KYUNGPOOK Math. J. 57(2017), 245-250
https://doi.org/10.5666/KMJ.2017.57.2.245
pISSN 1225-6951 eISSN 0454-8124

Approximation of Pompeiu's Point

Ick-Soon Chang and Hwan-Yong Shin*
Department of Mathematics, Chungnam National University, 99 Daehagno, Yuseong-gu, Daejeon 34134, Republic of Korea
e-mail : ischang@cnu.ac.kr *and* hyshin31@cnu.ac.kr

Abstract. In this paper, we obtain the refined stability of Pompeiu's points which extends a result of Huang and Li [8].

1. Introduction

The Hyers–Ulam stability problem was originated by Ulam [16] in 1940. Concerning a group homomorphism, Ulam posted the question asking how likely to an automorphism a function should behave in order to guarantee the existence of an automorphism near such functions. Ulam's question was partially solved by Hyers [9] in the case of approximately additive functions and when the groups in the question are Banach spaces.

Bourgin [2] and Aoki [1] treated this problem for approximate additive mappings controlled by unbounded functions. In [14], Rassias provided a generalization of Hyers's theorem for linear mappings which allows the Cauchy difference to be unbounded. In 1994, Găvruta [7] generalized these theorems for approximate additive mappings controlled by these unbounded Cauchy difference with regular conditions. During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the Hyers–Ulam stability and generalized Hyers–Ulam stability to a number of functional equations and mappings [3, 4, 11, 15].

In 1954, Hyers and Ulam [10] considered the stability of differentiable expressions and proved the following theorem.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be n-times differentiable in a neighborhood N of a point η. Suppose that $f^{(n)}(\eta) = 0$ and $f^{(n)}(x)$ changes sign at η. Then, for all $\epsilon > 0$, there exists a $\delta > 0$ such that for every function $g : \mathbb{R} \to \mathbb{R}$ which is n-times differentiable in N and satisfies $|f(x) - g(x)| < \delta$ for all $x \in N$, there exists a point $\xi \in N$ such that $g^{(n)}(\xi) = 0$ and $|\xi - \eta| < \varepsilon$.*

---

* Corresponding Author.
Received April 12, 2016; revised Jun 5, 2017; accepted Jun 7, 2016.
2010 Mathematics Subject Classification: 39B82, 39B22, 26A24.
Key words and phrases: Pompeiu's point, Hyers–Ulam stability, Mean value theorem.

Many mathematicians investigated the stability results of various mean value points by using Theorem 1.1. In 2003, Das, Riedel and Sahoo [5] gave a stability result for Flett's point. But there are some errors in the proof of Das et al.. In 2009, Lee, Xu and Ye [13] constructed a counter example to show the results of Das et al is incorrect and then they established the stability of Sahoo–Riedel's points and Flett's points. And in 2016, Kim and Shin [12] had the refined stability results of Sahoo–Riedel's point which are extensions of results of Lee et al.

Pompeiu [6] derived a variant of Lagrange's mean value theorem, now known as Pompeiu's mean value theorem.

Theorem 1.2. *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exists a point $\xi \in (x_1,x_2)$ such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi).$$

Such an intermediate point ξ will be called Pompeiu's point of the function f. The geometrical meaning of this is that the tangent at the point $(\xi, f(\xi))$ intersects on the y-axis at the same point as the secant line connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

In 2015, Huang and Li [8] obtained the following stability result of Pompeiu's points.

Theorem 1.3. *Let $f, h : [a,b] \to \mathbb{R}$ be differentiable and η be a Pompeiu's point of f. If f has second derivative at η with*

$$f''(\eta) \neq 0.$$

then corresponding to any $\varepsilon > 0$, there exists a $\delta > 0$ such that for every h satisfying $|h(x) - f(x)| < \delta$ for all $x \in [a,b]$, there exists a point $\xi \in (a,b)$ such that ξ is a Pompeiu's point of h with $|\xi - \eta| < \varepsilon$.

