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Delayed Dynamics of Prey-Predator System with Distinct Functional Responses

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Abstract. In this article, a mathematical model is proposed and analyzed to study the delayed dynamics of a system having a predator and two preys with distinct growth rates and functional responses. The equilibrium points of proposed system are determined and the local stability at each of the possible equilibrium points is investigated by its corresponding characteristic equation. The boundedness of the system is established in the absence of delay and the condition for existence of persistence in the system is determined. The discrete type gestational delay of predator is also incorporated on the system. Further it is proved that the system undergoes Hopf bifurcation using delay as bifurcation parameter. This study refers that time delay may have an impact on the stability of the system. Finally Computer simulations illustrate the dynamics of the system.

1. Introduction

The population dynamics of the predator and its prey brings to light diversity of patterns that have appeared in nature. Mathematical models have been designed to describe the predator-prey interaction. Analysis of the dynamical behavior of predator-prey systems is an area of interest for many researchers because of its complexity and challenging situation. The most noticeable element in the predator-prey relationship is functional response. Many of the predator-prey mod-

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els have functional response that depends on prey density and their properties is well understood. A recent proposal by biologists infer that the functional response depends on the ratio of prey and predator. This kind of functional response is said to be ratio-dependent. In the past decades, researchers mathematically modeled the predator-prey behaviour having ratio-dependent functional response(see Arditi [1], Akchaya [3] and Abrams [4] and references citied there in).

It is pointed out that qualitative analysis of food chain and multispecies models based on ratio-dependent approach exists in Kesh [24],Gakkar [14], Baek [7]. It has been documented in the study of Kuang [25], Hsu [19] and Xiao [35] that ratio dependent models produce richer and more reasonable dynamics. Jost and Ellner [21] proposed and analysed a two species model with ratio dependent III functional response. Agarwal [2] generalized the three species model (one prey-two predators) with ratio-dependent III functional response.There are enormous numbers of food chain models consisting of two or more species with unique functional responses. The system representing the interaction between three species shows complex dynamical behavior. For further reference see Gakkar [12,13,15], Kumar [27], Beak [6], Samantha [31], Tripathi [33], Fan [10], Patra[29], Freedman[9]. The interaction of species involving persistence and extinction have been the area of interest for the researchers Dubey [8], Kar [22,23], Naji [28].

The literature survey above infers that most models have same growth rates and functional response. But this is biologically unrealistic in nature. The reality is that predation happens on different preys in a number of consumption ways. To describe this happening, two different types of functional response are necessary. And it is also well-known that growth rate of different species is different. Sahoo [32] proposed that a real prey-predator model is constructed with different growth rates and different functional responses. So, in this paper, two prey species, one with Verhulst [34] logistic growth equation and other with Richards [30] growth equation is taken into account along with two types of functional responses namely Holling type I and Ratio-dependent III functional response.

This paper is organized as follows. We start in section 2 by defining the mathematical model of three species population which consists of two preys and one predator. The non linear system of differential equations that govern this system is introduced. Section 3 deals with the determination of equilibrium points and their existence conditions. In section 4, we analyzed dynamical behavior of these equilibrium points. Global stability and Persistence of the system is studied in section 5. In section 6, analysis of the model in presence of discrete delay is discussed. In section 7 is equipped with numerical simulation and discuss the problem.

2. Mathematical Model

Mathematical model considered is based on the predator-prey system with Holling type I and Ratio dependent type III functional response .The predator exhibits a Holling type I response to one prey and a Ratio dependent type III response to the other prey.

$$\begin{aligned}
\frac{dX}{dT} &= RX\left(1-\frac{X}{K}\right)-\lambda_1 XZ,\\
\frac{dY}{dT} &= SY\left(1-\left(\frac{Y}{L}\right)^{\beta}\right)-\frac{\lambda_2 Y^2 Z}{aZ^2+Y^2},\\
\frac{dZ}{dT} &= b_1\lambda_1 XZ+b_2\frac{\lambda_2 Y^2 Z}{aZ^2+Y^2}-cZ,
\end{aligned}\tag{2.1}$$

where X, Y denote population densities of prey and Z denote population density of the predator. In model (2.1) R and S are the intrinsic growth rate of two prey species, K and L are their carrying capacities, c is mortality rate of the predator,β is intraspecific competition factor, λ_1 and λ_2 denote prey species searching efficiency of the predator, a is the half-saturation co-efficient, b_1 and b_2 are the conversion factors denoting the number of newly born predators for each captured of first and second prey respectively.

In order to minimize the number of parameters involved with the model system, it is extremely useful to write the system in non-dimensionalized form. For this purpose we introduce the variables X, Y, Z and T as follows $x \to \frac{X}{K}, y \to \frac{Y}{L}, z \to \frac{\sqrt{a}Z}{L}$ and $t \to TRS$.

In terms of the non-dimensionalized variables the model system (2.1) becomes

$$\begin{aligned}
\frac{dx}{dt} &= rx(1-x)-c_1xz,\\
\frac{dy}{dt} &= sy\left(1-(y)^{\beta}\right)-\frac{c_2y^2z}{y^2+z^2},\\
\frac{dz}{dt} &= w_1c_1xz+w_2\frac{c_2y^2z}{y^2+z^2}-ez,
\end{aligned}\tag{2.2}$$

where $r=\frac{1}{S}, s=\frac{1}{R}, c_1=\frac{\lambda_1 L}{\sqrt{a}RS}, c_2=\frac{\lambda_2}{\sqrt{a}RS}, e=\frac{c}{RS}, w_1=\frac{b_1K\sqrt{a}}{L}, w_2=b_2\sqrt{a}$.

Definition 2.1. The solution of $\dot{x}=f(t,x)$ is said to be *uniformly bounded* if $\exists c>0$ and for every $0<a<c, \exists M=M(a)>0$ such that $||x(t_0)||\le a \Rightarrow ||x(t)||\le M, \forall t\ge t_0\ge 0$.

