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Inverse Eigenvalue Problems with Partial Eigen Data for Acyclic Matrices whose Graph is a Broom

Debashish Sharma*
Department of Mathematics, Gurucharan College, Silchar-788004, Assam, India
e-mail : debashish0612@gmail.com

Mausumi Sen
Department of Mathematics, National Institute of Technology, Silchar-788010, Assam, India
e-mail : senmausumi@gmail.com

Abstract. In this paper, we consider three inverse eigenvalue problems for a special type of acyclic matrices. The acyclic matrices considered in this paper are described by a graph called a broom on $n+m$ vertices, which is obtained by joining m pendant edges to one of the terminal vertices of a path on n vertices. The problems require the reconstruction of such a matrix from given partial eigen data. The eigen data for the first problem consists of the largest eigenvalue of each of the leading principal submatrices of the required matrix, while for the second problem it consists of an eigenvalue of each of its trailing principal submatrices. The third problem has an eigenvalue and a corresponding eigenvector of the required matrix as the eigen data. The method of solution involves the use of recurrence relations among the leading/trailing principal minors of $\lambda I - A$, where A is the required matrix. We derive the necessary and sufficient conditions for the solutions of these problems. The constructive nature of the proofs also provides the algorithms for computing the required entries of the matrix. We also provide some numerical examples to show the applicability of our results.

1. Introduction

Problems concerning the reconstruction of specially structured matrices from given information on some or all of their eigenvalues or eigenvectors, are of special interest in mathematics. Such problems are called Inverse eigenvalue problems (IEPs). The type of given eigen data and the structure of the required matrices

* Corresponding Author.
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determine the level of difficulty of the problems. M. T. Chu in [1] gave a detailed characterization of various classes of inverse eigenvalue problems. Some special types of inverse eigenvalue problems were studied in [4, 5, 6, 14, 17]. Many of these problems considered the matrices to be tridiagonal, Jacobi matrix, or arrow matrix. Eigenvalue problems for matrices with prescribed graphs have also been studied in [2, 3, 8, 11, 12]. Sharma and Sen in [15, 16] studied some IEPs where the matrices to be constructed were described by some special graphs. In particular, acyclic matrices whose graph is a path and matrices whose graph is a broom were studied. IEPs occur in various problems in mechanical vibrations, control theory, pole assignment problems and graph theory [1, 7, 11, 13].

In this paper, we study three IEPs which require the reconstruction of matrices whose graph is a broom. The eigen data in each of the three problems are different and so are the structures of the matrices to be constructed. For the first problem, the given eigen data consists of the largest eigenvalue of each of the leading principal submatrices of the required matrix. In the second problem, an eigenvalue of each of the trailing principal submatrices of the required matrix is given in the eigen data. The third problem has an eigenvalue and a corresponding eigenvector of the required matrix as the eigen data. We adopt a suitable scheme of labelling the vertices of the broom in order to express the corresponding matrices in a special form which simplifies the computation.

2. Preliminaries

Mathematically, we can define a graph with sets. Throughout this paper we assume that the graphs under consideration are free of multiple edges or loops and are undirected. With this assumption we have the following description of a graph : Let V be a finite set and let P be the set of all subsets of V which have two distinct elements. Let $E \subset P$. Then $G = (V, E)$ is said to be a *graph* with vertex set V and edge set E. The vertex set of a graph G is denoted by $V(G)$ and the edge set is denoted by $E(G)$. If $u, v \in V$ and $\{u, v\} \in E$ then we say that uv is an *edge* and u and v are called *adjacent* vertices. The degree of a vertex u is the number of edges which are incident on u. A vertex of degree one is called a *pendant vertex*. A *path* of a graph G is a sequence of distinct vertices $v_1, v_2, \ldots, v_n$ such that consecutive vertices are adjacent. A graph is said to be *connected* if there exists a path between every pair of its vertices. A *cycle* is a connected graph in which each vertex is adjacent to exactly two other vertices. A connected graph without any cycles is called a *tree*.

Given an $n \times n$ symmetric matrix A, the graph of A, denoted by $G(A)$, has vertex set $V(G) = \{1, 2, 3, \ldots, n\}$ and edge set $\{ij : i \neq j, a_{ij} \neq 0\}$. For a graph G with n vertices, $S(G)$ denotes the set of all $n \times n$ symmetric matrices which have G as their graph. A matrix whose graph is a tree is called an *acyclic* matrix. We have considered a special tree, namely broom B_{n+m} (Figure 1), which can be obtained by joining m pendant vertices to one of the terminal vertices of a path on n vertices.

