where $\varphi(s,t)$ and $\omega(s,t)$ are $C^\infty-$functions of $s$ and $t$. Since $C(s,t)-\gamma(s)$ is the surface normal of $S_{\pm}^1(s)$ and $C(s,t)$ is non-isotropic then $\psi(s,t)=0$ and

$$g(C(s,t)-\gamma(s),C(s,t)-\gamma(s))=\varphi(s,t)^2+\omega(s,t)^2=r(s)^2\tag{2.6}$$

and by differentiating (2.6) with respect to $s$ and $t$ we get

$$\varphi_t(s,t)\varphi(s,t)+\omega_t(s,t)\omega(s,t)=0,\tag{2.7}$$

(2.8) $$\varphi_s(s,t)\,\varphi(s,t) + \omega_s(s,t)\,\omega(s,t) = r'(s)\,r(s)\,,$$

(2.9) $$g\left(C(s,t) - \gamma(s)\,, C_s(s,t)\right) = 0,$$

(2.10) $$g\left(C(s,t) - \gamma(s)\,, C_t(s,t)\right) = 0,$$

and also we find the funcions $\varphi(s,t)$ and $\omega(s,t)$ are

$$\varphi(s,t) = r(s)\cos(t)\ ,\ \ \omega(s,t) = r(s)\sin(t)$$

by using (2.6), (2.7) and (2.8). Thus, we give the following corollary.

**Corollary 2.2.** *Let $\gamma(s)$ be an admissible curve. Then the position vector of canal surface with isotropic radious vector and centered curve $\gamma(s)$ is*

(2.11) $$C(s,t) = \gamma(s) + r(s)\cos(t)N(s) + r(s)\sin(t)B(s)\,.$$

The natural basis $\{C_s, C_t\}$ are given by

(2.12) $$\begin{aligned} C_s &= T + \{r'\cos(t) - r\tau\sin(t)\}\,N + \{r'\sin(t) + r\tau\cos(t)\}\,B, \\ C_t &= -r\sin(t)N + r\cos(t)B. \end{aligned}$$

From (2.4) and (2.12), the components $h_{ij}$ and $g_i$ are

$$h_{11} = \left(r'(s)\right)^2 + r^2(s)\,\tau^2(s)\,, \quad h_{12} = r^2(s)\,\tau(s)\,, \quad h_{22} = r^2(s)\,,$$
$$g_1 = 1, \quad g_2 = 0.$$

Thus, the first fundamental form of canal surface is

$$I_C = \left(1 + \left(r'(s)\right)^2 + r^2(s)\,\tau^2(s)\right) du^2 + 2r^2(s)\,\tau(s)\,dudv + r^2(s)\,dv^2.$$

By using (2.4), the second derivations of (2.12)

(13) $$\begin{aligned} C_{ss} &= \left\{\kappa + (r'' - r\tau^2)\cos(t) - (2r'\tau + r\tau')\sin(t)\right\} N \\ &+ \left\{(2r'\tau + r\tau')\cos(t) + (r'' - r\tau^2)\sin(t)\right\} B \\ C_{tt} &= -r\cos(t)N - r\sin(t)B \\ C_{st} &= -(r'\sin(t) + r\tau\cos(t))N + (r'\cos(t) - r\tau\sin(t))B \end{aligned}$$

and the unit normal vector

$$\eta(s,t) = -\cos(t)N(s) - \sin(t)B(s)$$

coefficients $L_{ij}$ are

$$L_{11} = -\left\{\kappa\left(s\right)\cos(t) + r''\left(s\right) - r\left(s\right)\tau^{2}\left(s\right)\right\}, \quad L_{12} = r\left(s\right)\tau\left(s\right), \quad L_{22} = r\left(s\right)$$

so the second fundamental form is

$$II_{C} = -\left\{\kappa\left(s\right)\cos(t) + r''\left(s\right) - r\left(s\right)\tau\left(s\right)^{2}\right\}du^{2} + 2r\left(s\right)\tau\left(s\right)dudv + r\left(s\right)dv^{2}.$$

The Gauss curvature and mean curvature of a non-isotropic canal surface in the Galilean space are given by

$$K\left(s,t\right) = \frac{r''\left(s\right) - \kappa\left(s\right)\cos(t)}{r\left(s\right)}, \quad H\left(s,t\right) = \frac{1}{2r\left(s\right)}.$$

In the case $K\left(s,t\right) = 0$, the centered curve has to be planar and there are two K-flat canal surfaces for $r(s) = c_1 s + c_2$ and $r(s) = c$. Thus, we conclude the following cases.

