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Extremal Problems for $\mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})$

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Abstract. We classify the extreme and exposed symmetric bilinear forms of the unit ball of the space of symmetric bilinear forms on $\mathbb{R}^2$ with hexagonal norms. We also show that every extreme symmetric bilinear forms of the unit ball of the space of symmetric bilinear forms on $\mathbb{R}^2$ with hexagonal norms is exposed.

1. Introduction

We write B_E for the closed unit ball of a real Banach space E and the dual space of E is denoted by E^*. $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. $x \in B_E$ is called an *exposed point* of B_E if there is a $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. We denote by $extB_E$ and $expB_E$ the sets of extreme and exposed points of B_E, respectively. A mapping $P : E \to \mathbb{R}$ is a continuous 2-homogeneous polynomial if there exists a continuous bilinear form L on the product $E \times E$ such that $P(x) = L(x, x)$ for every $x \in E$. We denote by $\mathcal{L}({}^2E)$ the Banach space of all continuous bilinear forms on E endowed with the norm $\|L\| = \sup_{\|x\|=\|y\|=1} |L(x, y)|$. $\mathcal{L}_s({}^2E)$ denotes the subspace of $\mathcal{L}({}^2E)$ of all continuous symmetric bilinear forms on E. $\mathcal{P}({}^2E)$ denotes the Banach space of all continuous 2-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [7].

In 1998, Choi *et al.* ([2], [3]) characterized the extreme points of the unit ball of $\mathcal{P}({}^2l_1^2)$ and $\mathcal{P}({}^2l_2^2)$. In 2007, Kim [11] classified the exposed 2-homogeneous

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polynomials on $\mathcal{P}(^2 l_p^2)$ $(1 \leq p \leq \infty)$. Kim ([13], [15], [19]) classified the extreme, exposed, smooth points of the unit ball of $\mathcal{P}(^2 d_*(1,w)^2)$, where $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal norm of weight w. In 2009, Kim [12] classified the extreme, exposed, smooth points of the unit ball of $\mathcal{L}_s(^2 l_\infty^2)$. Kim ([14], [16], [17], [18]) classified the extreme, exposed, smooth points of the unit balls of $\mathcal{L}_s(^2 d_*(1,w)^2)$ and $\mathcal{L}(^2 d_*(1,w)^2)$.

We refer to ([1–6], [8–25] and references therein) for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces. Let $0 < w < 1$ be fixed. We denote $\mathbb{R}^2$ with the hexagonal norm of weight w by

$$\mathbb{R}^2_{h(w)} := \{(x,y) \in \mathbb{R}^2 : \|(x,y)\|_{h(w)} := \max\{|y|,\ |x| + (1-w)|y|\}\ \}.$$

Recently, Kim [20] characterized the extreme points of the unit ball of $\mathcal{L}(^2\mathbb{R}^2_{h(w)})$. In this paper, we classify the extreme and exposed symmetric bilinear forms of the unit ball of $\mathcal{L}_s(^2\mathbb{R}^2_{h(w)})$. We also show that every extreme symmetric bilinear form of the unit ball of $\mathcal{L}_s(^2\mathbb{R}^2_{h(w)})$ is exposed.

2. The Extreme Points of the Unit Ball of $\mathcal{L}_s(^2\mathbb{R}^2_{h(w)})$

Let $0 < w < 1$ and $T((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w)})$ for some reals a, b, c. For simplicity we will write $T((x_1,y_1),\ (x_2,y_2)) = (a,b,c)$.

Theorem 2.1. *Let* $0 < w < 1$ *and* $T((x_1,y_1),\ (x_2,y_2)) := (a,b,c) \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w)})$. *Then,*

$$\|T\| = \max\{|a|, |a|w + |c|, |aw^2 - b|, |aw^2 + b| + 2w|c|\}.$$

Proof. By substituting $((x_1,y_1),(x_2,y_2))$ in T for $((x_1,-y_1),(x_2,-y_2))$, we may assume that $c \geq 0$. Since $\{(\pm1,0),(w,\ \pm1),(-w,\pm1)\}$ is the set of all extreme points of the unit ball of $\mathbb{R}^2_{h(w)}$ and T is bilinear,

$$\|T\| = \max\{|T((\pm1,0),(\pm1,0))|, |T((\pm1,0),(w,\ \pm1))|, |T((w,\ \pm1),(w,\ \pm1))|\}.$$