The structure of a matrix $A \in S(B_{n+m})$ will depend on the way of labelling the

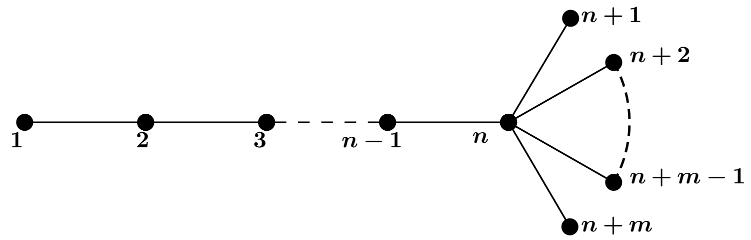

Figure 1: Broom B_{n+m}

$n + m$ vertices from $1, 2, \ldots, n + m$ as the rows and columns of A will be indexed by the vertices of B_{n+m}. The broom is formed by adjoining m pendant vertices to one of the terminal vertices of a path on n vertices. We label the vertices of the broom in such a way that the vertices numbered from 1 to n are the vertices of this path on n vertices. And then we label the pendant vertices adjacent to the vertex labelled as n, with the numbers from 1 to m. This system of labelling has been shown in Figure 1. Under this scheme of labelling, the matrix of a broom B_{n+m} will be of the following form :

$$A_{n+m} = \begin{pmatrix} a_1 & b_1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ b_1 & a_2 & b_2 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & b_2 & a_3 & \ddots & 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a_{n-1} & b_{n-1} & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & b_{n-1} & a_n & b_n & b_{n+1} & \ldots & b_{n+m-1} \\ 0 & 0 & 0 & \ldots & 0 & b_n & a_{n+1} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & b_{n+1} & 0 & a_{n+2} & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & b_{n+m-1} & 0 & 0 & \ldots & a_{n+m} \end{pmatrix}_{(n+m)\times(n+m)}$$

where the b_i's are non zero. Here, the matrix corresponding to the subgraph of B_{n+m} formed with the first n vertices is a tridiagonal matrix with all non-zero entries in the sub-diagonal and super diagonal. And the matrix corresponding to the subgraph formed with the vertex n and the pendants attached to it is an arrow matrix with non-zero off-diagonal entries in the first row and first column. If A_{n+m} is the adjacency matrix, as required in IEPB2 and IEPB3, then $a_i = 0$ for all $i = 1, 2, \ldots, n + m$ and $b_j = 1$ for all $j = 1, 2, \ldots, n + m - 1$.

Throughout this paper we shall use the following notation :

1. A_j will denote the $j \times j$ *leading* principal sub-matrix of the matrix A_{n+m}.

2. $P_j(\lambda) = det(\lambda I_j - A_j)$ for $j = 1, 2, \ldots, n+m$. For convenience of notations, $P_0(\lambda)$ is defined to be 1 for all λ.
3. $A_{j,n+m}$ denotes the $(n+m-j+1) \times (n+m-j+1)$ *trailing* principal sub-matrix of A_{n+m}.
4. $P_{j,n+m}(\lambda) = det(\lambda I_{n+m-j+1} - A_{j,n+m})$ for IEPB1 and $P_{j,n+m}(\lambda) = det(\lambda I_{n+m-j+1} - (A_{j,n+m} + D_{j,n+m}))$ for IEPB2 and IEPB3.

3. IEPs to be Studied

- **IEPB1** Given $n+m$ real numbers $\lambda_1,\ \lambda_2,\ \cdots, \lambda_n,\ \lambda_{n+1},\ \cdots, \lambda_{n+m}$, find a matrix $A_{n+m} \in S(B_{n+m})$ such that its diagonal entries are equal to a and λ_j is the largest eigenvalue of A_j i.e. of the $j \times j$ leading principal sub-matrix of A_{n+m}.
- **IEPB2** Given $n+m$ distinct real numbers $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n,\ \lambda_{n+1},\ \cdots, \lambda_{n+m}$ and the adjacency matrix A_{n+m} of B_{n+m}, find a diagonal matrix $D_{n+m} = diag(a_1, a_2, \ldots, a_{n+m})$ such that λ_j is an eigenvalue of $A_{j,n+m} + D_{j,n+m}$, i.e. of the $(n+m-j+1) \times (n+m-j+1)$ trailing principal sub-matrix of $A_{n+m} + D_{n+m}$.
- **IEPB3** Given a real vector $X = (x_1, x_2, \ldots, x_{n+m})^T$, a non zero real number λ and the adjacency matrix A_{n+m} of B_{n+m}, find a diagonal matrix $D_{n+m} = diag(a_1, a_2, \ldots, a_{n+m})$ such that (λ, X) is an eigenpair of $A_{n+m} + D_{n+m}$.

