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Properties of Nowhere Dense Sets in GTSs

Subramanian Vadakasi and Vellapandi Renukadevi*
Centre for Research and Post Graduate Studies in Mathematics, Ayya Nadar Janaki Ammal College(Autonomous), Sivakasi-626 124, Tamil Nadu, India
e-mail: vadakasi.subramanian@yahoo.co.in *and* renu_siva2003@yahoo.com

Abstract. In this paper, we analyze some new properties of nowhere dense and strongly nowhere dense sets. Also, we discuss the images of nowhere dense and strongly nowhere dense sets. Further, we study the properties of weak Baire space in generalized topological spaces.

1. Introduction

The notion of a generalized topological space was introduced by Császár in [1]. Let X be any nonempty set. A family $\mu \subset exp(X)$ is a *generalized topology* [3] in X if $\emptyset \in \mu$ and $\bigcup_{t\in T} G_t \in \mu$ whenever $\{G_t \mid t \in T\} \subset \mu$ where $exp(X)$ is the power set of X. We call the pair (X, μ) as a *generalized topological space* (GTS) [3]. If $X \in \mu$, then the pair (X, μ) is called a *strong generalized topological space* [3] (sGTS). If $Y \subset X$, then the *subspace generalized topology* on Y is defined by, $\mu_Y = \{Y \cap U \mid U \in \mu\}$ and (Y, μ_Y) is called the *subspace GTS*. We use $\tilde{\mu}$ [2] to denote $\{U \in \mu \mid U \neq \emptyset\}$ and $\mu(x)$ [2] to denote $\{U \in \mu \mid x \in U\}$.

Let $A \subset X$. The *interior of* A [3] denoted by $i_\mu A$, is the union of all μ-open sets contained in A and the *closure of* A [3] denoted by $c_\mu A$, is the intersection of all μ-closed sets containing A. The elements in μ are called the μ-open sets, the complement of a μ-open is called the μ-closed sets where the complement of μ is denoted by $\mu^{'}$. Let $\{(X_\alpha, \mu_\alpha) \mid \alpha \in \Gamma\}$ be a family of pairwise disjoint strong GTSs. Put $X = \bigcup_{\alpha\in\Gamma} X_\alpha$ and $\mu = \{A \subset X \mid A \cap X_\alpha \in \mu_\alpha$ for each $\alpha \in \Gamma\}$. Then (X, μ) is a GTS, which is denoted by $\bigoplus_{\alpha\in\Gamma} X_\alpha$, and called the *generalized topological sum* (GT-sum) of $\{(X_\alpha, \mu_\alpha) \mid \alpha \in \Gamma\}$ [3].

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* Corresponding Author.
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2. Preliminaries

A subset A of a GTS (X,μ) is said to be a *μ-nowhere dense* [2] (resp. *μ-dense* [3] and *μ-codense* [3]) set if $i_\mu c_\mu A=\emptyset$ (resp. $c_\mu A=X$ and $c_\mu(X-A)=X$). A subset A of a GTS (X,μ) is said to be *μ-strongly nowhere dense* [2] if for any $V\in\tilde{\mu}$, there exists $U\in\tilde{\mu}$ such that $U\subset V$ and $U\cap A=\emptyset$. Every μ-strongly nowhere dense set is μ-nowhere dense set [2]. A subset A of a GTS (X,μ) is said to be a *μ-meager* [2] (resp. *μ-s-meager* [2]) set if $A=\bigcup\limits_{n\in\mathbb{N}}A_n$ and A_n is μ-nowhere dense (resp. μ-strongly nowhere dense) for all $n\in\mathbb{N}$ where $\mathbb{N}$ denote the set of all natural numbers. Also, every subset of a μ-meager set is a μ-meager set. A subset A of a GTS (X,μ) is said to be a *μ-second category* (μ-II category) [2] (resp. *μ-s-second category* (μ-s-II category) [2]) set if A is not a μ-meager (resp. μ-s-meager) set. A subset A of a GTS (X,μ) is of *μ-residual* [2] (resp. *μ-s-residual* [2]) if $X-A$ is a μ-meager (resp. μ-s-meager) set. A GTS (X,μ) is called a *Baire space* (BS) [2] if every $V\in\tilde{\mu}$ is of μ-II category set. A GTS (X,μ) is called a *weak Baire space* (wBS) [2] if each set $V\in\tilde{\mu}$ is of μ-s-II category in X. A space (X,μ) is a *strong Baire space* (sBS) [2] if $V_1\cap V_2\cap...\cap V_n$ is a μ-II category set for all $V_1,V_2,...,V_n\in\mu$ such that $V_1\cap V_2\cap...\cap V_n\neq\emptyset$. Also, every sBS is BS [2]. In [2], it is defined that $\mu^\star=\{\bigcup_t(U_1^t\cap U_2^t\cap U_3^t\cap...\cap U_{n_t}^t)\mid U_1^t,U_2^t,...,U_{n_t}^t\in\mu\}$ and $\mu^{\star\star}=\{A\mid A$ is a μ-II category set$\}\cup\{\emptyset\}$. If (X,μ) is not sGTS, then $\mu^\star$ is closed under finite intersection and $\mu\subset\mu^\star$, $\mu\subset\mu^{\star\star}$ if (X,μ) is BS [2].