It follows that, by symmetry of T,

$$\begin{aligned}\|T\| &= \max\{|T((1,0),\ (1,0))|,\ |T((1,0),\ (w,1))|,\ |T((1,0),\ (w,-1))|,\\ &\qquad |T((w,1),\ (w,1))|,\ |T((w,-1),\ (w,-1))|, |T((w,1),\ (w,-1))|\}\\ &= \max\{|a|, |a|w + c, |aw^2 - b|, |aw^2 + b| + 2wc\}.\end{aligned}$$

□

Note that if $\|T\| = 1$, then $|a| \leq 1, |b| \leq 1$ and $|c| \leq 1$. Let

$$\begin{aligned} Norm(T) = \{&((x_1,y_1),(x_2,y_2)) \in \{((1,0),\ (1,0)),((1,0),\ (w,1)),((1,0),\ (w,-1)),\\ &((w,1),\ (w,1)),((w,-1),\ (w,-1)),((w,1),\ (w,-1))\} :\\ &|T((x_1,y_1),(x_2,y_2))| = \|T\|\}.\end{aligned}$$

We call $Norm(T)$ *the norming set* of T.

Theorem 2.2. *Let* $0 < w < 1$ *and* $T((x_1, y_1),\ (x_2, y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})$ *with* $\|T\| = 1$*. Then,* T *is extreme if and only if* $Norm(T)$ *has exactly three elements.*

Proof. Without loss of generality we may assume that $a, c \geq 0$.

($\Leftarrow$): We have 20 cases as follows:

Case 1: $Norm(T) = \{((1, 0), (1, 0)), ((1, 0), (w, 1)), ((1, 0), (w, -1))\}$
Case 2: $Norm(T) = \{((1, 0), (1, 0)), ((1, 0), (w, 1)), ((w, 1), (w, 1))\}$
Case 3: $Norm(T) = \{((1, 0), (1, 0)), ((1, 0), (w, 1)), ((w, -1), (w, -1))\}$
Case 4: $Norm(T) = \{((1, 0), (1, 0)), ((1, 0), (w, 1)), ((w, 1), (w, -1))\}$
Case 5: $Norm(T) = \{((1, 0), (1, 0)), ((1, 0), (w, -1)), ((w, 1), (w, 1))\}$
Case 6: $Norm(T) = \{((1, 0), (1, 0)), ((1, 0), (w, -1)), ((w, -1), (w, -1))\}$
Case 7: $Norm(T) = \{((1, 0), (1, 0)), ((1, 0), (w, -1)), ((w, 1), (w, -1))\}$
Case 8: $Norm(T) = \{((1, 0), (1, 0)), ((w, 1), (w, 1)), ((w, -1), (w, -1))\}$
Case 9: $Norm(T) = \{((1, 0), (1, 0)), ((w, 1), (w, 1)), ((w, 1), (w, -1))\}$
Case 10: $Norm(T) = \{((1, 0), (1, 0)), ((w, -1), (w, -1)), ((w, 1), (w, -1))\}$
Case 11: $Norm(T) = \{((1, 0), (w, 1)), ((1, 0), (w, -1)), ((w, 1), (w, 1))\}$
Case 12: $Norm(T) = \{((1, 0), (w, 1)), ((1, 0), (w, -1)), ((w, -1), (w, -1))\}$
Case 13: $Norm(T) = \{((1, 0), (w, 1)), ((1, 0), (w, -1)), ((w, 1), (w, -1))\}$
Case 14: $Norm(T) = \{((1, 0), (w, 1)), ((w, 1), (w, 1)), ((w, -1), (w, -1))\}$
Case 15: $Norm(T) = \{((1, 0), (w, 1)), ((w, 1), (w, 1)), ((w, 1), (w, -1))\}$
Case 16: $Norm(T) = \{((1, 0), (w, 1)), ((w, -1), (w, -1)), ((w, 1), (w, -1))\}$
Case 17: $Norm(T) = \{((1, 0), (w, -1)), ((w, 1), (w, 1)), ((w, -1), (w, -1))\}$
Case 18: $Norm(T) = \{((1, 0), (w, -1)), ((w, 1), (w, 1)), ((w, 1), (w, -1))\}$
Case 19: $Norm(T) = \{((1, 0), (w, -1)), ((w, -1), (w, -1)), ((w, 1), (w, -1))\}$
Case 20: $Norm(T) = \{((w, 1), (w, 1)), ((w, -1), (w, -1)), ((w, 1), (w, -1))\}$.