Theorem 2.2. *All the solutions of the system (2.2) with positive initial condition* (x_0, y_0, z_0) *are uniformly bounded within a region* Γ *where*

$$\Gamma=\left\{(x,y,z)\in R_+^3 : 0\le x\le 1, 0\le L\le \frac{w_1 r}{\delta}+\epsilon,\ \textit{for any } \epsilon>0\right\}.$$

Proof. Since the densities of population can never be negative, obviously the solutions $x(t), y(t)$ and $z(t)$ are positive for all $t \geq 0$. From the first equation of model (2.2), we have

$$\frac{dx}{dt} \leq rx(1-x).$$

This gives $\lim_{t\to\infty} \sup x(t) \leq 1$. Consider $L = w_1 x + w_2 y + z$. Then

$$\frac{dL}{dt} = w_1 \frac{dx}{dt} + w_2 \frac{dy}{dt} + \frac{dz}{dt}. \tag{2.3}$$

Substituting (2.2) in equation (2.3), we get

$$\frac{dL}{dt} = w_1 rx(1-x) + w_2 sy(1-y^{\beta}) - ez,$$
$$\frac{dL}{dt} \leq w_1 rx + w_2 sy - ez \leq w_1 r - \delta L.$$

where $\delta = \min(rw_1, sw_2, e)$. Therefore

$$\frac{dL}{dt} + \delta L \leq w_1 r.$$

Applying Birkoff [5] Lemma on differential inequalities then, we have

$$0 \leq L(x,y,z) \leq \frac{w_1 r}{\delta}\left(1 - e^{-\delta t}\right) + \frac{w(x(0), y(0), z(0))}{e^{\delta t}}.$$

Thus for $t \to \infty$ we have $0 \leq L(x,y,z) \leq \frac{w_1 r}{\delta}$. Thus all solutions of system (2.2) enter into the region

$$\Gamma = \left\{(x,y,z) \in R_+^3 : 0 \leq x \leq 1, 0 \leq L \leq \frac{w_1 r}{\delta} + \epsilon, \text{ for any } \epsilon > 0\right\}. \qquad \square$$

3. Existence of Equilibrium Points with Feasibility Condition

It can be checked that the system (2.2) has six non-negative equilibrium and two of them namely $E_0(0,0,0), E_1(1,0,0)$ is always exists. We show that the existence of other equilibrium as follows

Existence of $E_2(\tilde{x}, \tilde{y}, 0)$.

Here $\tilde{x}, \tilde{y}$ are the positive solutions of the following algebraic equations

$$r(1-x) = 0, \tag{3.1}$$

$$s(1-y^{\beta}) = 0. \tag{3.2}$$

Solving (3.1) and (3.2) we get $\tilde{x} = 1, \tilde{y} = 1$.

Existence of $E_3(\overline{x}, 0, \overline{z})$.

Here $\overline{x}, \overline{z}$ are the positive solutions of the following algebraic equations

$$r(1-x) - c_1 z = 0, \tag{3.3}$$

$$w_1 c_1 x - e = 0. \tag{3.4}$$

Solving (3.3) and (3.4) we get

$$\overline{x} = \frac{e}{w_1 c_1}, \overline{z} = \frac{r(w_1 c_1 - e)}{w_1 c_1^2}.$$

We see that $E_3(\overline{x}, 0, \overline{z})$ exists if $w_1 c_1 > e$.

Existence of $E_4(0, \hat{y}, \hat{z})$

Here $\tilde{y}, \hat{z}$ are the positive solution of the following algebraic equations

$$s(1-y^\beta) - \frac{\lambda_2 y z}{z^2 + y^2} = 0, \tag{3.5}$$

$$\frac{w_2 c_2 y^2}{z^2 + y^2} - e = 0. \tag{3.6}$$

Solving (3.5) and (3.6) we get

$$\hat{y} = \left[\frac{w_2 s - c_2\sqrt{e(w_2 c_2 - e)}}{w_2 s}\right]^{1/\beta}, \hat{z} = \sqrt{\frac{w_2 c_2 - e}{e}}\ \hat{y}.$$

We see that the equilibrium $E_4(0, \hat{y}, \hat{z})$ exists if $w_2 s > (c_2\sqrt{e(w_2 c_2 - e)})$.

Existence of $E_5(x^*, y^*, z^*)$

Here (x^*, y^*, z^*) is the positive solution of the system of algebraic equation given below:

$$r(1-x) - c_1 z = 0, \tag{3.7}$$

$$s(1-y^\beta) - \frac{c_2 . y . z}{z^2 + y^2} = 0, \tag{3.8}$$

$$w_1 c_1 x + \frac{w_2 c_2 y^2}{z^2 + y^2} - e = 0. \tag{3.9}$$

Solving (3.7), (3.8) and (3.9) we get

$$x^* = \frac{(s w_1 w_2 c_1 - r w_1) \pm \sqrt{(s w_1 w_2 c_1 - r w_1)^2 + 4 r w_1 (e + s w_2 e)}}{2 r w_1},$$

$$y^*(1 - y^{*\beta}) = \frac{r(1-x^*)(e - w_1 c_1 x^*)}{s w_2 c_1},$$

$$z^* = \frac{r(1-x^*)}{c_1}.$$

4. Dynamical Behaviour

We shall examine the stability of the system (2.2), the variational matrix relating to every equilibrium steady state is measured.

$$E(x,y,z) = \begin{pmatrix} r-2rx-c_1z & 0 & -c_1x \\ 0 & s-s(\beta+1)y^{\beta}-\dfrac{2c_2yz^3}{(z^2+y^2)^2} & \dfrac{-c_2y^2(y^2-z^2)}{(z^2+y^2)^2} \\ w_1c_1z & \dfrac{2w_2c_2yz^3}{(z^2+y^2)^2} & -e+w_1c_1x+\dfrac{w_2c_2y^2(y^2-z^2)}{(z^2+y^2)^2} \end{pmatrix}$$

Theorem 4.1. *The trivial equilibrium point E_0 is stable in z direction and unstable in $x-y$ direction.*

Proof. The variational matrix for the equilibrium point at $E_0(0,0,0)$ is

$$E_0 = \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & -e \end{pmatrix}$$

The eigen values of E_0 are $\lambda_1 = r, \lambda_2 = s$ and $\lambda_3 = -e$. Clearly, two of the eigen values are positive and one of them is negative. Therefore the equilibrium point E_0 is stable in z direction and unstable in $x-y$ direction. This completes the proof. □

Theorem 4.2. *The equilibrium point E_1 is stable in $x-z$ direction and unstable in y direction, if $w_1c_1 < e$. Otherwise unstable in $y-z$ direction and stable in x direction.*

Proof. The variational matrix for the equilibrium point at $E_1(1,0,0)$ is

$$E_1 = \begin{pmatrix} -r & 0 & -c_1 \\ 0 & s & 0 \\ 0 & 0 & w_1c_1-e \end{pmatrix}$$