4. Solution of IEPB1

For solving IEPB1, we analyze the expressions for the successive leading principal minors of $\lambda I_{n+m} - A_{n+m}$ and apply Cauchy's interlacing theorem([8, 9]). The following Lemmas will be necessary for solving the problem. The main result of this section is given as Theorem 4.4.

Lemma 4.1. *Let $P(\lambda)$ be a monic polynomial of degree n with all real eigenvalues and λ_1 and λ_n be the minimal and maximal zero of P respectively.*

1. *If $\mu < \lambda_1$, then $(-1)^n P(\mu) > 0$*
2. *If $\mu > \lambda_n$, then $P(\mu) > 0$*

The proof immediately follows after expressing the polynomial as a product of its linear factors.

Lemma 4.2. *The characteristic polynomials of the $j \times j$ leading principal submatrices of A_{n+m} satisfy the following recurrence relations :*

1. $P_1(\lambda) = \lambda - a$

2. $P_j(\lambda) = (\lambda - a)P_{j-1}(\lambda) - b_{j-1}^2 P_{j-2}(\lambda), \ \ 2 \leq j \leq n+1$
3. $P_{n+j}(\lambda) = (\lambda - a)P_{n+j-1}(\lambda) - b_{n+j-1}^2 (\lambda - a)^{j-1} P_{n-1}(\lambda), \ \ 2 \leq j \leq m$

This follows by computing the determinants of the successive leading principal submatrices of $\lambda I_{n+m} - A_{n+m}$.

Lemma 4.3. *For any $\lambda \in \mathbb{R}$, $P_{j-1}(\lambda)$ and $P_j(\lambda)$ cannot be simultaneously zero, for $2 \leq j \leq n+1$.*

Proof. If $P_1(\lambda) = 0$, then $P_2(\lambda) = (\lambda - a)P_1(\lambda) - b_1^2 = -b_1^2 \neq 0$. Thus $P_1(\lambda)$ and $P_2(\lambda)$ cannot be simultaneously zero. Now if for $3 \leq j \leq n+1$, $P_j(\lambda) = 0$ and $P_{j-1}(\lambda) = 0$, then from the second recurrence relation in Lemma 4.3, we have $P_{j-2}(\lambda) = 0$. This will ultimately lead to $P_2(\lambda) = 0$ and $P_1(\lambda) = 0$ which is not possible as argued above. $\square$

Theorem 4.4. *The IEPB1 has a solution if and only if $\lambda_1 < \lambda_2 < \cdots < \lambda_{n+m}$. The solution is given by*

$$\begin{aligned} a &= \lambda_1 \\ b_{j-1} &= \sqrt{\frac{(\lambda_j - \lambda_1)P_{j-1}(\lambda_j)}{P_{j-2}(\lambda_j)}}, \ 2 \leq j \leq n+1 \\ b_{n+j-1} &= \sqrt{\frac{P_{n+j-1}(\lambda_{n+j})}{(\lambda_{n+j} - \lambda_1)^{j-2} P_{n-1}(\lambda_{n+j})}}, \ 2 \leq j \leq m \end{aligned}$$

The solution is unique except for the signs of the off-diagonal entries.

Proof. We first suppose that the problem has a solution. We find the expressions for a and b_i, $i = 1, 2, \ldots, n+m$. Since λ_1 is an eigenvalue of A_1, so $P_1(\lambda_1) = 0$. This gives $a = \lambda_1$. Now for $2 \leq j \leq n+1$, λ_j is an eigenvalue of A_j, so

$$\begin{aligned} & P_j(\lambda_j) = 0 \\ & \Rightarrow (\lambda_j - \lambda_1)P_{j-1}(\lambda_j) - b_{j-1}^2 P_{j-2}(\lambda_j) = 0 \\ & \Rightarrow b_{j-1}^2 = \frac{(\lambda_j - \lambda_1)P_{j-1}(\lambda_j)}{P_{j-2}(\lambda_j)} \end{aligned}$$

