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# An Enhanced Chebyshev Collocation Method Based on the Integration of Chebyshev Interpolation

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Abstract. In this paper, we develop an enhanced Chebyshev collocation method based on an integration scheme of the generalized Chebyshev interpolations for solving stiff initial value problems. Unlike the former error embedded Chebyshev collocation method (CCM), the enhanced scheme calculates the solution and its truncation error based on the interpolation of the derivative of the true solution and its integration. In terms of concrete convergence and stability analysis, the constructed algorithm turns out to have the $7^{th}$ convergence order and the A-stability without any loss of advantages for CCM. Throughout a numerical result, we assess the proposed method is numerically more efficient compared to existing methods.

## 1. Introduction

In this article, we consider a numerical integration scheme for stiff initial value problems (IVPs) given by $\phi'(t) = f(t, \phi(t))$, $t \in (t_0, t_f]$; $\phi(t_0) = \phi_0$, where $f(t, \phi(t))$ is assumed to satisfy all the necessary conditions for the existence of a unique solution. The usage of solutions of the stiff IVPs are gradually increased by mathematicians and scientists to simulate many phenomena in astronomy, chemical reaction, electrical engineering, computational fluid dynamics, and so on. (See [2, 3, 4, 5, 8, 14, 16, 18, 19]). It is important to construct an efficient and stable implicit method for solving the stiff IVPs for the following reasons: (1) explicit methods designed for non-stiff problems are forced to use very small step sizes due to a stability constraint and (2) one can choose the step size based only on accuracy considerations without any consideration on stability constraints provided the scheme is $A$-stable.

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The variable time step size is one of important issues of implicit methods. It is usually obtained by a difference between the numerical solution and an extra lower order one (usually refer to an estimation of the local truncation error). One important issue of this pair of method is to reduce overall computational cost. For example, Radau methods [9] construct the lower order solution using a combination of the intra step values of the numerical solution without any extra costs for function evaluations. As another example, the Chebyshev collocation methods [20] calculate the lower order solution using a two step backward differentiation formula. These approaches require one more extra function evaluation and one more calculation for a linear system in each integration step.

One of aims of this article is to develop an efficient scheme to resolve these issues for which we use elegant properties of the Chebyshev series introduced by Lanczos [17] for constructing numerical integration schemes and extensively used in an interpolation theory and quadrature rules (for example, see [10, 11, 12, 13]). Clenshaw [6] and Clenshaw and Norton [7] extended the use of Chebyshev series to solving IVPs. After then, it was extensively studied and used to construct a collocation method(CM) for solving IVPs (for example, see [6, 7, 21]). We observe that the authors of [11] have developed an automatic Chebyshev quadrature rule based on the Fast Fourier Transform(FFT) by using a generalized Chebyshev interpolation. The motivation of this article is based on the algorithm([11]) for the generalized Chebyshev interpolation procedure increasing the number of sample points more moderately than doubling so that all zeros of the lower degree Chebyshev interpolation are contained in those of the higher degree one. Most recently, the authors developed an error embedded Chebyshev collocation method [15] by using the algorithm for the sample points and the Chebyshev interpolation polynomials for the solution $\phi$, which turns out to have the 6th order convergence and an almost L-stability.

Our main contribution of this article is to make a new enhanced method for the algorithm in [15] in terms of both the convergence order and the stability. To achieve the purposes, the new method couples above mentioned techniques - the Chebyshev interpolation procedure for the derivative of $\phi$ instead of the solution $\phi$, which is used in [15], and the collocation method(CM) based on it. We prove analytically that the suggested algorithm has a convergence order 7 and it is $A$-stable using a concrete stability analysis. We show numerically that the number of integration steps is dramatically reduced and a larger time step size is allowed regardless of stiffness compared with existing implicit schemes.

This paper is organized as follows. In Sec. 2, we derives a concrete algorithm. Convergence and stability analysis for the algorithm is given in Sec. 3. In Sec. 4, we present several numerical results to validate our analysis.