The eigen values of E_1 are $\lambda_1 = -r, \lambda_2 = s$ and $\lambda_3 = w_1c_1 - e$. If $w_1c_1 < e$, in this case two of the eigen values are negative and one of them is positive. Therefore the equilibrium point E_1 is stable in $x-z$ direction and unstable in y direction. But if $w_1c_1 > e$ it is unstable in $y-z$ direction and stable in x direction. This completes the proof. □

Theorem 4.3. *The equilibrium point E_2 is locally asymptotically stable if $w_1c_1 + w_2c_2 < e$. Otherwise unstable in z direction and stable in $x-y$ direction.*

Proof. The variational matrix for the equilibrium point at $E_2(1,1,0)$ is

$$E_2 = \begin{pmatrix} -r & 0 & -c_1 \\ 0 & -s\beta & -c_2 \\ 0 & 0 & w_1c_1 + w_2c_2 - e \end{pmatrix}$$

The eigen values of E_2 are $\lambda_1 = -r, \lambda_2 = -s\beta$ and $\lambda_3 = w_1c_1 + w_2c_2 - e$. If $w_1c_1 + w_2c_2 < e$ in this case all the eigen values are negative. Therefore the equilibrium point E_2 is locally asymptotically stable. But if $w_1c_1 + w_2c_2 > e$ it is unstable in z direction but stable in $x - y$ direction. This completes the proof. □

Theorem 4.4. *The equilibrium point E_3 is locally asymptotically stable if satisfy the condition $p_1 > 0, p_3 > 0$ and $p_1p_2 - p_3 > 0$ otherwise unstable.*

Proof. The variational matrix for the equilibrium point at $E_3(\overline{x}, 0, \overline{z})$ is

$$E_3 = \begin{pmatrix} -\dfrac{re}{w_1c_1} & 0 & -\dfrac{e}{w_1} \\ 0 & s & 0 \\ \dfrac{r(w_1c_1 - e)}{c_1} & 0 & 0 \end{pmatrix}$$

The corresponding characteristic equation for E_3 is $\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0$, where

$$p_1 = \frac{re - sw_1c_1}{w_1c_1},$$
$$p_2 = \frac{(s + re)(w_1c_1) - re(s + e)}{w_1c_1},$$
$$p_3 = \frac{sre(1 + e - w_1c_1)}{w_1c_1}.$$

By Routh-Hurwitz criteria if $p_1 > 0, p_3 > 0$ and $p_1p_2 - p_3 > 0$ then E_3 is locally asymptotically stable, otherwise it is unstable. □

Theorem 4.5. *The equilibrium point E_4 is locally asymptotically stable if and only if $A^* + B^* + C^* < 0$ and $\Delta > 0$ otherwise unstable.*

Proof. The variational matrix for the equilibrium point at $E_4(0, \hat{y}, \hat{z})$ is

$$E_4 = \begin{pmatrix} A^* & 0 & 0 \\ 0 & B^* & \dfrac{-c_2\hat{y}^2(\hat{y}^2 - \hat{z}^2)}{(\hat{y}^2 + \hat{z}^2)^2} \\ w_1c_1\hat{z} & \dfrac{2w_2c_2\hat{y}\hat{z}^3}{(\hat{y}^2 + \hat{z}^2)^2} & C^* \end{pmatrix}$$

where

$$A^* = r - c_1\hat{z},$$
$$B^* = s - s(\beta + 1)\hat{y}^\beta - \frac{2c_2\hat{y}\hat{z}^3}{(\hat{y}^2 + \hat{z}^2)^2},$$
$$C^* = \frac{w_2c_2\hat{y}^2(\hat{y}^2 - \hat{z}^2)}{(\hat{y}^2 + \hat{z}^2)^2} - e.$$

Here

$$\hat{y} = \left[\frac{w_2 s - \sqrt{e(w_2 c_2 - e)}}{w_2 s}\right]^{1/\beta},$$
$$\hat{z} = \sqrt{\frac{w_2 c_2 - e}{e}}\ \hat{y}.$$

The corresponding characteristic equation for E_4 is $\lambda^3 + q_1\lambda^2 + q_2\lambda + q_3 = 0$ where

$$\begin{aligned}
q_1 &= -(\text{trace of } E_4) = -(A^* + B^* + C^*),\\
q_2 &= A^*B^* + B^*C^* + A^*C^* + D,\\
q_3 &= -(\text{Det of } E_4) = -(A^*(B^*C^* + D)),\\
D &= \frac{2w_2c_2^2\hat{y}^3\hat{z}^3}{(\hat{y}^2 + \hat{z}^2)^4}.
\end{aligned}$$

Also

$$\begin{aligned}
\Delta &= q_1q_2 - q_3\\
&= -(A^* + B^* + C^*)(A^*B^* + B^*C^* + A^*C^* + D) - (-(A^*(B^*C^* + D)))\\
&= A^*(B^*C^* + D) - (A^* + B^* + C^*)(A^*B^* + B^*C^* + A^*C^* + D).
\end{aligned}$$

We notice that

(i) $A^* + B^* + C^* < 0 \Rightarrow q_1 > 0$,

(ii) $q_3 > 0$ for all parameters,

(iii) $\Delta = q_1q_2 - q_3 > 0$.

By using Routh-Hurwitz criteria, the theorem is proved. □

The variational matrix for the equilibrium point at $E_5(x^*, y^*, z^*)$

$$E_5 = \begin{pmatrix} a_{11} & 0 & a_{13}\\ 0 & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

where

$$\begin{aligned}
a_{11} &= r - 2rx^* - c_1z^*, & a_{13} &= -c_1x^*,\\
a_{22} &= s - s(\beta+1){y^*}^{\beta} - \frac{2c_2y^*{z^*}^3}{({z^*}^2 + {y^*}^2)^2}, & a_{23} &= \frac{-c_2{y^*}^2({y^*}^2 - {z^*}^2)}{({z^*}^2 + {y^*}^2)^2},\\
a_{31} &= w_1c_1z^*, & a_{32} &= \frac{2w_2c_2y^*{z^*}^3}{({z^*}^2 + {y^*}^2)^2},\\
a_{33} &= -e + w_1c_1x^* + \frac{w_2c_2{y^*}^2({y^*}^2 - {z^*}^2)}{({z^*}^2 + {y^*}^2)^2}.
\end{aligned}$$