In this paper, using the similar idea of [12], we prove the refined stability of Pompeiu's mean value points which extends Theorem 1.3.

2. Stability of Pompeiu's Mean Value Points

We now present a main theorem, which is a stability of Pompeiu's mean value points for real-valued differentiable functions on an interval $[a,b]$ which does not contain 0.

Theorem 2.1. *Let $f, g, : [a,b] \to \mathbb{R}$ be differentiable and η be a Pompeiu's point of f. Assume that*

$$f(x) - xf'(x) - \frac{af(b) - bf(a)}{a - b}$$

changes sign at η and $\varepsilon > 0$ and neighborhood $N \subset (\frac{1}{b}, \frac{1}{a})$ of $\frac{1}{\eta}$ be given. If there exists a $\delta > 0$ such that for every h satisfying $|h(\frac{1}{x}) - f(\frac{1}{x})| < \delta$ for all $x \in N \cup \{\frac{1}{b}, \frac{1}{a}\}$, then there exists a point $\xi \in N$ such that ξ is a Pompeiu's point of h with $|\xi - \eta| < \varepsilon$.

Proof. Without loss of generality, we shall assume $a, b > 0$. Define a real valued function F on the interval $[\frac{1}{b}, \frac{1}{a}]$ by

$$F(x) = xf(\frac{1}{x}).$$

Since f is differentiable of $[a, b]$ and 0 is not in $[a, b]$, we see that F is differentiable on $(\frac{1}{b}, \frac{1}{a})$ and $F'(x) = f(\frac{1}{x}) - \frac{1}{x} f'(\frac{1}{x})$. Let $\varepsilon > 0$ be given and let $N \subset (\frac{1}{b}, \frac{1}{a})$ be any neighborhood of $\frac{1}{\eta}$. Consider the auxiliary function $G_F(x) : [\frac{1}{b}, \frac{1}{a}] \to \mathbb{R}$ corresponding to F defined by

$$G_F(x) = F(x) - \frac{F(\frac{1}{b}) - F(\frac{1}{a})}{\frac{1}{b} - \frac{1}{a}}(x - \frac{1}{a}) \tag{2.1}$$

for all $x \in [\frac{1}{b}, \frac{1}{a}]$. Evidently, $G_F(x)$ is differentiable on $[\frac{1}{b}, \frac{1}{a}]$. Further, we have

$$G'_F(x) = F'(x) - \frac{F(\frac{1}{b}) - F(\frac{1}{a})}{\frac{1}{b} - \frac{1}{a}} = f(\frac{1}{x}) - \frac{1}{x} f'(\frac{1}{x}) - \frac{af(b) - bf(a)}{a - b}.$$

Since η is the Pompeiu's point of f, we get $G'_F(\frac{1}{\eta}) = 0$. Thus, it follows from the assumption that there exists a neighborhood $(\frac{1}{\eta} - r, \frac{1}{\eta} + r) \subset N$ of η such that $G'_F(x)$ changes sign at $\frac{1}{\eta}$ in $(\frac{1}{\eta} - r, \frac{1}{\eta} + r) \subset N$ for some $r > 0$ with $\frac{1}{\eta} - r > a$. Then, it follows from Theorem 1.1 that there exists a $\overline{\delta} > 0$ such that for any differentiable function H on $[\frac{1}{b}, \frac{1}{a}]$ with $|H(x) - G_F(x)| < \overline{\delta}$ for x in $(\frac{1}{\eta} - r, \frac{1}{\eta} + r)$, there exists a point $\zeta \in (\frac{1}{\eta} - r, \frac{1}{\eta} + r)$ satisfying $H'(\zeta) = 0$ and $|\zeta - \frac{1}{\eta}| < \frac{1}{b^2}\varepsilon$. Now, let us define differentiable functions H and G_H by