## 2. Derivation of Algorithm

The aim of this section is to derive a concrete algorithm for an acceralated error embedded collocation method based on the CM and a generalized Chebyshev polynomials. The target problem is described by

$$\phi'(t) = f(t, \phi(t)), \quad t \in (t_0, t_f]; \quad \phi(t_0) = \phi_0, \tag{2.1}$$

where $f(t, \phi(t))$ is assumed to satisfy all the necessary conditions for the existence of a unique solution. We assume that two approximations $\tilde{\phi}_k$ and $\phi_k$ for the solution $\phi(t_k)$ for all times $t_k \le t_m$ with a given time $t_m$ are already calculated, where $\tilde{\phi}_k$ are expressed in terms of the local truncation error $e_k$ for the error $E_k := \phi(t_k) - \phi_k$ such that $\tilde{\phi}_k := \phi_k + e_k$. We then try to find the next approximations $\tilde{\phi}_{m+1}$ and $\phi_{m+1}$ at time $t_{m+1}$ so that $\tilde{\phi}_{m+1} := \phi_{m+1} + e_{m+1}$ with the estimation $e_{m+1}$ for the local truncation error of $E_{m+1} = \phi(t_{m+1}) - \phi_{m+1}$, where the solution $\phi(t)$ satisfies the IVP given by

$$\phi'(t) = f(t, \phi(t)), \quad t \in [t_m, t_{m+1}]; \quad \phi(t_m) = \phi_m + E_m. \tag{2.2}$$

To apply the Chebyshev series, we introduce the change of variable $t(s) = t_m + \frac{h}{2}(1+s)$ which changes the computational domain $[t_m, t_{m+1}]$ to the reference domain $[-1, 1]$, where $h := t_{m+1} - t_m$ is the time step size. Then, from (2.2), one can see that the function $\bar{\phi}(s) := \phi(t(s))$ satisfies the IVP given by

$$\begin{cases} \bar{\phi}'(s) = \dfrac{h}{2} f\big(t(s), \bar{\phi}(s)\big), & s \in [-1, 1], \\ \bar{\phi}(-1) = \phi(t_m) = \phi_m + E_m. \end{cases} \tag{2.3}$$

## 2.1. Chebyshev interpolation polynomials

For solving the problem (2.3), we begin with introducing the Chebyshev-Gauss Lobatto (CGL) points such that

$$\eta_j := \cos\left(\frac{4-j}{4}\pi\right), \quad j = 0, \cdots, 4. \tag{2.4}$$

which are the zeros of the polynomial $\omega_5(s)$ defined by $\omega_5(s) = T_5(s) - T_3(s) = 2(s^2 - 1)U_4(s)$, where $T_k$ and $U_k$ are the Chebyshev polynomials of the first and second kind, respectively. Also, we let

$$\nu_j := \cos\Big(\big(\frac{j}{4} + \frac{3}{8}\big)\pi\Big), \quad j = 0, 1, \tag{2.5}$$

which are zeros of the Chebyshev polynomial $T_2(s) - \cos(3\pi/4)$ of degree 2, introduced by [10, 11]. Further, let $\{\tau_j\}$ be a rearrangement of the points $\eta_j$ and $\nu_j$ defined by (2.4) and (2.5), respectively, so that

$$-1 = \tau_0 < \tau_1 < \cdots < \tau_6 = 1. \tag{2.6}$$

From now on, let $N$ be a fixed number which is either 4 or 6. Depending on $N$, let $s_j^N$ be abscissae defined by

$$s_j^N := \begin{cases} \eta_j, & N = 4, \\ \tau_j, & N = 6, \end{cases} \tag{2.7}$$

where $\eta_j$ and $\tau_j$ are defined by (2.4) and (2.6), respectively. We now are ready to introduce the generalized Chebyshev interpolation polynomials for approximating the derivative $\bar{\phi}'$ by using the abscissae $s_j^N$ defined by (2.7). Let $p_4(s)$ be the Chebyshev interpolation polynomial of degree 4 given by

$$p_4(s) = \sum_{k=0}^{4}{}'' a_k^4 T_k(s), \quad s \in [-1,1] \tag{2.8}$$

where the coefficients $a_k^4$ are determined to satisfy the interpolation conditions

$$p_4(s_j^4) = \bar{\phi}'(s_j^4), \quad j = 0, \cdots, 4, \tag{2.9}$$

and given as follows [11, 13]

$$a_k^4 = \frac{1}{2} \sum_{j=0}^{4}{}'' \bar{\phi}'(s_j^4) \cos\Big(k s_j^4\Big), \quad 0 \le k \le 4. \tag{2.10}$$