Then corresponding characteristic equation becomes

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0. \tag{4.1}$$

where

$$\begin{aligned}
A_1 &= -(a_{11} + a_{22} + a_{33}) \\
&= 2rx^* + c_1 z^* + s(\beta+1){y^*}^{\beta} + \frac{2c_2 y^* {z^*}^3}{({z^*}^2 + {y^*}^2)^2} \\
&\quad + e - \left(r + s + w_1 c_1 x^* + \frac{w_2 c_2 {y^*}^2 ({y^*}^2 - {z^*}^2)}{({z^*}^2 + {y^*}^2)^2} \right) \\
A_2 &= a_{22}a_{33} - a_{23}a_{32} + a_{11}a_{22} + a_{11}a_{33} - a_{13}a_{31} \\
&= \left[(r - 2rx^* - c_1 z^*) \left(s - s(\beta+1){y^*}^{\beta} - \frac{2c_2 y^* {z^*}^3}{({z^*}^2 + {y^*}^2)^2} \right) \right] \\
&\quad + \left[\left(s - s(\beta+1){y^*}^{\beta} - \frac{2c_2 y^* {z^*}^3}{({z^*}^2 + {y^*}^2)^2} \right) \cdot \left(-e + w_1 c_1 x^* + \frac{w_2 c_2 {y^*}^2 ({y^*}^2 - {z^*}^2)}{({z^*}^2 + {y^*}^2)^2} \right) \right] \\
&\quad + \left[(r - 2rx^* - c_1 z^*) \cdot \left(-e + w_1 c_1 x^* + \frac{w_2 c_2 {y^*}^2 ({y^*}^2 - {z^*}^2)}{({z^*}^2 + {y^*}^2)^2} \right) \right] \\
&\quad + \left[\left(\frac{2w_2 c_2^2 {y^*}^3 {z^*}^3 ({y^*}^2 - {z^*}^2)}{({z^*}^2 + {y^*}^2)^4} \right) \right] + (w_1 c_1^2 x^* z^*) \\
A_3 &= \det(E^*) \\
&= a_{11}a_{32}a_{23} - a_{11}a_{22}a_{33} + a_{13}a_{22}a_{31} \\
&= [r - 2rx^* - c_1 z^*] \left[\frac{-2w_2 c_2^2 {y^*}^3 {z^*}^3 ({y^*}^2 - {z^*}^2)}{({z^*}^2 + {y^*}^2)^4} \right] \\
&\quad - [r - 2rx^* - c_1 z^*] \cdot \left[\left(s - s(\beta+1){y^*}^{\beta} - \frac{2c_2 y^* {z^*}^3}{({z^*}^2 + {y^*}^2)^2} \right) \right. \\
&\quad \left. \cdot \left(-e + w_1 c_1 x^* + \frac{w_2 c_2 {y^*}^2 ({y^*}^2 - {z^*}^2)}{({z^*}^2 + {y^*}^2)^2} \right) \right] \\
&\quad + [w_1 c_1^2 x^* z^*] \cdot \left[\left(s - s(\beta+1){y^*}^{\beta} - \frac{2c_2 y^* {z^*}^3}{({z^*}^2 + {y^*}^2)^2} \right) \right]
\end{aligned}$$

By Routh-Hurwitz criterion it follows that all eigenvalues of characteristic equation of (4.1) has negative real parts if and only if

$$A_1 > 0, A_3 > 0 \text{ and } A_1 A_2 - A_3 > 0. \tag{4.2}$$

Hence the positive equilibrium point $E_5(x^*, y^*, z^*)$ is asymptotically stable. Now we state the following theorem.

Theorem 4.6. *The equilibrium point is $E_5(x^*, y^*, z^*)$ locally asymptotically stable if and only if the inequalities of (4.2) are satisfied.*

5. Global Stability and Persistence

Theorem 5.1. *The interior equilibrium E_2 is globally asymptotically stable in the interior of the quadrant of the $x-y$ plane.*

Proof. Let $H_1(x,y) = \dfrac{1}{xy}$. Clearly $H_1(x,y)$ is positive in the interior of the positive quadrant of $x-y$ plane. Let $h_1(x,y) = rx(1-x)$ and $h_2(x,y) = sy(1-y^{\beta})$. Then

$$\Delta(x,y) = \frac{\partial}{\partial x}(h_1H_1) + \frac{\partial}{\partial y}(h_2H_1) = \frac{-r}{y} - \frac{\beta s y^{\beta-1}}{x} < 0.$$

By using Bendixson-Dulac criteria, we note that $\Delta(x,y)$ remains the same sign and is not identically zero in the interior of the positive quadrant of the $x-y$ plane. This completes the proof. □

We shall now prove that E_3 is globally asymptotically stable.

Theorem 5.2. *The interior equilibrium E_3 is globally asymptotically stable in the interior of the quadrant of the $x-z$ plane.*

Proof. Let $H_2(x,z) = \dfrac{1}{xz}$. Clearly $H_2(x,z)$ is positive in the interior of the positive quadrant of $x-z$ plane. Let $h_1(x,z) = rx(1-x) - c_1xz$ and $h_2(x,z) = w_1c_1xz - ez$. Then

$$\Delta(x,z) = \frac{\partial}{\partial x}(h_1H_2) + \frac{\partial}{\partial z}(h_2H_2) = \frac{-r}{z} < 0.$$

By using Bendixson-Dulac criteria, we note that $\Delta(x,z)$ remains the same sign and is not identically zero in the interior of the positive quadrant of the $x-z$ plane. This completes the proof. □

We shall now prove that E_4 is globally asymptotically stable.

Theorem 5.3. *The interior equilibrium E_4 is globally asymptotically stable in the interior of the quadrant of the $y-z$ plane.*

Proof. Let $H_3(y,z) = \dfrac{1}{yz}$. Clearly $H_3(y,z)$ is positive in the interior of the positive quadrant of $y-z$ plane. Let $h_1(y,z) = sy(1-y^{\beta}) - \dfrac{c_2y^2z}{z^2+y^2}$ and $h_2(y,z) = z\left[-e + \dfrac{w_2c_2y^2}{z^2+y^2}\right]$. Then

$$\Delta(y,z) = \frac{\partial}{\partial y}(h_1H_3) + \frac{\partial}{\partial z}(h_2H_3) = -\left[\frac{\beta s y^{\beta-1}}{z} + \frac{c_2(z^2-y^2) + 2w_2c_2yz}{(z^2+y^2)^2}\right] < 0.$$

By using Bendixson-Dulac criteria, we note that $\Delta(y,z)$ remains the same sign and is not identically zero in the interior of the positive quadrant of the $y-z$ plane. This completes the proof. □

Definition 5.4. A population is said to be uniformly persistent if there exists an $\delta > 0$, independent of $x(0) > 0$ such that $\lim_{t\to\infty} \inf x(t) > \delta$.