We will consider each case.

Case 1: $Norm(T) = \{((1, 0), (1, 0)), ((1, 0), (w, 1)), ((1, 0), (w, -1))\}$

Note that T does not exist in case 1.

Case 2: $Norm(T) = \{((1, 0), (1, 0)), ((1, 0), (w, 1)), ((w, 1), (w, 1))\}$

Then $T = (1, (1-w)^2, 1-w)$ for all $0 < w < 1$. Note that $T = (1, (1-w)^2, 1-w)$ is extreme for all $0 < w < 1$. Indeed, let $T_1 = (1+\epsilon, (1-w)^2+\delta, 1-w+\gamma), T_2 = (1-\epsilon, (1-w)^2-\delta, 1-w-\gamma)$ be such that $\|T_1\| = 1 = \|T_2\|$ for some $\epsilon, \delta, \gamma \in \mathbb{R}$. Since $|T_i((1, 0), (1, 0))| \leq 1, |T_i((1, 0), (w, 1))| \leq 1, |T_i((w, 1), (w, 1))| \leq 1$, we have $0 = \epsilon = \delta = \gamma$.

Case 3: $Norm(T) = \{((1, 0), (1, 0)), ((1, 0), (w, 1)), ((w, -1), (w, -1))\}$

Then $T = (1, -3w^2 + 2w - 1, 1 - w)$ for all $0 < w \leq \frac{1}{2}$. Note that $T = (1, -3w^2 + 2w - 1, 1 - w)$ is extreme for all $0 < w \leq \frac{1}{2}$. Indeed, let $T_1 = (1 + \epsilon, -3w^2 + 2w - 1 + \delta, 1 - w + \gamma), T_2 = (1 - \epsilon, -3w^2 + 2w - 1 - \delta, 1 - w - \gamma)$ be such that $\|T_1\| = 1 = \|T_2\|$ for some $\epsilon, \delta, \gamma \in \mathbb{R}$. Since $|T_i((1, 0), (1, 0))| \leq$

$1, |T_i((1,0),(w,1))| \leq 1, |T_i((w,-1),(w,-1))| \leq 1$, we have

$$\begin{aligned} \epsilon &= 0 \\ w\epsilon + \gamma &= 0 \\ w^2\epsilon + \delta - 2w\gamma &= 0, \end{aligned}$$

which show that $0 = \epsilon = \delta = \gamma$.

Case 4: $Norm(T) = \{((1,0),(1,0)),((1,0),(w,1)),((w,1),(w,-1))\}$

Then $T = (1, w^2-1, 1-w)$ for all $w \geq \frac{1}{2}$. Note that $T = (1, w^2-1, 1-w)$ is extreme for all $w \geq \frac{1}{2}$. Indeed, let $T_1 = (1+\epsilon, w^2-1+\delta, 1-w+\gamma), T_2 = (1-\epsilon, w^2-1-\delta, 1-w-\gamma)$ be such that $\|T_1\| = 1 = \|T_2\|$ for some $\epsilon, \delta, \gamma \in \mathbb{R}$. Since $|T_i((1,0),(1,0))| \leq 1, |T_i((1,0),(w,1))| \leq 1, |T_i((w,1),(w,-1))| \leq 1$, we have

$$\begin{aligned} \epsilon &= 0 \\ w\epsilon + \gamma &= 0 \\ w^2\epsilon - \delta &= 0, \end{aligned}$$

which show that $0 = \epsilon = \delta = \gamma$.

Case 5: $Norm(T) = \{((1,0),(1,0)),((1,0),(w,-1)),((w,1),(w,1))\}$

Note that T does not exist in case 5.

Case 6: $Norm(T) = \{((1,0),(1,0)),((1,0),(w,-1)),((w,-1),(w,-1))\}$

Note that T does not exist in case 6.

Case 7: $Norm(T) = \{((1,0),(1,0)),((1,0),(w,-1)),((w,1),(w,-1))\}$

Note that T does not exist in case 7.