Here, the summation symbol with the double prime denotes a sum whose first and last terms are halved. Also, let $p_6(s)$ be the Chebyshev interpolation polynomial of degree 6 satisfying the interpolation conditions

$$p_6(s_j^6) = \bar{\phi}'(s_j^6), \quad j = 0, \cdots, 6. \tag{2.11}$$

Then, Hassegawa et. al. [11] proved that $p_6(s)$ can be found by using the Newton form

$$p_6(s) = p_4(s) + \sum_{k=1}^{2} b_k \Big(T_{4-k}(s) - T_{4+k}(s)\Big), \tag{2.12}$$

where the coefficients $b_k$ are determined to satisfy the conditions

$$p_6(\nu_j) = \bar{\phi}'(\nu_j), \quad j = 0, 1, \tag{2.13}$$

where $\nu_j$ are defined in (2.5). Define $a_k^6$ by

$$a_k^6 = \begin{cases} a_k^4, \quad 0 \le k < 2, \\ a_k^4 + b_{4-k}, \quad 2 \le k < 4, \\ \frac{a_4^4}{2}, \quad k = 4, \\ -b_{k-4}, \quad 4 < k \le 6. \end{cases} \tag{2.14}$$

By combining (2.13) with (2.12), the Chebyshev interpolation polynomial $p_6(s)$ can be written in term of $a_k^6$ as follows:

$$p_6(s) = \sum_{k=0}^{6}{}' a_k^6 T_k(s), \quad s \in [-1,1], \tag{2.15}$$

where the summation symbol with the prime denotes a sum whose first term is halved. By using the interpolation conditions (2.9) and (2.11), the interpolation polynomial $p_N(s)$ for both (2.8) and (2.15) can be expressed in terms of Lagrange polynomials of order $N$, $l_k^N(s)$ such that

$$l_k^N(s) := \frac{q_N(s)}{(s-t_k^N)\dot{q}_N(s_k^N)}, \quad q_N(s) := \prod_{j=0}^{N}(s-s_j^N), \tag{2.16}$$

where $\dot{q}_N(s) = \frac{dq_N(s)}{ds}$. With the basis function $l_k^N(s)$ of (2.16), the derivative $\bar{\phi}'(s)$ for the solution $\bar{\phi}(s)$ of (2.3) can be approximated by the Chebyshev interpolation polynomial $p_N(s)$ such that

$$p_N(s) = \sum_{k=0}^{N} \bar{\phi}'(s_k^N) l_k^N(s) = \frac{h}{2}\sum_{k=0}^{N} f\big(t(s_k^N), \bar{\phi}(s_k^N)\big) l_k^N(s) \tag{2.17}$$

whose error $\rho_N(s)$ is given by the relation by

$$\bar{\phi}'(s) = p_N(s) + \rho_N(s). \tag{2.18}$$

### 2.2. Approximation scheme for solving (2.3)

Now, we are ready to drive an accurate scheme for approximating $\phi(t_{m+1}) = \bar{\phi}(1)$ based on the equations (2.17) and (2.18). Integrating both sides of (2.18) and combining the result with (2.17) yield the way how the intra-step values $\bar{\phi}(s_k^N)$, $k = 1, \cdots, N$, are determined by collocating the residual $r_N(s)$, which is

$$r_N(s) := \bar{\phi}(s) - \bar{\phi}(-1) - \frac{h}{2}\sum_{k=0}^{N} f\Big(t(s_k^N), \bar{\phi}(s_k^N)\Big)\int_{-1}^{s} l_k^N(\xi)d\xi. \tag{2.19}$$

By using the initial condition of (2.3) and collocating the residual $r_N(s)$ of (2.19) at $N$-points $s_j^N$, $j = 1, \cdots, N$, we have the nonlinear discrete system, for $1 \le j \le N$,

$$\bar{\phi}(s_j^N) - \frac{h}{2}\sum_{k=1}^{N} f\Big(t(s_k^N), \bar{\phi}(s_k^N)\Big)a_{jk}^N = \tilde{\phi}_m + \frac{h}{2} f\Big(t(s_0^N), \tilde{\phi}_m\Big)a_{j0}^N + r_j^{N,m}, \tag{2.20}$$

where

$$\begin{aligned} a_{jk}^N &:= \int_{-1}^{s_j^N} l_k^N(\xi)d\xi, \\ r_j^{N,m} &:= \int_{-1}^{s_j^N} \rho_N(\xi)d\xi + \frac{h}{2}\Big(f\Big(t(s_0^N), \phi(t_m)\Big) - f\Big(t(s_0^N), \tilde{\phi}_m\Big)\Big)a_{j0}^N. \end{aligned} \tag{2.21}$$

**Remark 2.1.** The quantities $r_j^{N,m}$ in (2.21) can be neglected because (i) $\phi(t_m)-\tilde{\phi}_m$ is the error induced from the previous time interval $[t_{m-1}, t_m]$ and is quite small (see Theorem 3.1), (ii) the first term of $r_j^{N,m}$ is the main part of $r_j^{N,m}$, which are the errors induced from the truncation error $\rho_N(s)$ for the interpolation polynomial and are quite small (see Theorem 2.1 of [15]).