Biologically persistence means the survival of all population in future time. Mathematically, persistence of a system means that strictly positive solution does not have omega limit points on the boundary of non-negative cone.

We examine the permanence of the system (2.2) we shall use average lyapunov function Gard [11] and Hafbaucer [17]. This method was first applied by Hutson [20] to ecological problems. Let the average Lyapunov function for the system (2.2) be $\sigma(x) = x^p y^q z^r$ where p, q, r be positive constants. Clearly in the interior of R^3_+, we have

$$\begin{aligned}\Psi(x) &= \frac{\dot{\sigma}(x)}{\sigma(x)} = p\frac{\dot{x}}{x} + q\frac{\dot{y}}{y} + r\frac{\dot{z}}{z}\\ &= p[r(1-x) - c_1 z] + q\left[s(1-y^\beta) - \frac{c_2 yz}{y^2+z^2}\right] + r\left[w_1 c_1 x + w_2\frac{c_2 y^2}{y^2+z^2} - e\right].\end{aligned}$$

Then E_2, E_3, E_4 exists. Further there are no orbits in the interior of $x-y$ plane, $x-z$ plane, and $y-z$ plane. Thus to prove the uniform persistence of the system, it is enough to show that $\Psi(x) > 0$ in the domain of D of R^3_+, where

(5.1)

$$D \equiv \left\{(x,y,z); x>0, y>0, z>0, \beta y^{\beta-1}(z^2+y^2)^2 + \frac{c_2 z((z^2-y^2)+2w_2 z)}{s} > 0\right\}.$$

For a suitable choice of p, q and $r > 0$. That is one that has satisfy the following conditions

$$\begin{aligned}\Psi(E_0) &= pr + qs - re > 0,\\ \Psi(E_1) &= qs + rw_1c_1 - re > 0,\\ \Psi(E_2) &= r(w_1c_1) + rw_2c_2 - re > 0,\\ \Psi(E_3) &= qs > 0,\\ \Psi(E_4) &= p\left[r - c_1\sqrt{\frac{w_2c_2-e}{e}}\left[\sqrt{\frac{w_2s - \sqrt{e(w_2c_2-e)}}{w_2s}}\right]^{1/\beta}\right].\end{aligned}$$

We note that by increasing p to sufficiently large value, $\Psi(E_0)$ can be made positive. Hence we state the following theorem.

Theorem 5.5. *Let the hypotheses of theorems 5.1, 5.2 and 5.3 hold, and then the system (2.2) is uniformly persistent if the following inequalities hold*

$$w_1c_1 + w_2c_2 > e,$$

$$r - c_1\sqrt{\frac{w_2c_2-e}{e}}\left(\sqrt{\frac{w_2s - \sqrt{e(w_2c_2-e)}}{w_2s}}\right)^{1/\beta} > 0.$$

6. Model with Discrete Delay

We apply differential equations for any system involving time delay. Time delay may arise naturally or artificially. Delay differential equations involves complex dynamics compared to ordinary differential equations as time delay may cause stability fluctuations. without time delay a real system may not be well established. The application of time-delay in realistic models is detailed in the books of Gopalsamy [16], Kuang [26].

In this section, we analyze the model system (2.2) with delay τ (discrete time delay in the predator response function). Then the model system (2.2) takes the following form

$$\begin{aligned}\frac{dx}{dt} &= rx(1-x) - c_1xz,\\ \frac{dy}{dt} &= sy(1-(y)^{\beta}) - \frac{c_2y^2z}{y^2+z^2},\\ \frac{dz}{dt} &= w_1c_1x(t-\tau)z + w_2\frac{c_2y^2(t-\tau)z}{y^2(t-\tau)+z^2(t-\tau)} - ez,\end{aligned} \tag{6.1}$$

with the initial densities

$$x(\theta) \geq 0, y(0) \geq 0, z(0) \geq 0, \theta \in (-\tau, 0), \tau \neq 0. \tag{6.2}$$

The main purpose of this section is to study the stability behavior of $E_5(x^*, y^*, z^*)$ in the presence of discrete delay ($\tau \neq 0$). Now to prove the stability behavior of $E_5(x^*, y^*, z^*)$ for the system (6.1), first we linearize the system (6.1) by using following transformation

$$\begin{aligned}x(t) &= x^* + u(t),\\ y(t) &= y^* + v(t),\\ z(t) &= z^* + w(t).\end{aligned}$$

The linear system is given by

$$\begin{aligned}\dot{u}(t) &= a_{11}u(t) + a_{13}w(t),\\ \dot{v}(t) &= a_{22}v(t) + a_{23}w(t),\\ \dot{w}(t) &= c_{31}u(t-\tau) + c_{32}v(t-\tau) + c_{33}w(t-\tau),\end{aligned}$$

where

$$\begin{aligned}&a_{11} = -rx^*, \qquad a_{13} = -c_1x^*,\\ &a_{22} = -s\beta {y^*}^{\beta} - \frac{c_2y^*z^*({z^*}^2 - {y^*}^2)}{({z^*}^2+{y^*}^2)^2}, \qquad a_{23} = \frac{-c_2{y^*}^2({y^*}^2 - {z^*}^2)}{({z^*}^2+{y^*}^2)^2},\\ &c_{31} = w_1c_1z^*, \qquad c_{32} = \frac{2w_2c_2y^*{z^*}^3}{({z^*}^2+{y^*}^2)^2}, \qquad c_{33} = \frac{-2w_2c_2{z^*}^2{y^*}^2}{({z^*}^2+{y^*}^2)^2}.\end{aligned}$$

We look for solution of the model (6.1) of the form $A(\tau) = \rho e^{-\lambda\tau}, \rho \neq 0$. This leads to the characteristic equation

$$\Delta(\lambda,\tau) = (\lambda^3 + l_1\lambda^2 + l_2\lambda) + (l_3\lambda^2 + l_4\lambda + l_5)e^{-\lambda\tau} = 0, \tag{6.3}$$

where

$$\begin{aligned} &l_1 = -a_{11} - a_{22}, \qquad l_2 = a_{11}a_{22}, \qquad l_3 = -c_{33},\\ &l_4 = a_{22}c_{33} + a_{11}c_{33} - a_{13}c_{31} - a_{23}c_{32},\\ &l_5 = a_{13}a_{22}c_{31} + a_{23}a_{11}c_{32} - a_{11}a_{22}c_{33}. \end{aligned}$$