**Theorem 2.3.** *Let M be a canal surface in Galilean 3-space. Then followings are true.*

***i.*** *There is no minimal canal surface.*

***ii.*** *The Gauss and mean curvatures of canal surface satisfy the relation*

$$K\left(s,t\right) + 2H\left(s,t\right)\left\{\kappa\left(s\right)\cos(t) - r''\left(s\right)\right\} = 0,$$

***iii.*** *M is a K−flat canal surface if and only if M is a elliptic cone and its position vector is*

$$\begin{aligned} C(s,t) &= (s,(c_1 s + c_2)(c_3\cos(t) \mp \sqrt{1-(c_3)^2}\sin(t) \\ &\quad ,(c_1 s + c_2)(\mp\sqrt{1-(c_3)^2}\cos(t) - c_3\sin(t))), \end{aligned}$$

*where* $c_1 \neq 0,\ c_2 \in IR,\ c_3 \in [0,1]$, *see Figure 1(a).*

***iv.*** *M is a K−flat tubular surface if and only if M is a elliptic cyclinder and its position vector is*

$$C(s,t) = (s, c_1 c_2\cos(t) \mp c_1\sqrt{1-(c_2)^2}\sin(t), \mp c_1\sqrt{1-(c_2)^2}\cos(t) - c_1 c_2\sin(t)),$$

*where* $c_1 \in IR^{+},\ c_2 \in [0,1]$, *see Figure 1(b).*

***v.*** *All the tubes are surface with constant mean curvature.*

On the other hand, a surface is said to be a Weingarten surface if its Gauss and mean curvatures satisfy the Jacobi condition $\Phi\left(H,K\right) = K_t H_s - H_t K_s = 0$. Thus, we can give also following theorem.

**Theorem 2.4.** *Let $M$ be a canal surface in Galilean 3-space. Then $M$ is a Weingarten surface if and only if $M$ is either a tubular surface or a surface of revolution with* $r(s) = \pm c_3 e^{-\left(\frac{c_2+s}{c_1}\right)}\left(e^{2\left(\frac{c_2+s}{c_1}\right)}+1\right)$ *and* $r(s) = c_1 s + c_2$, *where* $c_1 \neq 0$, $c_2 \in IR$ *and* $c_3 \in IR^+$.

*Proof.* Let us assume that $M$ be a Weingarten surface. Differentiating $K(s,t)$ and $H(s,t)$ with respect to $s$ and $t$ gives

$$K_s(s,t) = \frac{r'''(s)\, r(s) - r''(s)\, r'(s) + (\kappa(s)\, r'(s) - \kappa'(s)\, r(s))\cos(t)}{r(s)^2}$$

$$K_t(s,t) = \frac{r''(s) + \kappa(s)\sin(t)}{r(s)}$$

and

$$H_s(s,t) = -\frac{r'(s)}{2r(s)},\ H_t(s,t) = 0.$$

From the Jacobi equation $\Phi(H,K) = K_t H_s - H_t K_s = 0$, we get

$$\frac{1}{r(s)^3}\left\{\begin{array}{c} r''(s)\, r'''(s)\, r(s) - r''(s)\, r''(s)\, r'(s) \\ +\kappa(s)\left\{\begin{array}{c} r'''(s)\, r(s) - r''(s)\, r'(s) \\ +(\kappa(s)\, r'(s) - \kappa'(s)\, r(s))\cos(t) \end{array}\right\}\sin(t) \\ +r''(s)(\kappa(s)\, r'(s) - \kappa'(s)\, r(s))\cos(t) \end{array}\right\} = 0.$$

Since $\{1, \sin(t), \cos(t)\}$ is linearly independent then

$$\begin{array}{rcl} r''(s)\{r'''(s)\, r(s) - r''(s)\, r'(s)\} &=& 0, \\ \kappa(s)(r'''(s)\, r(s) - r''(s)\, r'(s)) &=& 0, \\ \kappa(s)(\kappa(s)\, r'(s) - \kappa'(s)\, r(s)) &=& 0, \\ r''(s)(\kappa(s)\, r'(s) - \kappa'(s)\, r(s)) &=& 0. \end{array} \tag{14}$$

If $\kappa(s)$ is non-zero constant then from $(14)_3$, $r(s)$ is non-zero constant. If $\kappa(s) = 0$ then from $(14)_1$, either $r(s) = c_1 s + c_2$ or

$$r(s) = \pm c_3 e^{-\left(\frac{c_2+s}{c_1}\right)}\left(e^{2\left(\frac{c_2+s}{c_1}\right)}+1\right).$$

It is easy to see that the Jacobi equation $\Phi(H,K) = 0$ satisfies in each case of ($\kappa(s)$ and $r(s)$ are non-zero constants), ($\kappa(s) = 0$, and $r(s) = c_1 s + c_2$) and ($\kappa(s) = 0$, and $r(s) = \pm c_3 e^{-\left(\frac{c_2+s}{c_1}\right)}\left(e^{2\left(\frac{c_2+s}{c_1}\right)}+1\right)$), for the necessary part. Thus proof is completed. □

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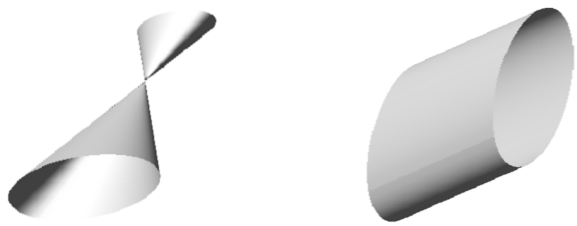

(a) (b)

Figure 1: Some Galilean Canal Surfaces. For (a); $c_1 = c_2 = 1, c_3 = 1/2$, for (b); $c_1 = 1, c_2 = 1/2$.

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