Throughout this paper, $\mathcal{D}(\mu)$ (resp. $\mathcal{N}(\mu)$) denotes the set of all dense (resp. nowhere dense) sets in (X,μ). The notations X_3,X_4,X_5 are mean the sets $\{a,b,c\},\{a,b,c,d\},\{a,b,c,d,e\}$, respectively. The following lemmas will be useful in the sequel.

Lemma 2.1.([2, Property 2.2]) *Let (X,μ) satisfy the condition: if $V_1,V_2\in\mu$ and $V_1\cap V_2\neq\emptyset$, then $i_\mu(V_1\cap V_2)\neq\emptyset$. Then the set $A\subset X$ is μ-strongly nowhere dense if and only if A is a μ-nowhere dense set.*

Lemma 2.2.([2, Property 2.3]) *Let (X,μ) be a GTS and $A\subset X$ be a μ-strongly nowhere dense set. Then any subset of A is μ-strongly nowhere dense set.*

Lemma 2.3.([2, Property 2.4]) *If $\mathcal{A}$ is a finite family of μ-strongly nowhere dense sets, then $\bigcup\mathcal{A}$ is a μ-strongly nowhere dense set.*

Lemma 2.4.([3, Proposition 3.4]) *Let (X,μ) be a GTS and let $A\subset X$. Then $A\in\mathcal{N}(\mu)$ if and only if $c_\mu A$ is μ-codense in X.*

Lemma 2.5.([3, Lemma 6.12]) *Let (X,μ) be the GT-sum of $\{(X_\alpha,\mu_\alpha):\alpha\in\Gamma\}$. If $A\in\mathcal{D}(\mu)$, then $A\cap X_\alpha\in\mathcal{D}(\mu_\alpha)$ for any $\alpha\in\Gamma$.*

3. Properties of Nowhere Dense Sets

In this section, we discuss the properties of two types of nowhere dense sets in generalized topological spaces. Also, we study the properties of $\mu^\star$, $\mu^{\star\star}-$nowhere dense and $\mu^\star$, $\mu^{\star\star}-$strongly nowhere dense sets in generalized topological spaces.

Theorem 3.1. *Let (X, μ) be a GTS and let $A \subset Y \subset X$. If Y is a μ-s-meager set in X, then A is a μ-s-meager set in X.*

Proof. Suppose Y is a μ-s-meager set in X. Then $Y = \bigcup_{n\in\mathbb{N}} B_n$ where B_n is a μ-strongly nowhere dense set. Now, $A = A \cap Y = \bigcup_{n\in\mathbb{N}} (A \cap B_n)$. Put $A_n = A \cap B_n$ for $n \in \mathbb{N}$. Then A_n is a μ-strongly nowhere dense set for $n \in \mathbb{N}$, by Lemma 2.2. Therefore, A is a μ-s-meager set in X. $\square$

The following Corollary 3.2 follows from Theorem 3.1 and the easy proof of the following Theorem 3.3 is omitted.