Similarly λ_{n+j} is an eigenvalue of A_{n+j} and so $P_{n+j}(\lambda_{n+j}) = 0$. Thus,

$$b_{n+j-1}^2 = \frac{P_{n+j-1}(\lambda_{n+j})}{(\lambda_{n+j} - \lambda_1)^{j-2} P_{n-1}(\lambda_{n+j})}$$

Now, for the problem to have a solution, the quantity in RHS of the above expressions for b_i^2 must be positive. First, we claim $\lambda_j - \lambda_1 > 0$ for all $j = 2, 3, \ldots, n$. By Cauchy's interlacing theorem([8, 9]), the eigenvalues of a symmetric matrix and those of any of its principal submatrix interlace each other. Since each λ_j is the largest eigenvalue of the $j \times j$ leading principal submatrix of A_{n+m}, so it follows that

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n+m-1} \le \lambda_{n+m} \tag{4.1}$$

Thus $\lambda_j \ge \lambda_{j-1}$ for all $j = 2, 3, \ldots, n+m$. Hence $\lambda_j - \lambda_1 \ge 0$ for all $j = 2, 3, \ldots, n+m$. Next we show that the inequality (4.1) is strict. If $\lambda_{j-1} = \lambda_j$ for some j with $1 \le j \le n+1$, then we have $P_{j-1}(\lambda_j) = 0 = P_j(\lambda_j)$ but this is not possible according to Lemma 4.3. Thus $\lambda_1 < \lambda_2 < \cdots < \lambda_{n+1}$. Now if $\lambda_{n+j-1} = \lambda_{n+j}$ for some j with $2 \le j \le m$, then $P_{n+j-1}(\lambda_{n+j}) = 0 = P_{n+j}(\lambda_{n+j})$ and by the fourth recurrence relation from Lemma 4.2, we get $P_{n-1}(\lambda_{n+j}) = 0$. This implies that λ_{n+j} is a root of $P_{n-1}(\lambda)$ but λ_{n-1} is the largest root of $P_{n-1}(\lambda)$ and so the inequality (4.1) will imply that $\lambda_{n-1} = \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+j}$. Thus we end up with $P_{n-1}(\lambda_{n+j}) = 0$ and $P_n(\lambda_{n+j}) = 0$ but this is not possible as they cannot be simultaneously zero, by Lemma 4.3. Hence the inequality (4.1) must be strict.

Now since $\lambda_1 < \lambda_2 < \cdots < \lambda_{n+m}$ and also each since λ_i is the largest root of $P_i(\lambda)$, so by Lemma 4.1, we have $P_{j-2}(\lambda_j) > 0$, $P_{j-1}(\lambda_j) > 0$, $P_{n-1}(\lambda_{n+1}) > 0, P_n(\lambda_{n+1}) > 0$. Hence, the expressions for b_i^2 are all positive. Thus the solution of IEPB1 is given by

$$\begin{aligned} a &= \lambda_1 \\ b_{j-1} &= \sqrt{\frac{(\lambda_j - \lambda_1)P_{j-1}(\lambda_j)}{P_{j-2}(\lambda_j)}},\ 2 \le j \le n+1 \\ b_{n+j-1} &= \sqrt{\frac{P_{n+j-1}(\lambda_{n+j})}{(\lambda_{n+j} - \lambda_1)^{j-2}P_{n-1}(\lambda_{n+j})}},\ 2 \le j \le m \end{aligned} \tag{4.2}$$

Conversely if $\lambda_1 < \lambda_2 < \cdots < \lambda_{n+m}$, then by the above argument, the expressions for b_i^2 are all positive, so that the IEPB1 has a solution. □

5. Solution of IEPB2

Lemma 5.1. *The characteristic polynomials of the $(n+m-j+1) \times (n+m-j+1)$ trailing principal submatrices of $A_{n+m} + D_{n+m}$ satisfy the following recurrence relations :*

1. $P_{j,n+m}(\lambda) = (\lambda - a_j)P_{j+1,n+m}(\lambda) - P_{j+2,n+m}(\lambda),\ 1 \le j \le n-1$
2. $P_{n,n+m}(\lambda) = (\lambda - a_n)\prod_{i=1}^{m}(\lambda - a_{n+i}) - \sum_{j=1}^{m}\prod_{i=1, i\neq j}^{m}(\lambda - a_{n+i})$
3. $P_{n+j,n+m}(\lambda) = \prod_{i=j}^{m}(\lambda - a_{n+i}),\ 1 \le j \le m$