Case 8: $Norm(T) = \{((1,0),(1,0)),((w,1),(w,1)),((w,-1),(w,-1))\}$

Then $T = (1, 1-w^2, 0)$ for all $0 < w < 1$. Note that $T = (1, 1-w^2, 0)$ is extreme for all $0 < w < 1$. Indeed, let $T_1 = (1+\epsilon, 1-w^2+\delta, \gamma), T_2 = (1-\epsilon, 1-w^2-\delta, -\gamma)$ be such that $\|T_1\| = 1 = \|T_2\|$ for some $\epsilon, \delta, \gamma \in \mathbb{R}$. Since $|T_i((1,0),(1,0))| \leq 1, |T_i((w,1),(w,1))| \leq 1, |T_i((w,-1),(w,-1))| \leq 1$, we have

$$\begin{aligned} \epsilon &= 0 \\ w^2\epsilon + \delta + 2w\gamma &= 0 \\ w^2\epsilon + \delta - 2w\gamma &= 0, \end{aligned}$$

which show that $0 = \epsilon = \delta = \gamma$.

Case 9: $Norm(T) = \{((1,0),(1,0)),((w,1),(w,1)),((w,1),(w,-1))\}$

Note that T does not exist in case 9.

Case 10: $Norm(T) = \{((1,0),(1,0)),((w,-1),(w,-1)),((w,1),(w,-1))\}$

Then $T = (1, w^2-1, w)$ for all $0 < w \leq \frac{1}{2}$. Note that $T = (1, w^2-1, w)$ is extreme for all $0 < w \leq \frac{1}{2}$. Indeed, let $T_1 = (1+\epsilon, w^2-1+\delta, w+\gamma), T_2 = (1-\epsilon, w^2-1-\delta, w-\gamma)$ be such that $\|T_1\| = 1 = \|T_2\|$ for some $\epsilon, \delta, \gamma \in \mathbb{R}$. Since $|T_i((1,0),(1,0))| \leq 1, |T_i((w,-1),(w,-1))| \leq 1, |T_i((w,1),(w,-1))| \leq 1$, we have

$$\begin{aligned} \epsilon &= 0 \\ w^2\epsilon + \delta - 2w\gamma &= 0 \\ w^2\epsilon - \delta &= 0, \end{aligned}$$

which show that $0=\epsilon=\delta=\gamma$.

Case 11: $Norm(T)=\{((1,0),(w,1)),((1,0),(w,-1)),((w,1),(w,1))\}$

Then $T=(0,1-2w,1)$ for all $0<w\leq\frac{1}{2}$. Note that $T=(0,1-2w,1)$ is extreme for all $0<w\leq\frac{1}{2}$. Indeed, let $T_1=(\epsilon,1-2w+\delta,1+\gamma),T_2=(-\epsilon,1-2w-\delta,1-\gamma)$ be such that $\|T_1\|=1=\|T_2\|$ for some $\epsilon,\delta,\gamma\in\mathbb{R}$. Since $|T_i((1,0),(w,1))|\leq 1,|T_i((1,0),(w,-1))|\leq 1,|T_i((w,1),(w,1))|\leq 1$, we have

$$\begin{aligned} w\epsilon+\gamma &= 0\\ w\epsilon-\gamma &= 0\\ w^2\epsilon+\delta+2w\gamma &= 0, \end{aligned}$$

which show that $0=\epsilon=\delta=\gamma$.

Case 12: $Norm(T)=\{((1,0),(w,1)),((1,0),(w,-1)),((w,-1),(w,-1))\}$

Note that T does not exist in case 12.

Case 13: $Norm(T)=\{((1,0),(w,1)),((1,0),(w,-1)),((w,1),(w,-1))\}$

Note that T does not exist in case 13.

Case 14: $Norm(T)=\{((1,0),(w,1)),((w,1),(w,1)),((w,-1),(w,-1))\}$

Then $T=(\frac{2w-1}{2w^2},\frac{1-2w}{2},\frac{1}{2w})$ for all $w\geq\frac{1}{2}$. Note that $T=(\frac{2w-1}{2w^2},\frac{1-2w}{2},\frac{1}{2w})$ is extreme for all $w\geq\frac{1}{2}$. Indeed, let $T_1=(\frac{2w-1}{2w^2}+\epsilon,\frac{1-2w}{2}+\delta,\frac{1}{2w}+\gamma),T_2=(\frac{2w-1}{2w^2}-\epsilon,\frac{1-2w}{2}-\delta,\frac{1}{2w}-\gamma)$ be such that $\|T_1\|=1=\|T_2\|$ for some $\epsilon,\delta,\gamma\in\mathbb{R}$. Since $|T_i((1,0),(w,1))|\leq 1,|T_i((w,1),(w,1))|\leq 1,|T_i((w,-1),(w,-1))|\leq 1$, we have