Corollary 3.2. *Let (X, μ) be a GTS and let $A \subset Y \subset X$. Then the following hold.*

(a) *If A is of μ-s-II category set in X, then Y is of μ-s-II category set in X.*

(b) *If A is a μ-s-residual set in X, then Y is a μ-s-residual set in X.*

(c) *If A is μ-s-residual, then the closure of A is μ-s-residual.*

Theorem 3.3. *Let (X, μ) be a GTS. Then the following hold.*

(a) *If G_n is μ-s-meager for each $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} G_n$ is μ-s-meager.*

(b) *If F_n is μ-s-residual for each $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} F_n$ is μ-s-residual.*

Lemma 3.4. *Let (X, μ) be a GTS and Y be a μ-dense subset of X. If $A \subset Y \subset X$ is μ-strongly nowhere dense in X, then A is μ_Y-strongly nowhere dense in Y.*

Proof. Suppose Y is a μ-dense subset of X. Let $U \in \tilde{\mu}_Y$. Then $U = G \cap Y$ where $G \in \tilde{\mu}$. By hypothesis, there exists $W \in \tilde{\mu}$ such that $W \subset G$ and $W \cap A = \emptyset$. This implies $W \cap Y \in \tilde{\mu}_Y$ such that $W \cap Y \subset G \cap Y$ and $W \cap Y \cap A = \emptyset$. If $V = W \cap Y$, then there exists $V \in \tilde{\mu}_Y$ such that $V \subset U$ and $V \cap A = \emptyset$. Therefore, A is μ_Y-strongly nowhere dense in Y. $\square$

The following Example 3.5 shows that the condition dense on Y cannot be dropped in Lemma 3.4.

Example 3.5. Consider the GTS (X_4, μ) where $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Let $Y = \{a, c, d\}$. Then $\mu_Y = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. If $A = \{c, d\}$, then A is a μ-strongly nowhere dense set. For $V = \{c\}$, there is no $U \in \tilde{\mu}_Y$ such that $U \subset V$ and $U \cap A = \emptyset$. Thus, A is not a μ_Y-strongly nowhere dense set in Y.

Theorem 3.6. *Let (X, μ) be a GTS and Y be a μ-dense subset of X. If $A \subset Y \subset X$ is a μ-s-meager set in X, then A is a μ_Y-s-meager set in Y.*

Proof. Suppose A is a μ-s-meager set in X. Then $A = \bigcup_{n\in\mathbb{N}} A_n$ where each A_n is a μ-strongly nowhere dense set in X. By Lemma 3.4, each A_n is a μ_Y-strongly nowhere dense set in Y. Therefore, A is a μ_Y-s-meager set in Y. $\square$

Corollary 3.7. *Let (X, μ) be a GTS and Y be a μ-dense subset of X. Then the following hold.*

(a) *If $A \subset Y \subset X$ is a μ_Y-s-II category set in Y, then A is a μ-s-II category set in X.*

(b) *If $A \subset Y \subset X$ is μ-s-residual in X, then A is μ_Y-s-residual in Y.*

Proof. (a) Follows from Theorem 3.6.
(b) Suppose A is a μ-s-residual set in X. Then $X - A$ is a μ-s-meager set in X. By hypothesis and Theorem 3.6, $X - A$ is a μ_Y-s-meager set in Y. This implies $Y - A$ is a μ_Y-s-meager set in Y, by Theorem 3.1. Therefore, A is a μ_Y-s-residual set in Y. □

Theorem 3.8. *Let (X, μ) be a GTS and $A \subset X$. If A is a μ-strongly nowhere dense set in X, then A is μ-codense (that is, $X - A \in \mathcal{D}(\mu)$) in X.*

The following Example 3.9 shows that the converse of the above Theorem 3.8 is not be true and the Corollary 3.10 below follows from Lemma 2.3 and Theorem 3.8.