This can be proved by computing the determinants of the successive trailing principal submatrices of $\lambda I_{n+m} - (A_{n+m} + D_{n+m})$. The following theorem gives the solution of IEPB2 :

Theorem 5.2. *The IEPB2 has a solution if and only if* $P_{j+1,n+m}(\lambda_j) \neq 0$ *for all* $j = 1, 2, \ldots, n-1$. *The solution, if it exists, is unique and is given by*

$$\begin{aligned} a_{n+j} &= \lambda_{n+j},\ m \geq j \geq 1 \\ a_n &= \lambda_n - \frac{\sum_{j=1}^m \prod_{i=1,i\neq j}^m (\lambda_n - \lambda_{n+i})}{\prod_{i=1}^m (\lambda_n - \lambda_{n+i})} \\ a_j &= \lambda_j - \frac{P_{j+2,n+m}(\lambda_j)}{P_{j+1,n+m}(\lambda_j)},\ n-1 \geq j \geq 1 \end{aligned} \tag{5.1}$$

Proof. We first suppose that the IEPB2 has a solution. We derive the explicit expressions for all the diagonal elements of $A_{n+m} + D_{n+m}$. Since λ_{n+m} is an eigenvalue of $A_{n+m,n+m} + D_{n+m,n+m}$, so $P_{n+m,n+m}(\lambda_{n+m}) = 0$. Using the third recurrence relation from Lemma 4.1, we get $(\lambda_{n+m} - a_{n+m}) = 0$ which gives $a_{n+m} = \lambda_{n+m}$. Similarly, since λ_{n+m-1} is an eigenvalue of $A_{n+m-1,n+m} + D_{n+m-1,n+m}$, so $(\lambda_{n+m-1} - a_{n+m-1})(\lambda_{n+m-1} - \lambda_{n+m}) = 0$. Also, $\lambda_{n+m-1} \neq \lambda_{n+m}$. So we get $a_{n+m-1} = \lambda_{n+m-1}$. Proceeding this way it can be shown that

$$a_{n+j} = \lambda_{n+j},\ m \geq j \geq 1 \tag{5.2}$$

Next, since λ_n is an eigenvalue of $A_{n,n+m} + D_{n,n+m}$, so from the second recurrence relation in Lemma 4.1, we have

$$\begin{aligned} & P_{n,n+m}(\lambda_n) = 0 \\ & \Rightarrow (\lambda_n - a_n) \prod_{i=1}^m (\lambda_n - a_{n+i}) - \sum_{j=1}^m \prod_{i=1,i\neq j}^m (\lambda_n - a_{n+i}) = 0 \\ & \Rightarrow a_n = \lambda_n - \frac{\sum_{j=1}^m \prod_{i=1,i\neq j}^m (\lambda_n - \lambda_{n+i})}{\prod_{i=1}^m (\lambda_n - \lambda_{n+i})} \end{aligned}$$

which gives

$$a_n = \lambda_n - \frac{\sum_{j=1}^m \prod_{i=1,i\neq j}^m (\lambda_n - \lambda_{n+i})}{\prod_{i=1}^m (\lambda_n - \lambda_{n+i})} \tag{5.3}$$

which exists as the given eigenvalues are distinct. Again, since λ_j is an eigenvalue of $A_{j,n+m} + D_{j,n+m}$, so $P_{j,n+m}(\lambda_j) = 0$ and so from the first recurrence relation of Lemma 4.1, we get

$$a_j = \lambda_j - \frac{P_{j+2,n+m}(\lambda_j)}{P_{j+1,n+m}(\lambda_j)},\ n-1 \geq j \geq 1 \tag{5.4}$$

which exists only if $P_{j+1,n+m}(\lambda_j) \neq 0$ for all $j = 1, 2, \ldots, n-1$.