For convenience, let us define matrix as

$$\mathcal{A}_N := \left(a_{jk}^N\right), \quad 1 \le j, k \le N, \tag{2.22}$$

where $a_{jk}$ is defined in (2.21), and vectors as

$$\begin{aligned} \boldsymbol{\alpha}^N &:= \left[\boldsymbol{\alpha}_1^N, \cdots, \boldsymbol{\alpha}_N^N\right]^T, \quad \boldsymbol{\gamma}^{N,m} := \left[r_1^{N,m}, \cdots, r_N^{N,m}\right]^T, \\ \mathbf{b}_N &:= \left[a_{10}^N, \cdots, a_{N0}^N\right]^T, \quad \boldsymbol{F}(\boldsymbol{\alpha}^N) := \left[f\left(t_{m1}, \boldsymbol{\alpha}_1^N\right), \cdots, f\left(t_{mN}, \boldsymbol{\alpha}_N^N\right)\right]^T, \end{aligned} \tag{2.23}$$

Then, based on Remark 2.1, instead of solving (2.20) for $\bar{\phi}(s_k^N)$, we will approximate it by $\boldsymbol{\alpha}_k^N$ which satisfies the nonlinear discrete Chebyshev collocation system

$$\boldsymbol{\alpha}^N = \frac{h}{2}\mathcal{A}_N \boldsymbol{F}(\boldsymbol{\alpha}^N) + \tilde{\phi}_m + \frac{h}{2} f(t_m, \tilde{\phi}_m)\mathbf{b}_N. \tag{2.24}$$

Solving the nonlinear system (2.24), we obtain, in particular, the required approximation value at the final point on the interval, $\phi(t_{m+1}) = \bar{\phi}(s_N^N) \approx \boldsymbol{\alpha}_N^N$. Recall that the approximation $\tilde{\phi}_m$ consists of the approximation $\phi_m$ and its truncation error $e_m$ for $\phi(t_m)$ and $E_m = \phi(t_m) - \phi_m$, respectively, and both $\phi_m$ and $e_m$ are assumed to be already calculated at the previous time $t_m$. Also, observe that the solution $\boldsymbol{\alpha}_N^N$ of (2.24) is dependent of them. Hence, it is needed to find formulae for both $\phi_{m+1}$ and $e_{m+1}$, where $e_{m+1}$ must be an approximation for the error $E_{m+1} = \phi(t_{m+1}) - \phi_{m+1}$. The Newton formula (2.12) between $p_6(s)$ and $p_4(s)$ shows that the error $\rho_4(s)$ for the Chebyshev interpolation polynomial $p_4(s)$ can be estimated by using the difference between $p_6(s)$ and $p_4(s)$. Therefore, if we let $\phi_{m+1} := \boldsymbol{\alpha}_4^4$, then the Newton formula yields a way how the error $E_{m+1} := \phi(t_{m+1}) - \phi_{m+1}$ is approximated by subtracting $\boldsymbol{\alpha}_4^4$ from $\boldsymbol{\alpha}_6^6$. Hence, we would like to propose our algorithm as follows.

$$\begin{cases} \phi_{m+1} = \boldsymbol{\alpha}_4^4, \\ e_{m+1} = \boldsymbol{\alpha}_6^6 - \boldsymbol{\alpha}_4^4, \quad m \ge 0, \end{cases} \tag{2.25}$$

with initial conditions $\phi_0 = \phi(t_0)$ and $e_0 = 0$.