Example 3.9. Consider the GTS (X_4, μ) where $\mu = \{\emptyset, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Let $A = \{a, d\}$. Then A is a μ-codense set. If $V = \{a, c\}$, then there is no $U \in \tilde{\mu}$ such that $U \subset V$ and $U \cap A = \emptyset$. Thus, A is not a μ-strongly nowhere dense set in X.

Corollary 3.10. *Let (X, μ) be a GTS. If $\mathcal{A}$ is a finite family of μ-strongly nowhere dense set, then $\bigcup \mathcal{A}$ is μ-codense.*

The following Theorem 3.11 follows from the definition of $\mu^\star$ and Lemma 2.1. Observe that for a sBS (X, μ), $\mu^{\star\star}$ is closed under finite intersections. Hence we have the Theorem 3.13 and Theorem 3.14 below follows from Theorem 3.13.

Theorem 3.11. *Let (X, μ) be a GTS. Then the following hold.*

(a) *$A \subset X$ is a $\mu^\star$-nowhere dense set if and only if A is a $\mu^\star$-strongly nowhere dense set.*

(b) *$A \subset X$ is $\mu^\star$-meager if and only if it is $\mu^\star$-s-meager.*

(c) *$A \subset X$ is $\mu^\star$-residual if and only if it is $\mu^\star$-s-residual.*

(d) *$A \subset X$ is of $\mu^\star$-II category if and only if it is of $\mu^\star$-s-II category.*

Theorem 3.12. *Let (X, μ) be a GTS. Then the following are equivalent.*

(a) *$(X, \mu^\star)$ is a sBS.*

(b) *$(X, \mu^\star)$ is a BS.*

(c) *$(X, \mu^\star)$ is a wBS.*

Proof. It is enough to prove (c) $\Rightarrow$ (a). Let $V_1, V_2, ..., V_n \in \mu^\star$ such that $V_1 \cap V_2 \cap ... \cap V_n \neq \emptyset$. By the definition of $\mu^\star$, $V_1 \cap V_2 \cap ... \cap V_n \in \tilde{\mu^\star}$. Then $V_1 \cap V_2 \cap ... \cap V_n$ is of $\mu^\star$-s-II category. By Theorem 3.11, $V_1 \cap V_2 \cap ... \cap V_n$ is of $\mu^\star$-II category. Therefore, $(X, \mu^\star)$ is a sBS. □

Theorem 3.13. *Let (X, μ) be a sBS. Then the following hold.*

(a) *$A \subset X$ is a $\mu^{\star\star}$-nowhere dense set if and only if A is a $\mu^{\star\star}$-strongly nowhere dense set.*

(b) *$A \subset X$ is $\mu^{\star\star}$-meager if and only if it is $\mu^{\star\star}$-s-meager.*

(c) *$A \subset X$ is $\mu^{\star\star}$-residual if and only if it is $\mu^{\star\star}$-s-residual.*

(d) *$A \subset X$ is of $\mu^{\star\star}$-II category if and only if it is of $\mu^{\star\star}$-s-II category.*

Theorem 3.14. *Let (X, μ) be a sBS. Then the following are equivalent.*

(a) *$(X, \mu^{\star\star})$ is a sBS.*

(b) *$(X, \mu^{\star\star})$ is a BS.*

(c) *$(X, \mu^{\star\star})$ is a wBS.*

4. Images of Nowhere Dense Sets in GTSs

In this section, we discuss the behavior of nowhere dense sets in GTS under feebly (μ, λ)-continuous, feebly (μ, λ)-open and (μ, λ)-open mappings. Let (X, μ) and (Y, λ) be two GTSs. A function $f : (X, \mu) \to (Y, \lambda)$ is called *feebly (μ, λ)-continuous* [4] if $i_\mu(f^{-1}(B)) \neq \emptyset$ for every $B \subset Y$ with $i_\lambda(B) \neq \emptyset$. A function $f : (X, \mu) \to (Y, \lambda)$ is called *(μ, λ)-open* (resp. *feebly (μ, λ)-open*) if $f(U) \in \lambda$ for each $U \in \mu$ (resp. $i_\lambda(f(U)) \neq \emptyset$ for each $U \in \tilde{\mu}$) [3].