$$\begin{aligned} \epsilon+\gamma &= 0\\ w^2\epsilon+\delta+2w\gamma &= 0\\ w^2\epsilon+\delta-2w\gamma &= 0, \end{aligned}$$

which show that $0=\epsilon=\delta=\gamma$.

Case 15: $Norm(T)=\{((1,0),(w,1)),((w,1),(w,1)),((w,1),(w,-1))\}$

Note that T does not exist in case 15.

Case 16: $Norm(T)=\{((1,0),(w,1)),((w,-1),(w,-1)),((w,1),(w,-1))\}$

Then $T=(\frac{1}{2w},\frac{w-2}{2},\frac{1}{2})$ for all $w\geq\frac{1}{2}$. Note that $T=(\frac{1}{2w},\frac{w-2}{2},\frac{1}{2})$ is extreme for all $w\geq\frac{1}{2}$. Indeed, let $T_1=(\frac{1}{2w}+\epsilon,\frac{w-2}{2}+\delta,\frac{1}{2}+\gamma),T_2=(\frac{1}{2w}-\epsilon,\frac{w-2}{2}-\delta,\frac{1}{2}-\gamma)$ be such that $\|T_1\|=1=\|T_2\|$ for some $\epsilon,\delta,\gamma\in\mathbb{R}$. Since $|T_i((1,0),(w,1))|\leq 1,|T_i((w,-1),(w,-1))|\leq 1,|T_i((w,1),(w,-1))|\leq 1$, we have

$$\begin{aligned} w\epsilon+\gamma &= 0\\ w^2\epsilon+\delta-2w\gamma &= 0\\ w^2\epsilon-\delta &= 0, \end{aligned}$$

which show that $0=\epsilon=\delta=\gamma$.

Case 17: $Norm(T)=\{((1,0),(w,-1)),((w,1),(w,1)),((w,-1),(w,-1))\}$

Note that T does not exist in case 17.

Case 18: $Norm(T)=\{((1,0),(w,-1)),((w,1),(w,1)),((w,1),(w,-1))\}$

Note that T does not exist in case 18.

Case 19: $Norm(T) = \{((1,0),(w,-1)),((w,-1),(w,-1)),((w,1),(w,-1))\}$

Note that T does not exist in case 19.

Case 20: $Norm(T) = \{((w,1),(w,1)),((w,-1),(w,-1)),((w,1),(w,-1))\}$.

Note that T does not exist in case 20.

$(\Rightarrow)$: By the argument of $(\Leftarrow)$, it is enough to show that if $Norm(T)$ has at most two elements, then T is not extreme. For an example, let

$$Norm(T) = \{((1,0),(1,0)),((1,0),(w,1))\}.$$

We will show that T is not extreme. Notice that

$$|T((1,0),(1,0))| = 1 = |T((1,0),(w,1))|, |T((w,1),(w,1))| < 1, |T((w,1),(w,-1))| < 1.$$

Hence, $a = 1, c = 1-w, |w^2 - b| < 1, |w^2 + b| + 2w(1-w) < 1$. Let $\delta > 0$ such that $|w^2 - b| + \delta < 1, |w^2 + b| + 2w(1-w) + \delta < 1$. Let $T_1 = (1, b+\delta, 1-w)$ and $T_2 = (1, b-\delta, 1-w)$. By Theorem 2.1, $\|T_i\| = 1$ for $i = 1,2$. Since $T_i \neq T, T = \frac{1}{2}(T_1 + T_2)$, T is not extreme. For the other cases, we may show that if $Norm(T)$ has at most two elements, then T is not extreme using Theorem 2.1. Hence, we will omit the proofs. Therefore, we complete the proof. □

Now we are in position to describe all the extreme points of the unit ball of $\mathcal{L}_s(^2\mathbb{R}^2_{h(w)})$.