In the algorithm (2.25), even though the quantity $e_{m+1}$ is an approximation for the local truncation error of the approximation $\phi_{m+1}$, one can consider it as an estimation for the local truncation error for the higher order approximation

$\boldsymbol{\alpha}_6^6 = \phi_{m+1} + e_{m+1} = \tilde{\phi}_{m+1}$, which is obtained by adding two equations in (2.25). So, we will use this estimated error $e_{m+1}$ for the variable time step size. As discussed in Introduction, comparing to most existing implicit schemes, we perform a different procedure to obtain the estimation error $e_{m+1}$ and the lower order solution $\boldsymbol{\alpha}_4^4$ , which uses the same type of nonlinear equation (2.24) for the higher order solution $\boldsymbol{\alpha}_6^6$. As a technique for solving the nonlinear system (2.24), we will use the simplified Newton iteration([9, 15]) and the eigenvalue decomposition for the matrix $\mathcal{A}_N$, which can be found in [15]. Finally, we note that the scheme is also applicable to a system of ordinary differential equations of the form

$$\boldsymbol{\phi}'(t) = \boldsymbol{f}(t, \boldsymbol{\phi}(t)), \quad t \in (t_0, t_f]; \quad \boldsymbol{\phi}(t_0) = \boldsymbol{\phi}_0, \tag{2.26}$$

where $\boldsymbol{\phi} : \mathbb{R}^d \to \mathbb{R}^d$ and $\boldsymbol{f} : [t_0, t_f] \times \mathbb{R}^d \to \mathbb{R}^d$.

## 3. Convergence and Stability Analysis

In this section, we will analyze the actual error

$$\mathcal{E}_m = \phi(t_m) - \tilde{\phi}_m, \quad m = 1, 2, \cdots$$

between the exact solution $\phi$ of (2.1) and the approximate solution $\tilde{\phi}_m$ obtained by (2.25). The following convergence analysis can be generalized in a straightforward way to a high dimension system provided some vector and matrix notations are applied. We begin this section by introducing functions $\mathcal{G}_j$ defined by

$$\mathcal{G}_j(x_0, \cdots, x_6) := x_j - x_0 - \frac{h}{2}\sum_{k=0}^{6} f(t(s_k^6), x_k)a_{jk}^6, \quad j = 1, \cdots, 6, \tag{3.1}$$

where $h = t_{m+1} - t_m$ and $a_{jk}^6$ are defined by (2.21). Then, for the two solutions $\phi(t)$ and $\boldsymbol{\alpha}^6$ of (2.1) and (2.24), respectively, in each interval $[t_m, t_{m+1}]$, if we define

$$\Phi := \left[\bar{\phi}(s_0^6), \bar{\phi}(s_1^6), \cdots, \bar{\phi}(s_6^6)\right]^T, \quad \Psi := \left[\boldsymbol{\alpha}_0^6, \boldsymbol{\alpha}_1^6, \cdots, \boldsymbol{\alpha}_6^6\right]^T, \tag{3.2}$$

where $\bar{\phi}(s_j^6) = \phi(t(s_j^6))$ and $\boldsymbol{\alpha}_0^6 := \tilde{\phi}_m = \phi_m + e_m$, then using the two equations (2.20) and (2.24) and applying the Taylor's series expansion for $\mathcal{G}_j(\Phi)$ about the vector $\Psi$ yield the way how the intra-step errors $\epsilon_k := \bar{\phi}(s_k^6) - \boldsymbol{\alpha}_k^6$ are determined by solving the system of equations given by

$$\begin{aligned} r_j^{6,m} &= \mathcal{G}_j(\Phi) = \mathcal{G}_j(\Psi) + \frac{\partial \mathcal{G}_j}{\partial x_0}\mathcal{E}_m + \sum_{k=1}^{6}\frac{\partial \mathcal{G}_j}{\partial x_k}\epsilon_k, \\ &= \frac{\partial \mathcal{G}_j}{\partial x_0}\mathcal{E}_m + \sum_{k=1}^{6}\frac{\partial \mathcal{G}_j}{\partial x_k}\epsilon_k, \quad j = 1, \cdots, 6, \end{aligned} \tag{3.3}$$

where $r_j^{6,m}$ is defined by (2.21) and

$$\frac{\partial \mathcal{G}_j}{\partial x_k} = \begin{cases} \delta_{kj} - \dfrac{h}{2} a_{jk}^6 f_\phi(t(s_k^6), \xi_k), & k \neq 0, \\ -1 - \dfrac{h}{2} a_{j0}^6 f_\phi(t(s_0^6), \xi_0), & k = 0, \end{cases} \tag{3.4}$$

for some intermediate points $\xi_k$ between $\bar{\phi}(s_k^6)$ and $\boldsymbol{\alpha}_k^6$. By applying the mean value theorem for the second term of $r_j^{6,m}$ in (2.21), we have