Theorem 4.1. *Let (X, μ) and (Y, λ) be two GTSs and $f : (X, \mu) \to (Y, \lambda)$ be a injective, (μ, λ)-open and feebly (μ, λ)-continuous mapping. If $A \in \mathcal{N}(\mu)$, then $f(A) \in \mathcal{N}(\lambda)$.*

Proof. Suppose $A \in \mathcal{N}(\mu)$. Then $X - c_\mu(A)$ is μ-dense and μ-open in X, by Lemma 2.4. Suppose $Y - c_\lambda(f(A))$ is not λ-dense in Y. Then there is a set $U \in \tilde{\lambda}$ such that $U \cap (Y - c_\lambda(f(A))) = \emptyset$. This implies that $U \subset Y - (Y - c_\lambda(f(A)))$ which implies that $U \subset c_\lambda(f(A))$. Suppose $x \notin f(c_\mu(A))$. Then $f^{-1}(x) \notin c_\mu(A)$. This implies that there exists a set $V \in \mu$ containing $f^{-1}(x)$ such that $V \cap A = \emptyset$ and so $f(V \cap A) = \emptyset$. Since f is one-one, $f(A) \cap f(V) = \emptyset$. Therefore, $x \notin c_\lambda(f(A))$, since f is (μ, λ)-open and so $c_\lambda(f(A)) \subset f(c_\mu(A))$. Hence $U \subset f(c_\mu(A))$. By hypothesis, $f^{-1}(U) \subset f^{-1}f(c_\mu(A)) = c_\mu A$. Take $G = i_\mu(f^{-1}(U))$. Then $G \neq \emptyset$ and $G \subset c_\mu(A)$, since f is feebly (μ, λ)-continuous. This implies that $G \cap (X - c_\mu(A)) = \emptyset$, which is a contradiction to $X - c_\mu(A)$ is μ-dense. Therefore, $Y - c_\lambda(f(A))$ is λ-dense in Y. Thus, $i_\lambda c_\lambda(f(A)) = \emptyset$. Hence $f(A) \in \mathcal{N}(\lambda)$. □

The following Example 4.2 shows that the conditions injective, (μ, λ)-open and feebly (μ, λ)-continuous on f cannot be dropped in Theorem 4.1.

Example 4.2.

(a) Consider the two GTSs (X_5, μ) and (X_4, λ) where $\mu = \{\emptyset, \{e\}, \{a, b\}, \{b, c\}, \{a, b, e\}, \{a, b, c\}, \{b, c, e\}, \{a, b, c, e\}\}$ and $\lambda = \{\emptyset, \{b\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$. Define a function $f : (X_5, \mu) \to (X_4, \lambda)$ by $f(a) = d, f(b) = c, f(c) = d, f(d) = a, f(e) = b$. Then f is (μ, λ)-open but not injective. Clearly, f is a feebly (μ, λ)-continuous mapping. Let $A = \{a\} \subset X_5$. Then $i_\mu c_\mu(A) = i_\mu(\{a, d\}) = \emptyset$ and so $A \in \mathcal{N}(\mu)$. But $i_\lambda(c_\lambda f(A)) = i_\lambda(\{a, c, d\}) = \{c, d\} \neq \emptyset$. Therefore, $f(A) \notin \mathcal{N}(\lambda)$.

(b) Consider the set X_4 with two generalized topologies μ, λ where $\mu = \{\emptyset, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ and $\lambda = \{\emptyset, \{a\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X_4\}$. Define a function $f : (X_4, \mu) \to (X_4, \lambda)$ by $f(a) = b$, $f(b) = c, f(c) = d, f(d) = a$. Then f is an injective and (μ, λ)-open mapping. Let $A = \{a\} \subset X_4$. Then $i_\lambda A \neq \emptyset$ and $i_\mu f^{-1}(A) = i_\mu(\{d\}) = \emptyset$. Therefore, f is not a feebly (μ, λ)-continuous mapping. Let $B = \{d\} \subset X_4$. Then $i_\mu c_\mu(B) = i_\mu(\{d\}) = \emptyset$ and so $B \in \mathcal{N}(\mu)$. But $i_\lambda c_\lambda(f(B)) = i_\lambda(\{a\}) = \{a\} \neq \emptyset$. Therefore, $f(B) \notin \mathcal{N}(\lambda)$.