The eigen values are the roots of the characteristic equation (6.3) of the system (6.1) that has infinitely many solutions. We wish to find periodic solution of the system (6.1), for the periodic solution eigenvalues will be purely imaginary. Substituting $\lambda = i\omega$ in equation (6.3) we get

$$[-i\omega^3 - l_1\omega^2 + il_2\omega] + [-l_3\omega^2 + il_4\omega + l_5]e^{-i\omega\tau} = 0$$

Comparing real and imaginary parts, we get

$$\begin{aligned} &l_1\omega^2 = (l_5 - l_3\omega^2)\cos\omega\tau + \omega l_4 \sin\omega\tau,\\ &l_2\omega - \omega^3 = -\omega l_4\cos\omega\tau + (l_5 - l_3\omega^2)\sin\omega\tau. \end{aligned}$$

Squaring and adding we get

$$\omega^6 + s_1\omega^4 + s_2\omega^2 + s_3 = 0, \tag{6.4}$$

where

$$s_1 = l_1^2 - 2l_2 - l_3^2,\ s_2 = l_2^2 + 2l_3l_5 - l_4^2,\ s_3 = -l_5^2.$$

Putting $\omega^2 = \delta$ equation becomes

$$f(\delta) = \delta^3 + s_1\delta^2 + s_2\delta + s_3 = 0. \tag{6.5}$$

Now equation (6.5) will be positive if $s_1 > 0, s_3 < 0$.

By Descartes rule of sign, the cubic equation (6.5), has at least one positive root. Consequently the stability criteria of the system for $\tau = 0$, will not necessarily ensure the stability of system for $\tau \neq 0$. The critical value of delay that is given as

$$\cos\omega\tau = \frac{\omega^4(l_4 - l_1l_3) + \omega^2(l_1l_5 - l_2l_4)}{(l_5 - l_3\omega^2)^2 + l_4^2\omega^2}$$

So corresponding to $\lambda = i\omega_0$ there exists τ_{0n}^* such that

$$\tau_{0n}^* = \frac{1}{\omega_0}\left[\cos^{-1}\left[\frac{\omega_0^4(l_4 - l_1l_3) + \omega_0^2(l_1l_5 - l_2l_4)}{(l_5 - l_3\omega_0^2)^2 + l_4^2\omega_0^2}\right]\right] + \frac{2n\pi}{\omega_0}, n = 0, 1, 2, 3, \ldots$$

Hopf Bifurcation

We observe that the condition's for Hopf bifurcation (Hale [18]) are satisfied yielding the required periodic solution, that is

$$\left[\frac{d(Re\lambda)}{d\tau}\right]_{\tau=\tau^*} \neq 0$$

This signifies that there exists at least one Eigen value with positive real part for $\tau > \tau^*$. Now, we show the existence of Hopf bifurcation near $E_5(x^*, y^*, z^*)$ by taking τ as bifurcating parameter.

Differentiating equation (6.3) with respect to τ,

$$\begin{aligned}
\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{3\lambda^2 + 2l_1\lambda + l_2}{\lambda(l_3\lambda^2 + l_4\lambda + l_5)e^{-\lambda\tau}} + \frac{2l_3\lambda + l_4}{\lambda(l_3\lambda^2 + l_4\lambda + l_5)} - \frac{\tau}{\lambda} \\
&= \frac{2\lambda^3 + l_1\lambda^2 - (l_3\lambda^2 + l_4\lambda + l_5)e^{-\lambda\tau}}{\lambda^2(l_3\lambda^2 + l_4\lambda + l_5)e^{-\lambda\tau}} + \frac{2l_3\lambda^2 + l_4\lambda}{\lambda^2(l_3\lambda^2 + l_4\lambda + l_5)} - \frac{\tau}{\lambda} \\
&= \frac{(2\lambda^3 + l_1\lambda^2)}{-\lambda^2(l_3\lambda^2 + l_4\lambda + l_5)e^{-\lambda\tau}} - \frac{1}{\lambda^2} + \frac{2l_3\lambda^2 + l_4\lambda}{\lambda^2(l_3\lambda^2 + l_4\lambda + l_5)} - \frac{\tau}{\lambda} \\
&= \frac{(2\lambda^3 + l_1\lambda^2)}{-\lambda^2(\lambda^3 + l_1\lambda^2 + l_2\lambda)} + \frac{l_3\lambda^2 - l_5}{\lambda^2(l_3\lambda^2 + l_4\lambda + l_5)} - \frac{\tau}{\lambda}.
\end{aligned}$$

Taking $\lambda = i\omega_0$ in the above equation, we get

$$\begin{aligned}
\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega_0} &= \frac{2(i\omega_0)^3 + l_1(i\omega_0)^2}{-(i\omega_0)^2((i\omega_0)^3 + l_1(i\omega_0)^3 + l_2(i\omega_0))} \\
&\quad + \frac{l_3(i\omega_0)^2 - l_5}{(i\omega_0)^2(i\omega_0)^2 + l_4(i\omega_0) + l_5} + \frac{\tau i}{\omega_0} \\
&= \left[\frac{(l_1\omega_0^2) + 2i\omega_0^3}{\omega_0^2[(l_1\omega_0^2) + i(\omega_0^3 - l_2\omega_0)]} \cdot \frac{(l_1\omega_0^2) - i(\omega_0^3 - l_2\omega_0)}{(l_1\omega_0^2) - i(\omega_0^3 - l_2\omega_0)}\right] \\
&\quad + \left[\frac{l_3\omega_0^2 + l_5}{\omega_0^2((l_5 - l_3\omega_3^2) + il_4\omega_0)} \cdot \frac{(l_5 - l_3\omega_3^2) - il_4\omega_0}{(l_5 - l_3\omega_0^2) - il_4\omega_0}\right] + \frac{\tau i}{\omega_0},
\end{aligned}$$

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega_0} = \left[\frac{(l_1)(l_1\omega_0^2) + 2\omega_0(\omega_0^3 - l_2\omega_0)}{[(l_1\omega_0^2)^2 + (\omega_0^3 - l_2\omega_0)^2]}\right] + \frac{(l_5)^2 - (l_3\omega_0^2)^2}{\omega_0^2[(l_5 - l_3\omega_0^2)^2 + l_4^2\omega_0^2]}.$$

Thus we obtain $Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega_0} > 0$.

Therefore transversality condition holds and hence Hopf bifurcation occurs at $\tau = \tau^*$. This signifies that there exits atleast or equal value with positive real part for $\tau > \tau^*$.