$$r_j^{6,m} = \int_{-1}^{s_j^6} \rho_6(\xi) d\xi + \frac{h}{2} \mathcal{E}_m f_\phi(t(s_0^6), \zeta_0) a_{j0}^6, \tag{3.5}$$

for some $\zeta_0$ between $\phi(t_m)$ and $\tilde{\phi}_m$. For convenience, let us define a matrix and a vector as

$$\begin{aligned} \mathcal{J} &= \Big(J_{ik}\Big), \quad J_{jk} := a_{jk}^6 f_\phi(t(s_k^6), \xi_k), \quad 1 \leq j, k \leq 6, \\ \boldsymbol{g} &= \Big[g_1, \cdots, g_6\Big]^T = \left[\int_{-1}^{s_1^6} \rho_6(\xi) d\xi, \cdots, \int_{-1}^{s_6^6} \rho_6(\xi) d\xi\right]^T. \end{aligned}$$

Then, the three equations (3.3), (3.4) and (3.5) gives a linear system

$$\Big(\boldsymbol{I}_6 - \frac{h}{2}\mathcal{J}\Big)\boldsymbol{\epsilon} = \mathcal{E}_m\Big(\boldsymbol{1} + \frac{h}{2}\Big(f_\phi(t(s_0^6), \xi_0) + f_\phi(t(s_0^6), \zeta_0)\Big)\boldsymbol{b}_6\Big) + \boldsymbol{g} \tag{3.6}$$

for $\boldsymbol{\epsilon} = \Big[\epsilon_1, \cdots, \epsilon_6\Big]^T$ and $\boldsymbol{b}_6 = \Big[a_{10}^6, \cdots, a_{60}^6\Big]^T$. By letting $\mathcal{B} = \boldsymbol{I}_6 - \frac{h}{2}\mathcal{J}$, one can get

$$\begin{aligned} \boldsymbol{\epsilon} &= \mathcal{B}^{-1}\Big[\mathcal{E}_m\Big(\boldsymbol{1} + \frac{h}{2}\Big(f_\phi(t(s_0^6), \xi_0) + f_\phi(t(s_0^6), \zeta_0)\Big)\boldsymbol{b}_6\Big) + \boldsymbol{g}\Big] \\ &= \mathcal{B}^{-1}\mathcal{E}_m + \frac{h}{2}\mathcal{B}^{-1}\mathcal{E}_m\Big(f_\phi(t(s_0^6), \xi_0) + f_\phi(t(s_0^6), \zeta_0)\Big)\boldsymbol{b}_6 + \mathcal{B}^{-1}\boldsymbol{g}, \end{aligned} \tag{3.7}$$

where $\boldsymbol{1} = [1, \cdots, 1]^T$. Note that $\mathcal{B}$ is nonsingular for sufficiently small $h$ and there is a constant $C$ independent of $h$ such that

$$||\mathcal{B}^{-1}|| \leq ||(I - \frac{h}{2}\mathcal{J})^{-1}|| \leq \frac{1}{1 - \frac{h}{2}||\mathcal{J}||} \leq C.$$

Using the fact $f_\phi$ is bounded and the theorem of Geometric series for $(\boldsymbol{I}_6 - \frac{h}{2}\mathcal{J})^{-1}$, the last component $E_{m+1}$ of the vector $\boldsymbol{\epsilon}$ in (3.7) can be estimated by

$$\begin{cases} \Big|\mathcal{E}_{m+1}\Big| \leq \Big(1 + Ch\Big)\Big|\mathcal{E}_m\Big| + D\delta_6, & m \geq 0, \\ E_0 = 0, \end{cases} \tag{3.8}$$

for some constants $C$ and $D$ independent of $h$, where

$$\delta_6 = \max_{1\le j\le 6}\left|\int_{-1}^{s_j^6}\rho_6(\xi)d\xi\right|. \tag{3.9}$$

Consequently, we get the following convergence theorem.