(c) Consider the set X_4 with two generalized topologies μ, λ where $\mu = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and $\lambda = \{\emptyset, \{a, d\}, \{c, d\}, \{a, c, d\}\}$. Define a function $f : (X_4, \mu) \to (X_4, \lambda)$ by $f(a) = d, f(b) = c, f(c) = b, f(d) = a$. Clearly, f is injective. Since each $A \subset X_4$ with $i_\lambda A \neq \emptyset$, $i_\mu f^{-1}(A) \neq \emptyset$, f is a feebly (μ, λ)-continuous mapping. But f is not a (μ, λ)-open. For, let $A = \{c\} \in \tilde{\mu}$. Then $f(A) = \{b\} \notin \tilde{\lambda}$. Let $B = \{b, d\} \subset X_4$. Then $i_\mu c_\mu(B) = i_\mu(\{b, d\}) = \emptyset$ and so $B \in \mathcal{N}(\mu)$. But $i_\lambda c_\lambda(f(B)) = i_\lambda(\{X_4\}) = \{a, c, d\} \neq \emptyset$. Therefore, $f(B) \notin \mathcal{N}(\lambda)$.

Corollary 4.3. *Let (X, μ) and (Y, λ) be two GTSs and $f : (X, \mu) \to (Y, \lambda)$ be a (μ, λ)-open, feebly (μ, λ)-continuous and injective mapping. Then the following hold.*

(a) *If $A \subset X$ is of μ-meager, then $f(A)$ is of λ-meager.*

(b) *If f is surjective and $B \subset X$ is μ-residual in X, then $f(B)$ is λ-residual in Y.*

(c) *If f is surjective and $C \subset Y$ is of λ-second category, then $f^{-1}(C)$ is of μ-second category.*

(d) *If A is a μ-strongly nowhere dense set, then $f(A) \in \mathcal{N}(\lambda)$ where $A \subset X$.*

Proof. (a) Let A be a μ-meager set in X. Then $A = \bigcup_{n \in \mathbb{N}} A_n$ where each A_n is μ-nowhere dense set in X. Then by hypothesis and Theorem 4.1, each $f(A_n)$ is

λ-nowhere dense set in Y. Thus, $f(A) = f(\bigcup_{n\in\mathbb{N}} A_n) = \bigcup_{n\in\mathbb{N}} f(A_n)$. Therefore, $f(A)$ is λ-meager in Y.

(b) Let B be a μ-residual set in X. Then $X - B$ is of μ-meager set in X. By (a), $f(X - B)$ is of λ-meager in Y. Now $f(X - B) = f(X) - f(B) = Y - f(B)$, since f is bijective map. Thus, $Y - f(B)$ is of λ-meager set in Y. Therefore, $f(B)$ is λ-residual in Y.

(c) Let $C \subset Y$ be a λ-second category set in Y. Suppose $f^{-1}(C)$ is of μ-meager set in X. By (a), $f(f^{-1}(C))$ is of λ-meager in Y. Since f is surjective, $f(f^{-1}(C)) = C$ and so C is of λ-meager in Y, which is not possible. Therefore, $f^{-1}(C)$ is of μ-second category in X.