Conversely, if $P_{j+1,n+m}(\lambda_j) \neq 0$ for all $j = 1, 2, \ldots, n-1$, then proceeding as above we can deduce equations (5.2), (5.3) and (5.4), and so the IEPB2 has a solution. $\square$

6. Solution of IEPB3

Lemma 6.1. *The entries of the matrix $A_{n+m} + D_{n+m}$ and of the vector X satisfy the following relations :*

1. $a_1 x_1 + x_2 = \lambda x_1$
2. $x_{j-1} + a_j x_j + x_{j+1} = \lambda x_j, \ 2 \leq j \leq n-1$
3. $x_{n-1} + a_n x_n + \sum_{j=1}^{m} x_{n+j} = \lambda x_n$
4. $x_n + a_{n+j} x_{n+j} = \lambda x_{n+j}, \ 1 \leq j \leq m$

It can be proved by equating the successive leading principal submatrices on both sides of $(A_{n+m} + D_{n+m})X = \lambda X$. The main result of this section is given in the following theorem :

Theorem 6.2. *The IEPB3 has a unique solution if and only if $x_j \neq 0$ for all $j = 1, 2, \ldots, n+m$. The unique solution is given by*

$$\begin{aligned}
a_1 &= \frac{1}{x_1}(\lambda x_1 - x_2) \\
a_j &= \frac{1}{x_j}(\lambda x_j - x_{j-1} - x_{j+1}), \ 2 \leq j \leq n-1 \\
a_n &= \frac{1}{x_n}\left(\lambda x_n - x_{n-1} - \sum_{j=1}^{m} x_{n+j}\right) \\
a_{n+j} &= \frac{1}{x_{n+j}}(\lambda x_{n+j} - x_n), \ 1 \leq j \leq m
\end{aligned}$$

Proof. First, we suppose that the IEPB3 has a unique solution i.e. there exists a unique diagonal matrix $D_{n+m} = diag(a_1, a_2, \ldots, a_{n+m})$ such that (λ, X) is an eigenpair of $A_{n+m} + D_{n+m}$. Thus, $(A_{n+m} + D_{n+m})X = \lambda X$. Equating the corresponding entries on both sides, we get as in Lemma 6.1,

$$\begin{aligned}
a_1 x_1 + x_2 &= \lambda x_1 \\
x_{j-1} + a_j x_j + x_{j+1} &= \lambda x_j, \ 2 \leq j \leq n-1 \\
x_{n-1} + a_n x_n + \sum_{j=1}^{m} x_{n+j} &= \lambda x_n \\
x_n + a_{n+j} x_{n+j} &= \lambda x_{n+j}, \ 1 \leq j \leq m
\end{aligned}$$

Each of the above equations is linear in a_j. Since the IEPB3 has a unique solution, so each of these linear equations must have unique solution. Hence the coefficient of each a_j must be non zero i.e. $x_j \neq 0$ for all $j = 1, 2 \ldots, n+m$. Thus, solving the above linear equations for each a_j, we get

$$
\begin{aligned}
a_1 &= \frac{1}{x_1}(\lambda x_1 - x_2) \\
a_j &= \frac{1}{x_j}(\lambda x_j - x_{j-1} - x_{j+1}),\ 2 \leq j \leq n-1 \\
a_n &= \frac{1}{x_n}\left(\lambda x_n - x_{n-1} - \sum_{j=1}^{m} x_{n+j}\right) \\
a_{n+j} &= \frac{1}{x_{n+j}}(\lambda x_{n+j} - x_n),\ 1 \leq j \leq m
\end{aligned} \tag{6.1}
$$

Conversely, if $x_j \neq 0$ for all $j = 1, 2, \ldots, n+m$, then proceeding as above we can arrive at the expressions as in (6.1), concluding that the IEPB3 has a unique solution. □

7. Numerical Examples

The results obtained in the preceding sections have been formulated as scripts in SCILAB 5.5.2 to solve the inverse eigenvalue problems for any eigen data entered by the user. The numerical examples here have been obtained by executing the corresponding scripts in SCILAB. The entries are correct up to 4 places of decimal.

Example 7.1. Given 7 real numbers $\lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = 3.5,\ \lambda_4 = 6.92,\ \lambda_5 = 8,\ \lambda_6 = 9.5,\ \lambda_7 = 12.4$, construct $A_{3+4} \in S(B_{3+4})$ such that λ_j is the largest eigenvalue of A_j.