Theorem 2.3. (a) *Let* $0 < w \leq \frac{1}{2}$. *Then,*

$$\begin{aligned} extB_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w)})} &= \{\pm(0,1,0), \pm(1,(1-w)^2, \pm(1-w)), \pm(1, 1-w^2, 0), \\ &\quad \pm(1, w^2-1, \pm w), \pm(0, 1-2w, \pm 1), \\ &\quad \pm(1, -3w^2+2w-1, \pm(1-w))\}. \end{aligned}$$

(b) *Let* $\frac{1}{2} < w < 1$. *Then,*

$$\begin{aligned} extB_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w)})} &= \{\pm(0,1,0), \pm(1,(1-w)^2, \pm(1-w)), \pm(1, 1-w^2, 0), \\ &\quad \pm(1, w^2-1, \pm(1-w)), \pm(\frac{1}{2w}, \frac{w-2}{2}, \pm\frac{1}{2}), \\ &\quad \pm(\frac{2w-1}{2w^2}, \frac{1-2w}{2}, \pm\frac{1}{2w})\}. \end{aligned}$$

Proof. It follows from the proof of Theorem 2.2. □

3. The Exposed Points of the Unit Ball of $\mathcal{L}_s(^2\mathbb{R}^2_{h(w)})$

Theorem 3.1. *Let* $0 < w < 1$ *and* $f \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w)})^*$ *and* $\alpha = f(x_1x_2), \beta = f(y_1y_2), \gamma = f(x_1y_2 + x_2y_1)$.

(a) *Let* $0 < w \leq \frac{1}{2}$. *Then,*

$$\begin{aligned}\|f\| &= \max\{|\beta|, |\alpha+(1-w)^2\beta|+(1-w)|\gamma|, |\alpha+(1-w^2)\beta|,\\ &\quad |\alpha-(1-w^2)\beta|+w|\gamma|, |\alpha-(3w^2-2w+1)\beta|+(1-w)|\gamma|,\\ &\quad (1-2w)|\beta|+|\gamma|\}.\end{aligned}$$

(b) *Let* $\frac{1}{2} < w < 1$. *Then,*

$$\begin{aligned}\|f\| &= \max\{|\beta|, |\alpha+(1-w)^2\beta|+(1-w)|\gamma|, |\alpha+(1-w^2)\beta|,\\ &\quad |\alpha-(1-w^2)\beta|+(1-w)|\gamma|, |(\frac{1}{2w})\alpha-(\frac{2-w}{2})\beta|+\frac{1}{2}|\gamma|,\\ &\quad |(\frac{2w-1}{2w^2})\alpha+(\frac{1-2w}{2})\beta|+\frac{1}{2w}|\gamma|\}.\end{aligned}$$

Proof. It follows from Theorem 2.3 and the fact that

$$\|f\| = \max_{T\in extB_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w)})}} |f(T)|.$$

□

Note that if $\|f\|=1$, then $|\alpha|\leq 1, |\beta|\leq 1, |\gamma|\leq \min\{1, 2w\}$.

Theorem 3.2.([17, Theorem 2.3]) *Let E be a real Banach space such that $extB_E$ is finite. Suppose that $x \in extB_E$ satisfies that there exists an $f\in E^*$ with $f(x) = 1 = \|f\|$ and $|f(y)| < 1$ for every $y \in extB_E\backslash\{\pm x\}$. Then, $x\in expB_E$.*

Now we are in position to describe all the exposed points of the unit ball of $\mathcal{L}_s(^2\mathbb{R}^2_{h(w)})$.

Theorem 3.3. *For $0 < w < 1$, $expB_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w)})} = extB_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w)})}$.*

Proof. We divide two cases.

Case 1: $0 < w \leq \frac{1}{2}$.

Claim: $T = (0, 1, 0)$ is exposed.

Let $f \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w)})^*$ be such that $\alpha = 0 = \gamma, \beta = 1$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w)})}\backslash\{\pm T\}$. By Theorem 3.2, T is exposed.

Claim: $T = (1, (1-w)^2, 1-w)$ is exposed.

Let $f \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w)})^*$ be such that $\alpha = \frac{1}{2} - \frac{(1-w)^2}{n}, \beta = \frac{1}{n}, \gamma = \frac{1}{2(1-w)}$ for a sufficiently large $n\in\mathbb{N}$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w)})}\backslash\{\pm T\}$. By Theorem 3.2, T is exposed.