**Theorem 3.1.(Convergence)** *For the final time $t_f$ and sufficiently small time step size $h$ with $mh \le t_f, m \ge 0$, the actual error $\mathcal{E}_m$ can be estimated by*

$$|\mathcal{E}_m| \le D(\exp(Ct_f)-1)\frac{\delta_6}{h}, \quad m \ge 0, \tag{3.10}$$

*where $C$ and $D$ are some constants.*

*Proof.* By mathematical induction, it is easy to show that the difference equation (3.8) can be solved by

$$|\mathcal{E}_m| \le D\frac{(1+Ch)^m-1}{Ch}\delta_6, \quad m \ge 0. \tag{3.11}$$

If $mh \le t_f$, then $1+Ch \le \exp(Ch)$ and $(1+Ch)^m \le \exp(Cmh) \le \exp(Ct_f)$. Therefore, inequality (3.11) provides the desired result. $\square$

**Remark 3.2.** *Combining Theorem 3.1 and Lemma 3.1 of [15], one can see that the approximation $\{\tilde{\phi}_m\}$ has the convergence of order 7.*

It must be noted that the proposed algorithm (2.24) uses the Chebyshev interpolation polynomial with the same sample points in [15] for the derivative of the solution $\phi$ instead the solution $\phi$, which is the main difference with the algorithm of [15].

For the stability analysis of the algorithm (2.25), we will try Dalquist's test problem $\phi'(t) = f(t,\phi(t)) = \lambda\phi(t)$. For Dahlquist's test problem, $f(t,\phi) = \lambda\phi$ is a linear function and hence the right-hand side of (2.24) becomes

$$\frac{h}{2}\Big(\mathcal{A}_N \otimes \boldsymbol{I}_d\Big)\boldsymbol{F}(\boldsymbol{\alpha}^N) + \mathbf{1}\otimes\tilde{\phi}_m + \frac{h}{2}\mathbf{b}_N \otimes f(t_m,\tilde{\phi}_m) = \frac{\lambda h}{2}\mathcal{A}_N\boldsymbol{\alpha}^N + \tilde{\phi}_m\Big(\mathbf{1} + \frac{\lambda h}{2}\mathbf{b}_N\Big).$$

Hence, the discrete Chebyshev collocation system (2.24) applied to Dalquist's test problem $\phi'(t) = f(t,\phi(t)) = \lambda\phi(t)$ becomes

$$\Big(\boldsymbol{I}_N - \frac{\lambda h}{2}\mathcal{A}_N\Big)\boldsymbol{\alpha}^N = \tilde{\phi}_m\Big(\mathbf{1} + \frac{\lambda h}{2}\mathbf{b}_N\Big) \tag{3.12}$$

with the unknown $\boldsymbol{\alpha}^N = [\boldsymbol{\alpha}_1^N, \cdots, \boldsymbol{\alpha}_N^N]^T$. Thus, by solving the linear system (3.12) with $N = 6$, the algorithm (2.25) becomes

$$\tilde{\phi}_{m+1} = \mathcal{S}(\lambda h)\tilde{\phi}_m, \tag{3.13}$$

where the stability function $\mathcal{S}(z)$ is the last component of the vector $\Big(\boldsymbol{I}_6 - \frac{z}{2}\mathcal{A}_6\Big)^{-1}\Big(\mathbf{1} + \frac{z}{2}\mathbf{b}_6\Big)$.

For the stability function $\mathcal{S}(z)$, the stability region of the method is defined by (see [9, p.16])

$$\Gamma := \{z \in \mathbb{C} : |\mathcal{S}(z)| < 1\}.$$

When the left-half complex plane is contained in $\Gamma$, the method is called $A$-stable [9, p.42].

Using a symbolic calculation with Mathematica, the explicit formula for the stability function $\mathcal{S}(z)$ can be obtained as follows.

$$R(z) = \frac{\mathcal{N}(z)}{\mathcal{D}(z)}, \tag{3.14}$$

where

$$\mathcal{N}(z) = \Big(1, \frac{1}{2}, \frac{76+\sqrt{2}}{672}, \frac{20+\sqrt{2}}{1344}, \frac{130+17\sqrt{2}}{107520}, \frac{38+11\sqrt{2}}{645120}, \frac{2+\sqrt{2}}{1290240}\Big),$$
$$\mathcal{D}(z) = \Big(1, -\frac{1}{2}, \frac{76+\sqrt{2}}{672}, -\frac{20+\sqrt{2}}{1344}, \frac{130+17\sqrt{2}}{107520}, -\frac{38+11\sqrt{2}}{645120}, -\frac{2+\sqrt{2}}{1290240}\Big),$$

where the notation $(c_0, \cdots, c_k)$ means the polynomial $\sum_{j=0}^{k} c_j z^j$ with respect to $z$. The corresponding stability region $\Gamma$ for $\mathcal{S}(z)$ is drawn in Fig. 1. From Fig. 1, it can

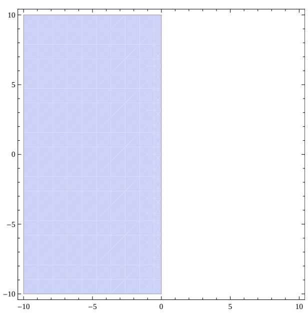


Figure 1: Stability Region

be noted that the proposed algorithm (2.25) is $A$-*stable*.