(d) The proof follows from Theorem 4.1 and the fact that every μ-strongly nowhere dense set is μ-nowhere dense set. $\square$

Theorem 4.4. *Let (X, μ) and (Y, λ) be two GTSs. Let $f : (X, \mu) \to (Y, \lambda)$ be a feebly (μ, λ)-continuous, injective mapping. If A is μ-codense, then $f(A)$ is λ-codense.*

Proof. Suppose $f(A)$ is not a λ-codense set in Y. Then there is $G \in \tilde{\lambda}$ such that $G \subset f(A)$ which implies $f^{-1}(G) \subset f^{-1}(f(A)) = A$, since f is injective. Thus, $i_\mu(f^{-1}(G)) \subset i_\mu(A)$. Since f is a feebly (μ, λ)-continuous, $i_\mu(f^{-1}(G)) \neq \emptyset$. This implies that $i_\mu A \neq \emptyset$ and hence A is not μ-codense. This completes the proof. $\square$

The following Example 4.5 shows that the conditions injective, feebly (μ, λ)-continuous on f cannot be dropped in Theorem 4.4.

Example 4.5.

(a) Consider the two GTSs (X_4, μ) and (X_3, λ) where $\mu = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ and $\lambda = \{\emptyset, \{a, b\}, \{a, c\}, \{b, c\}, X_3\}$. Define a function $f : (X_4, \mu) \to (X_3, \lambda)$ by $f(a) = c, f(b) = a, f(c) = b, f(d) = c$. Clearly, f is a feebly (μ, λ)-continuous but not injective mapping. Let $A = \{c, d\}$. Then $i_\mu A = \emptyset$ and hence A is μ-codense. But $c_\lambda(X_3 - f(A)) = c_\lambda(\{a\}) = \{a\} \neq X_3$. Hence $f(A)$ is not λ-codense.

(b) Consider the set X_4 with two generalized topologies μ, λ where $\mu = \{\emptyset, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}$ and $\lambda = \{\emptyset, \{a\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define a function $f : (X_4, \mu) \to (X_4, \lambda)$ by $f(a) = d, f(b) = c, f(c) = b, f(d) = a$. Then f is injective. Let $B = \{a, b\}$ with $i_\lambda B \neq \emptyset$. But $i_\mu(f^{-1}(B)) = i_\mu(\{c, d\}) = \emptyset$. Thus, f is not a feebly (μ, λ)-continuous mapping on X_4. Let $A = \{c, d\}$. Then A is a μ-codense set in X_4. But $c_\lambda(X_4 - f(A)) = c_\lambda(\{c, d\}) = \{b, c, d\} \neq X_4$. Hence $f(A)$ is not λ-codense.

Theorem 4.6. *Let (X, μ) and (Y, λ) be two GTSs. Let $f : (X, \mu) \to (Y, \lambda)$ be a feebly (μ, λ)-open mapping. If A is λ-codense, then $f^{-1}(A)$ is μ-codense.*

Proof. Suppose $f^{-1}(A)$ is not a μ-codense set in X. Then there is $G \in \tilde{\mu}$ such that $G \subset f^{-1}(A)$ which implies $f(G) \subset f(f^{-1}(A)) \subset A$. Thus, $i_\lambda(f(G)) \subset i_\lambda(A)$. Since

f is a feebly (μ, λ)-open, $i_\lambda(f(G)) \neq \emptyset$. This implies that $i_\lambda(A) \neq \emptyset$ and hence A is not λ-codense. This completes the proof. □

The following Example 4.7 shows that the condition feebly (μ, λ)-open on f cannot be dropped in Theorem 4.6.

Example 4.7. Consider the two GTSs (X_4, μ) and (X_3, λ) where $\mu = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ and $\lambda = \{\emptyset, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define a function $f : (X_4, \mu) \to (X_3, \lambda)$ by $f(a) = c, f(b) = a, f(c) = b, f(d) = c$. Clearly, f is not a feebly (μ, λ)-open mapping. Let $A = \{c\} \subset X_3$. Then $i_\lambda A = \emptyset$ and hence A is λ-codense. But $c_\mu(X_4 - f^{-1}(A)) = c_\mu(\{b, c\}) = \{b, c, d\} \neq X_4$. Hence $f^{-1}(A)$ is not μ-codense.