Claim: $(1, 1-w^2, 0)$ is exposed.

Let $f \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w)})^*$ be such that $\alpha = \frac{1}{2} - \frac{1-w^2}{n}, \beta = \frac{1}{n}, \gamma = 0$ for a sufficiently large $n\in\mathbb{N}$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w)})}\backslash\{\pm T\}$. By Theorem 3.2, T is exposed.

Claim: $(0, 1-2w, 1)$ is exposed.

Let $f \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})^*$ be such that $\alpha = 0 = \beta, \gamma = 1$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})} \backslash \{\pm T\}$. By Theorem 3.2, T is exposed.

Claim: $T = (1, w^2-1, w)$ is exposed.

First suppose that $0 < w < \frac{1}{2}$. Let $f \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})^*$ be such that $\alpha = 0, \beta = -1, \gamma = w$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})} \backslash \{\pm T\}$. By Theorem 3.2, T is exposed.

If $w = \frac{1}{2}$, Then $T = (1, -\frac{3}{4}, \frac{1}{2})$. By Theorem 2.3,

$$extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(\frac{1}{2})})} = \{\pm(1, \frac{1}{4}, \pm\frac{1}{2}), \pm(1, -\frac{3}{4}, \pm\frac{1}{2}), \pm(1, \frac{3}{4}, 0), (0, 0, \pm 1)\}.$$

Let $f \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})^*$ be such that $\alpha = \frac{1}{4}, \beta = -1, \gamma = 0$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})} \backslash \{\pm T\}$. By Theorem 3.2, T is exposed.

Claim: $T = (1, -3w^2 + 2w - 1, 1-w)$ for $0 < w < 1$ is exposed.

Let $f \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})^*$ be such that $\alpha = \frac{1}{2} - \frac{3w^2-2w+1}{n}, \beta = -\frac{1}{n}, \gamma = \frac{1}{2(1-w)}$ for a sufficiently large $n \in \mathbb{N}$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})} \backslash \{\pm T\}$. By Theorem 3.2, T is exposed.

Case 2: $\frac{1}{2} < w < 1$.

Claim: $T = (0, 1, 0)$ is exposed.

Let $f \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})^*$ be such that $\alpha = 0 = \gamma, \beta = 1$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})} \backslash \{\pm T\}$. By Theorem 3.2, T is exposed.

Claim: $T = (1, (1-w)^2, 1-w)$ is exposed.

Let $f \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})^*$ be such that $\alpha = w - \frac{(1-w)^2}{n}, \beta = \frac{1}{n}, \gamma = 1$ for a sufficiently large $n \in \mathbb{N}$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})} \backslash \{\pm T\}$. By Theorem 3.2, T is exposed.

Claim: $(1, 1-w^2, 0)$ is exposed.

Let $f \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})^*$ be such that $\alpha = \frac{1}{2} - \frac{1-w^2}{n}, \beta = \frac{1}{n}, \gamma = 0$ for a sufficiently large $n \in \mathbb{N}$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})} \backslash \{\pm T\}$. By Theorem 3.2, T is exposed.

Claim: $(\frac{2w-1}{2w^2}, \frac{1-2w}{2}, \pm\frac{1}{2w})$ is exposed.

Let $f \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})^*$ be such that $\alpha = 0 = \beta, \gamma = 2w$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})} \backslash \{\pm T\}$. By Theorem 3.2, T is exposed.

Claim: $T = (1, w^2-1, 1-w)$ is exposed.

Let $f \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})^*$ be such that $\alpha = w, \beta = -\frac{1}{1+w}, \gamma = 0$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})} \backslash \{\pm T\}$. By Theorem 3.2, T is exposed.

Claim: $T = (\frac{1}{2w}, \frac{w-2}{2}, \frac{1}{2})$ is exposed.

Let $f \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})^*$ be such that $\alpha = 0, \beta = 1, \gamma = w$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})} \backslash \{\pm T\}$. By Theorem 3.2, T is exposed. □

References

[1] R. M. Aron, Y. S. Choi, S. G. Kim and M. Maestre, *Local properties of polynomials on a Banach space*, Illinois J. Math., **45**(2001), 25–39.