## 4. Numerical Results

In this section, we test Van der Pol problem, which is widely used to give numerical evidences and the efficiency of a numerical method. It is a nonlinear stiff

IVP describing the behavior of vacuum tube circuits, proposed by B. Van der Pol in 1920's. The problem consists of a system of two equation given by

$$\begin{cases} \phi_1'(t) = \phi_2(t), \\ \phi_2'(t) = ((1-\phi_1(t)^2)\phi_2(t) - \phi_1(t))/\epsilon, \quad t \in (0,2] \end{cases} \tag{4.1}$$

with initial conditions $\phi_1(0) = 2, \quad \phi_2(0) = 0$. For numerical comparisons, we use two implicit methods, Radau5 [9] and CCM46 [15] and denote the proposed algorithm with ICCM46 simply. The efficiency for each method is measured with the computational time (cputime) and the number of function evaluations (nfeval) required to solve each problem. The cputime and nfeval are calculated by varying relative tolerances (Rtol) and absolute tolerances (Atol). All numerical simulations are executed with the software Visual Studio 2010 C++ under OS Windows 7. For numerical simulation, we set the damping parameter $\epsilon$ to $\epsilon = 10^{-6}$. The reference solution at the end of integration interval has been taken from the test set in [1],

$$\phi_1(2) = 0.1706167732170483 \times 10^1, \quad \phi_2(2) = -0.8928097010247975 \times 10^0.$$

We calculate cputime, nfeval, nstep and the relative $L_2$-norm error of numerical solution at the end time corresponding to given tolerance $(\text{Rtol}, \text{Atol}) = (10^{-n}, 10^{-n-2}), \quad n = 7, 8, 9, 10$. The numerical results are displayed in Fig. 2. Note that the calculated cputime is a average time of execute the numerical scheme 100 times. The Fig. 2 shows that the proposed scheme is more efficient than the other methods.

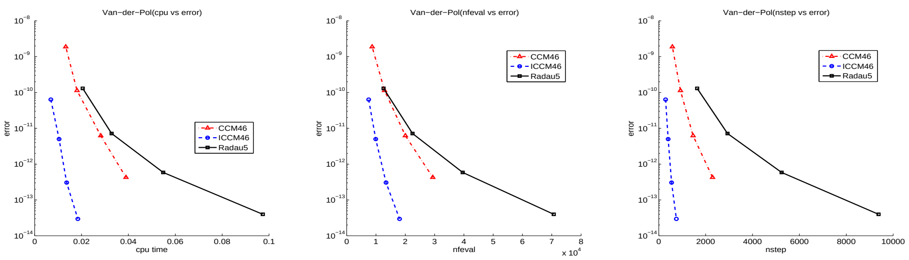


Figure 2: Comparisons of errors with function of CPU time, function of nstep and function of nfeval by varying tolerances(Rtol,Atol)=$(10^{-n}, 10^{-n-2})$, $n = 7, 8, 9, 10$

In Fig. 3, we display the step sizes of different numerical schemes to solve the Van der Pol Problem with $(\text{Rtol}, \text{Atol}) = (10^{-7}, 10^{-9})$. Fig. 3 shows that the proposed scheme uses much bigger step sizes than those for the other schemes.

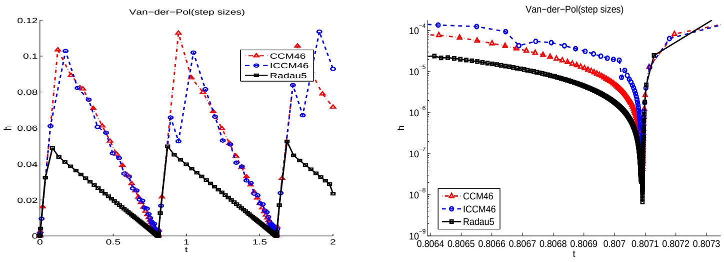


Figure 3: Step size in whole interval(left) and the detailed figures in the stiff region

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