Theorem 6.1. *If E_5 exists with the condition (4.2) and $\delta = \omega_0^2$ be positive root of (6.4), then there exists a $\tau = \tau^*$ such that*

(i) *E_5 is locally asymptotically stable for* $0 \leq \tau < \tau^*$

(ii) *E_5 is unstable for* $\tau > \tau^*$

(iii) *The system (6.1) undergoes a Hopf-bifurcation around E_5 at* $\tau = \tau^*$

$$\tau^* = \min h(\omega_0)$$

where

$$h(\omega_0) = \tau^*_{0n} = \frac{1}{\omega_0}\left[\cos^{-1}\left[\frac{\omega_0^4(l_4 - l_1 l_3) + \omega_0^2(l_1 l_5 - l_2 l_4)}{(l_5 - l_3\omega_0^2)^2 + l_4^2\omega_0^2}\right]\right] + \frac{2n\pi}{\omega_0}, n = 0, 1, 2, 3, \ldots$$

and the minimum taken over all positive ω_0 such that $\delta = \omega_0^2$ is a solution of (6.4).

7. Numerical Simulation

Analytical studies become complete only with the numerical justification of the results. Therefore, we assign some hypothetical data in order to verify the analytical findings. A qualitative analysis of the main features in the system is described by numerical simulations. The numerical simulation based on the analytical findings of the present model system is illustrated for the purpose of clear understanding of the complex dynamical behaviour of the system. It is obvious that changing the parameter value changes the numerical outcomes. So every different set of parameter gives unique results.

Let R_1 be the parameter set taken as $r = 1.5, s = 3.5, \beta = 2, c_1 = 1, c_2 = 9, w_1 = 3.5, w_2 = 0.06, e = 6.65$

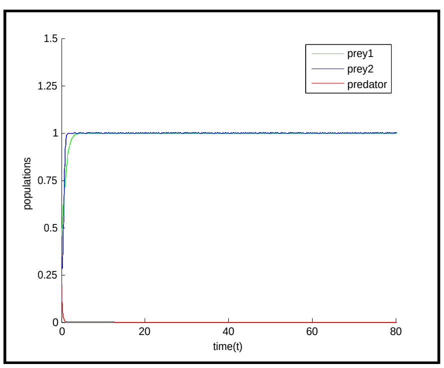


Fig. 1(a) Stable behaviour of x, y, z population in finite time

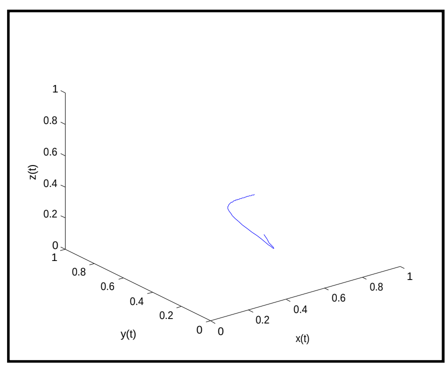


Fig. 1(b) Phase portrait of the system (2.2)

With the above parameter set, the equilibrium position E_2 is locally asymptotically stable which satisfying the condition $w_1 c_1 + w_2 c_2 < e$. In this case the prey species approaches the carrying capacity while the predator is driven to extinction (see Fig.1 (a)). Also phase portrait shows the solution tends to the boundary equilibrium point E_2 (see Fig.1 (b)).

Let R_2 be the parameter set taken as $r = 1.5, s = 3.5, c_1 = 8, c_2 = 9, w_1 = 3.5, w_2 = 0.06, e = 6.65$ with the above parameter set, varying the values of β and keep other parameter fixed. We observe that second prey species has extinction risk for lower values of β (see Fig.2 (a), 2(b)). If we increase the values of β, second prey species increase (see Fig 2(c)) and keep the population in desired level. Hence we concluded that survival of species depends upon the higher values of β.

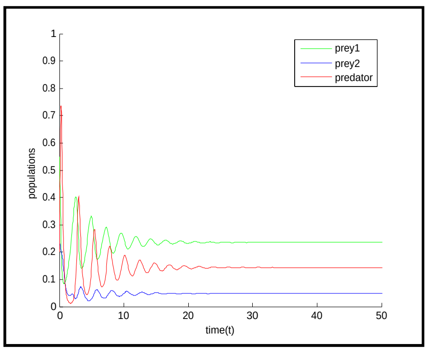


Fig. 2(a) Numerical Solution of system (2.2) with $\beta = 0.5$

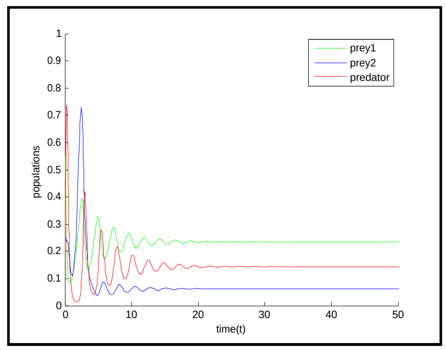


Fig. 2(b) Numerical Solution of system (2.2) with $\beta = 1$

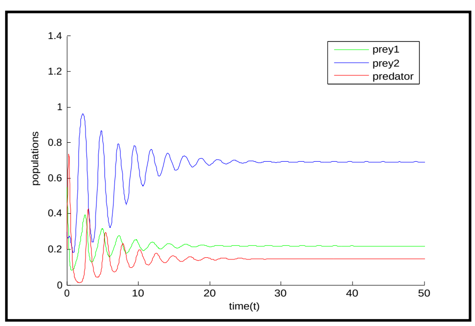


Fig. 2(c) Numerical Solution of system (2.2) with $\beta = 2$

Also phase portrait of the system (2) is plotted (see Fig.3 (a)-3(c)). From the Figure 3(a) and 3(b), we observe that first prey population has stable limit cycle while second prey population extinct for lower values of β. If $\beta > 1$ second prey and predator population has stable dynamics (see Fig 3(c)). Hence we concluded that population density depends on the values of β.