[2] Y. S. Choi, H. Ki and S. G. Kim, *Extreme polynomials and multilinear forms on l_1*, J. Math. Anal. Appl., **228**(1998), 467–482.

[3] Y. S. Choi and S. G. Kim, *The unit ball of $\mathcal{P}({}^2l_2^2)$*, Arch. Math.(Basel), **71**(1998), 472–480.

[4] Y. S. Choi and S. G. Kim, *Extreme polynomials on c_0*, Indian J. Pure Appl. Math., **29**(1998), 983–989.

[5] Y. S. Choi and S. G. Kim, *Smooth points of the unit ball of the space $\mathcal{P}({}^2l_1)$*, Results Math., **36**(1999), 26–33.

[6] Y. S. Choi and S. G. Kim, *Exposed points of the unit balls of the spaces $\mathcal{P}({}^2l_p^2)$ $(p = 1, 2, \infty)$*, Indian J. Pure Appl. Math., **35**(2004), 37–41.

[7] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer-Verlag, London (1999).

[8] S. Dineen, *Extreme integral polynomials on a complex Banach space*, Math. Scand., **92**(2003), 129–140.

[9] B. C. Grecu, *Geometry of 2-homogeneous polynomials on l_p spaces,* $1 < p < \infty$, J. Math. Anal. Appl., **273**(2002), 262–282 .

[10] B. C. Grecu, G. A. Munoz-Fernandez and J. B. Seoane-Sepulveda, *Unconditional constants and polynomial inequalities*, J. Approx. Theory, **161**(2009), 706–722.

[11] S. G. Kim, *Exposed 2-homogeneous polynomials on $\mathcal{P}({}^2l_p^2)$ $(1 \le p \le \infty)$*, Math. Proc. Royal Irish Acad., **107A**(2007), 123–129.

[12] S. G. Kim, *The unit ball of $\mathcal{L}_s({}^2l_\infty^2)$*, Extracta Math., **24**(2009), 17–29.

[13] S. G. Kim, *The unit ball of $\mathcal{P}({}^2d_*(1, w)^2)$*, Math. Proc. Royal Irish Acad., **111A**(2011), 79–94.

[14] S. G. Kim, *The unit ball of $\mathcal{L}_s({}^2d_*(1, w)^2)$*, Kyungpook Math. J., **53**(2013), 295–306.

[15] S. G. Kim, *Smooth polynomials of $\mathcal{P}({}^2d_*(1, w)^2)$*, Math. Proc. Royal Irish Acad., **113A**(2013), 45–58.

[16] S. G. Kim, *Extreme bilinear forms of $\mathcal{L}({}^2d_*(1, w)^2)$*, Kyungpook Math. J., **53**(2013), 625–638.

[17] S. G. Kim, *Exposed symmetric bilinear forms of $\mathcal{L}_s({}^2d_*(1, w)^2)$*, Kyungpook Math. J., **54**(2014), 341–347.

[18] S. G. Kim, *Exposed bilinear forms of $\mathcal{L}({}^2d_*(1, w)^2)$*, Kyungpook Math. J., **55**(2015), 119–126.

[19] S. G. Kim, *Exposed 2-homogeneous polynomials on the two-dimensional real predual of Lorentz sequence space*, Mediterr. J. Math., **13**(2016), 2827–2839.

[20] S. G. Kim, *The unit ball of $\mathcal{L}({}^2\mathbb{R}^2_{h(w)})$*, Bull. Korean Math. Soc., **54**(2017), 417–428.

[21] S. G. Kim and S. H. Lee, *Exposed 2-homogeneous polynomials on Hilbert spaces*, Proc. Amer. Math. Soc., **131**(2003), 449–453.

[22] J. Lee and K. S. Rim, *Properties of symmetric matrices*, J. Math. Anal. Appl., **305**(2005), 219–226.

[23] G. A. Munoz-Fernandez, S. Revesz and J. B. Seoane-Sepulveda, *Geometry of homogeneous polynomials on non symmetric convex bodies*, Math. Scand., **105**(2009), 147–160.

[24] G. A. Munoz-Fernandez and J. B. Seoane-Sepulveda, *Geometry of Banach spaces of trinomials*, J. Math. Anal. Appl., **340**(2008), 1069–1087.

[25] R. A. Ryan and B. Turett, *Geometry of spaces of polynomials*, J. Math. Anal. Appl., **221**(1998), 698–711.