$$H(x) = xh(\frac{1}{x})$$

and

$$G_H(x) = H(x) - \frac{H(\frac{1}{b}) - H(\frac{1}{a})}{\frac{1}{b} - \frac{1}{a}}(x - \frac{1}{a}).$$

for all $x \in [\frac{1}{b}, \frac{1}{a}]$. Then, it is easy to see that $G_H(x)$ is differentiable in N. And we have

$$\begin{aligned} |G_H(x)-G_F(x)| &\le \Big|H(x)-\frac{H(\frac{1}{b})-H(\frac{1}{a})}{\frac{1}{b}-\frac{1}{a}}(x-\frac{1}{a})-F(x)+\frac{F(\frac{1}{b})-F(\frac{1}{a})}{\frac{1}{b}-\frac{1}{a}}(x-\frac{1}{a})\Big| \\ &\le |H(x)-F(x)|+\Big|(x-\frac{1}{a})\Big(\frac{H(\frac{1}{b})-F(\frac{1}{b})}{\frac{1}{b}-\frac{1}{a}}\Big)\Big| \\ &\quad +\Big|(x-\frac{1}{a})\Big(\frac{H(\frac{1}{a})-H(\frac{1}{a})}{\frac{1}{b}-\frac{1}{a}}\Big)\Big| \\ &\le |h(\frac{1}{x})-f(\frac{1}{x})|+|h(b)-f(b))|+|h(a)-f(a)| \end{aligned}$$

for all $x \in (\frac{1}{\eta}-r, \frac{1}{\eta}+r) \subset N$. Let $\delta = \frac{a\overline{\delta}}{3}$ and $|h(\frac{1}{x})-f(\frac{1}{x})| < \delta$ for all $x \in N \cup \{\frac{1}{b}, \frac{1}{a}\}$. Then we have $|G_H(x)-G_F(x)| \le \frac{3\delta}{a} = \overline{\delta}$. Hence, there exists a point $\xi_0 \in (\frac{1}{\eta}-r, \frac{1}{\eta}+r)$ such that $G'_H(\xi_0)=0$ and $|\xi_0-\frac{1}{\eta}| < \dfrac{\varepsilon}{b^2}$. Define $\xi = \frac{1}{\xi_0}$ We note that $G'_H(\xi_0) = G'_H(\frac{1}{\xi}) = 0$ implies

$$h(\xi)-\xi h'(\xi) = \frac{\frac{1}{b}h(b)-\frac{1}{a}h(a)}{\frac{1}{a}-\frac{1}{b}} = \frac{ah(b)-bh(a)}{a-b},$$

from which it follows that ξ is a Pompeiu's point of h. Moreover,

$$|\xi-\eta| = \Big|\frac{1}{\xi_0}-\eta\Big| = \Big|\frac{\xi_0-\frac{1}{\eta}}{\xi_0\cdot\frac{1}{\eta}}\Big| \le b^2\Big|\xi_0-\frac{1}{\eta}\Big| < \varepsilon.$$

This completes the proof. □

Corollary 2.2. *Let $f, h : [a,b] \to \mathbb{R}$ be differentiable and η be a Pompeiu's point of f. Suppose f has second derivative at η with*

$$f''(\eta) \ne 0$$

and $\varepsilon > 0$ and neighborhood $N \subset (\frac{1}{b}, \frac{1}{a})$ of $\frac{1}{\eta}$ be given. If there exists a $\delta > 0$ such that for every h satisfying $|h(\frac{1}{x})-f(\frac{1}{x})| < \delta$ for all $x \in N \cup \{\frac{1}{b}, \frac{1}{a}\}$, then there exists a point $\xi \in N$ such that ξ is a Pompeiu's point of h with $|\xi-\eta| < \varepsilon$.