Solution. Since $\lambda_1 < \lambda_2 < \cdots < \lambda_7$, so by Theorem 4.4, we obtain the solution as

$$
A_{3+4} = \begin{pmatrix}
-1 & 3 & 0 & 0 & 0 & 0 & 0 \\
3 & -1 & 3.3541 & 0 & 0 & 0 & 0 \\
0 & 3.3541 & -1 & 7.0421 & 4.3303 & 5.4458 & 8.3496 \\
0 & 0 & 7.0421 & -1 & 0 & 0 & 0 \\
0 & 0 & 4.3303 & 0 & -1 & 0 & 0 \\
0 & 0 & 5.4458 & 0 & 0 & -1 & 0 \\
0 & 0 & 8.3496 & 0 & 0 & 0 & -1
\end{pmatrix}
$$

We compute the eigenvalues of the successive leading principle submatrices. The given eigenvalues are written in bold font to illustrate that the conditions of the problem are satisfied.

$\sigma(A_1) = \{\mathbf{-1}\}$

$\sigma(A_2) = \{-4, \mathbf{2}\}$
$\sigma(A_3) = \{-5.5, -1, \mathbf{3.5}\}$
$\sigma(A_4) = \{-8.92, -3.6675, 1.6675, \mathbf{6.92}\}$
$\sigma(A_5) = \{-10, -3.7557, -1, 1.7557, \mathbf{8}\}$
$\sigma(A_6) = \{-11.5, -3.8284, -1, -1, 1.8284, \mathbf{9.5}\}$
$\sigma(A_7) = \{-14.4, -3.8994, -1, -1, -1, 1.8994, \mathbf{12.4}\}$

Example 7.2. Given 7 real numbers $\lambda_1 = 9,\ \lambda_2 = 1,\ \lambda_3 = 2,\ \lambda_4 = 7,\ \lambda_5 = 4,\ \lambda_6 = 5,\ \lambda_7 = 6$, and the adjacency matrix A_{3+4} of B_{3+4}, find a diagonal matrix D_{3+4} such that λ_j is an eigenvalue of $A_{j,3+4} + D_{j,3+4}$.

Solution. By Theorem 5.2, we obtain the solution as

$$A_{3+4} + D_{3+4} = \begin{pmatrix} 8.8576 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1.7500 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3.2833 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 7.0000 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4.0000 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 5.0000 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 6.0000 \end{pmatrix}$$

We compute the spectra of the successive trailing principal submatrices of $A_{3+4} + D_{3+4}$. The given eigenvalues are written in bold font to show that the conditions of the problem are satisfied.

$\sigma(A_{1,3+4} + D_{1,3+4}) = \{0.9236, 2.6075, 4.3206, 5.3229, 6.3206, 7.3958, \mathbf{9}\}$
$\sigma(A_{2,3+4} + D_{2,3+4}) = \{\mathbf{1}, 2.6661, 4.3229, 5.3244, 6.3218, 7.3982\}$
$\sigma(A_{3,3+4} + D_{3,3+4}) = \{\mathbf{2}, 4.2952, 5.3046, 6.3063, 7.3772\}$
$\sigma(A_{4,3+4} + D_{4,3+4}) = \{4, 5, 6, \mathbf{7}\}$
$\sigma(A_{5,3+4} + D_{5,3+4}) = \{\mathbf{4}, 5, 6\}$
$\sigma(A_{6,3+4} + D_{6,3+4}) = \{\mathbf{5}, 6\}$
$\sigma(A_{7,3+4} + D_{7,3+4}) = \{\mathbf{6}\}$

Example 7.3. Given a real vector $X = (-1, 2, 3.5, 6.92, 8, 9.5, 12.4)^T$, a real number $\lambda = 2$ and the adjacency matrix A_{3+4} of B_{3+4}, find a diagonal matrix D_{3+4} such that (λ, X) is an eigenpair of $A_{3+4} + D_{3+4}$.

Solution. Since each component of the vector X is non-zero, so by Theorem 6.2, we obtain the solution as

$$A_{3+4} + D_{3+4} = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0.75 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -9.0914 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1.4942 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1.5625 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1.6316 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1.7177 \end{pmatrix}$$

8. Conclusion

The inverse eigenvalue problems considered in this paper fall into the category of partially described inverse eigenvalue problems, as the eigen data consists of one eigenvalue each of the leading/trailing principal submatrices of the required matrix. Such partially described problems may occur in various computations involving a complicated physical system where it is often impossible to obtain the entire spectral information. We are also studying similiar problems for more general trees instead of a broom, for example in case of a tree with only one diametrical path such that each non-terminal vertex is of degree four. The problems may get complicated for acyclic matrices whose graphs are trees with more than one diametrical path. Such problems can be challenging and interesting.

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