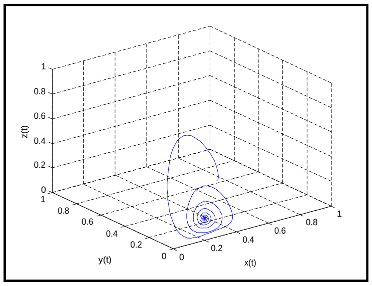


Fig. 3(a) Phase portrait of system (2.2) with $\beta = 0.5$

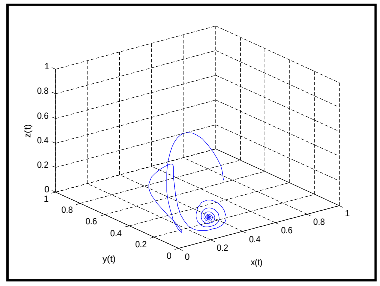


Fig. 3(b) Phase portrait of system (2.2) with $\beta = 1$

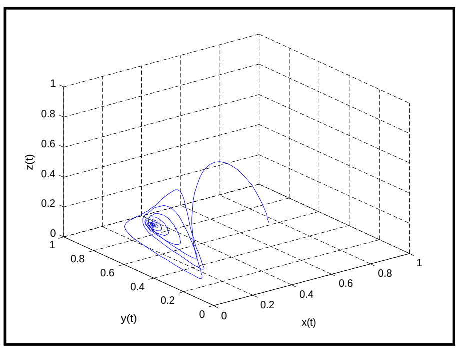


Fig. 3(c) Phase portrait of system (2.2) with $\beta = 2$

Let R_3 be the parameter set taken as $r = 1.5, s = 3.5, \beta = 1, c_1 = 6, c_2 = 9, w_1 = 3.5, w_2 = 0.06, e = 6.65$ with the above parameter set E_5 locally asymptotically stable. From Fig.4 (a) shows that x, y and z population approaches to their study state values of x^*, y^* and z^* respectively in finite time. The phase portrait of the system is shown in Fig 4(b) clearly the solution is stable spiral converging to E_5.

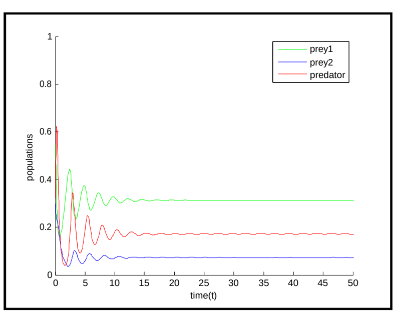


Fig. 4(a) Stable Solution of system (2.2)

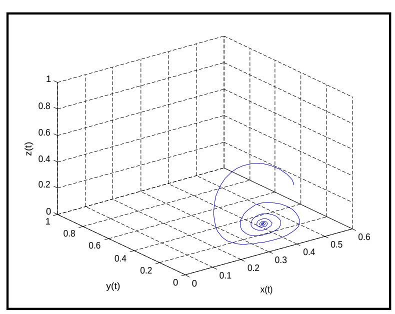
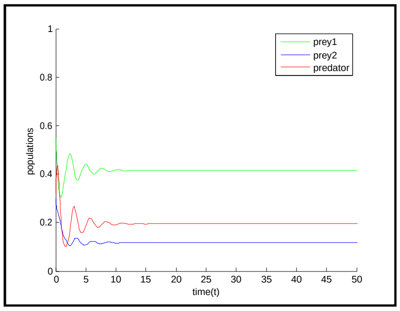


Fig. 4(b) Phase portrait of system (2.2)



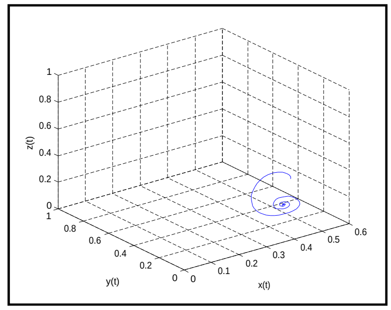
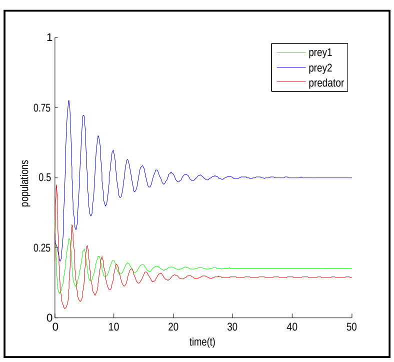


Fig. 5(a) Stable Solution of system (6.1) with $\tau = 8.5$

Fig. 5(b) Phase portrait of system (6.1) with $\tau = 8.5$

The stability criteria in the absence of delay $\tau = 0$ will not necessarily guarantee the stability of the system in presence of delay ($\tau \neq 0$). For the above choice of parameter set R_3 there is a unique positive root of the equation for which $\tau = \tau^* = 9.23$. Therefore By theorem 6.1, $E_5(x^*, y^*, z^*)$ loses its stability as τ passes through critical value of τ^*. We verify that $\tau = 8.5 < \tau, E_5$, is locally asymptotically stable (see Fig.5(a) and 5(b)), keeping other parameter fixed, if we take $\tau = 9.5 > \tau^*$, it is seen that E_5 is unstable and there is bifurcating periodic solution near E_5 (See Fig 6(b)), Periodic oscillations of x, y and z in finite time are shown in Fig 6(a).

Thus using the time delay as control, it is possible to break stable behaviour of system and drive it to an unstable state. Also it is possible to keep population at a desired level.



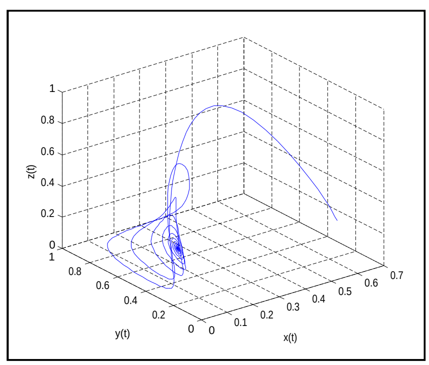


Fig. 6(a) Periodic Solution of system (6.1)

Fig. 6(b) Phase portrait of system (6.1)

8. Conclusion

In this paper, we studied dynamics of delayed two preys and predator system having distinct growth rate and functional response. In this system, discrete time delay in predator population is incorpated in the system. It is found that when time delay is absent, system is uniformly bounded which turn implies that the system well behaved. We examine the occurrence of possible equilibrium points and local stability of this equilibrium points are analyzed. The condition for persistence of system is determined. We have also shown the system has limit cycle oscillations, and stable coexisting dynamics of different growth rate from our analysis, it is observed that second prey species has extinction risk for lower values of β. Therefore survival depends on the growth rate and consumption rates. Finally time delay play a significant role on stability of the system. It breaks the stable behaviour of the system and drives it to unstable state.

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