Proof. Let $G_F : [\frac{1}{b}, \frac{1}{a}] \to \mathbb{R}$ be defined as in (2.1). Since η is a Pompeiu's point, we have

$$G'_F(\frac{1}{\eta}) = F'(\frac{1}{\eta}) - \frac{F(\frac{1}{b})-F(\frac{1}{a})}{\frac{1}{b}-\frac{1}{a}} = f(\eta)-\eta f'(\eta) - \frac{af(b)-bf(a)}{a-b} = 0.$$

Moreover, by the assumption that $f''(\eta) \ne 0$, we obtain that

$$G''_F(\frac{1}{\eta}) = F''(\frac{1}{\eta}) = \eta^3 f''(\eta) \ne 0,$$

which implies $G''_F(x)$ changes sign at $\frac{1}{\eta}$. By using Theorem 2.1, we complete the proof. □

We remark that if N is $(\frac{1}{b}, \frac{1}{a})$, then Corollary 2.2 is equal to Theorem 1.1.

Corollary 2.3. *Let $f, h : [a, b] \to \mathbb{R}$ be differentiable and η be a unique Pompeiu's point of f. Suppose*

$$af(b) = bf(a)$$

and $\varepsilon > 0$ and neighborhood $N \subset (\frac{1}{b}, \frac{1}{a})$ of $\frac{1}{\eta}$ are given. If there exists a $\delta > 0$ such that for every h satisfying $|h(\frac{1}{x}) - f(\frac{1}{x})| < \delta$ for all $x \in N \cup \{\frac{1}{b}, \frac{1}{a}\}$, then there exists a point $\xi \in N$ such that ξ is a Pompeiu's point of h with $|\xi - \eta| < \varepsilon$.

Proof. Let $G_F : [\frac{1}{b}, \frac{1}{a}] \to \mathbb{R}$ be defined as in (2.1). Suppose η is a unique Pompeiu's point of f. Then, by the fact $G_F(\frac{1}{a}) = G_F(\frac{1}{b})$ and $G'_F(\frac{1}{\eta}) = 0$, we obtain G_F changes sign at η. Using Theorem 2.1, the proof is done. □

Acknowledgements. The authors would like to thank the referees for giving useful suggestions and for the improvement of this manuscript. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (no. 2013R1A1A2A10004419).

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces,* J. Math. Soc. Japan, **2**(1950), 64–66.

[2] D. G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc., **57**(1951), 223–237.

[3] Y. J. Cho, T. M. Rassias, R. Saadati, *Stability of Functional Equations in Random Normed Spaces*, Springer Optimization and Its Applications, **86**, Springer, New York, 2013.

[4] S. Czerwik, *Stability of Functional Equations of Ulam–Hyers–Rassias Type*, Hadronic Press, Florida, 2003.

[5] M. Das, T. Riedel, P. K. Sahoo, *Hyers-Ulam stability of Flett's points*, Appl. Math. Lett., **16**(2003), 269–271.

[6] T. M. Flett, *A mean value theorem*, Math. Gazette, **42**(1958), 38–39.

[7] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184**(1994), 431–436

[8] J. Huang, Y. Li, *Hyers–Ulam stability of Pompeiu's point*, Kyungpook Math. J., **55**(2015), 103–107.

[9] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A., **27**(1941), 222–224.

[10] D. H. Hyers, S. M. Ulam, *On the stability of differential expressions*, Math. Mag., **28**(1954), 59–64.

[11] S. -M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer Optimization and Its Applications, **48**, Springer, New York, 2011.

[12] H. -M. Kim, H. -Y. Shin, *Approximation of almost Sahoo–Riedel's points by Sahoo–Riedel's points*, Aequationes Math., **90**(2016), 809815.

[13] W. Lee, S. Xu, F. Ye, *Hyers-Ulam stability of Sahoo-Riedel's point*, Appl. Math. Lett., **22**(2009), 1649–1652.

[14] T. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297–300.

[15] T. M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, London, 2003.

[16] S. M. Ulam, *Problems in Modern Mathematics*, Chapter 6, Wiley, New York 1964.