Theorem 4.8. *Let (X, μ) and (Y, λ) be two GTSs and $f : X \to Y$ be a feebly (μ, λ)-continuous, feebly (μ, λ)-open, injective mapping. Then A is a μ-strongly nowhere dense set in X if and only if $f(A)$ is a λ-strongly nowhere dense set in Y.*

Proof. Suppose A is a μ-strongly nowhere dense set in X. Let $U \in \tilde{\lambda}$. Then $i_\mu(f^{-1}(U)) \in \tilde{\mu}$. By hypothesis, there exists $G \in \tilde{\mu}$ such that $G \subset i_\mu(f^{-1}(U))$ and $G \cap A = \emptyset$. This implies that $i_\lambda(f(G)) \in \tilde{\lambda}$ and $i_\lambda(f(G)) \subset f(i_\mu(f^{-1}(U))) \subset U$ and $i_\lambda(f(G)) \cap f(A) = \emptyset$. Therefore, $f(A)$ is a λ-strongly nowhere dense set in Y. Conversely, let $G \in \tilde{\mu}$. Then $i_\lambda(f(G)) \in \tilde{\lambda}$. By hypothesis, there exists $W \in \tilde{\lambda}$ such that $W \subset i_\lambda(f(G))$ and $W \cap f(A) = \emptyset$. This implies that $i_\mu(f^{-1}(W)) \in \tilde{\mu}$ and $i_\mu(f^{-1}(W)) \subset f^{-1}(i_\lambda(f(G))) = G$ and $i_\mu(f^{-1}(W)) \cap A = \emptyset$. Therefore, A is a μ-strongly nowhere dense set in X. □

The following Example 4.9 shows that the conditions feebly (μ, λ)-continuous, feebly (μ, λ)-open, injective on f cannot be dropped in Theorem 4.8. The easy proof of the Corollary 4.10 below is omitted.

Example 4.9.

(a) Consider the two GTSs (X_4, μ) and (X_3, λ) where $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, d\}, \{a, b, d\}\}$ and $\lambda = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Define a function $f : (X_4, \mu) \to (X_3, \lambda)$ by $f(a) = a, f(b) = c, f(c) = b, f(d) = c$. Clearly, f is a feebly (μ, λ)-continuous, feebly (μ, λ)-open mapping. But f is not injective. Let $B = \{c, d\} \subset X_4$. Then B is a μ-strongly nowhere dense set in X_4. If $V = \{c\} \in \tilde{\lambda}$, then there is no $U \in \tilde{\lambda}$ such that $U \subset V$ and $U \cap f(B) = \emptyset$. Therefore, $f(B)$ is not a λ-strongly nowhere dense set in X_3.

(b) Consider the two GTSs (X_4, μ) and (X_3, λ) where $\mu = \{\emptyset, \{c, d\}, \{b, d\}, \{b, c, d\}\}$ and $\lambda = \{\emptyset, \{a\}, \{a, b\}\}$. Define a function $f : (X_4, \mu) \to (X_3, \lambda)$ by $f(a) = c, f(b) = b, f(c) = a, f(d) = a$. Then f is a feebly (μ, λ)-continuous, feebly (μ, λ)-open mapping. But f is not injective. Let $B = \{b\} \subset X_3$. Then B is a λ-strongly nowhere dense set in X_3. If $V = \{b, d\} \in \tilde{\mu}$, then there is no $U \in \tilde{\mu}$ such that $U \subset V$ and $U \cap f^{-1}(B) = \emptyset$. Therefore, $f^{-1}(B)$ is not a μ-strongly nowhere dense set in X_4.

(c) Consider the set X_4 with two generalized topologies μ, λ where $\mu = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$ and $\lambda = \{\emptyset, \{a\}, \{c\}, \{d\}, \{a,c\}, \{a,d\}, \{c,d\}, \{a,c,d\}\}$. Define a function $f : (X_4, \mu) \to (X_4, \lambda)$ by $f(a) = c, f(b) = a, f(c) = d, f(d) = b$. Then f is an injective and feebly (μ, λ)-open mapping. If $A = \{d\} \in \tilde{\lambda}$, then $i_\mu(f^{-1}(A)) = \emptyset$. Therefore, f is not feebly (μ, λ)-continuous. Let $B = \{c, d\} \subset X_4$. Then B is a μ-strongly nowhere dense set in X_4. If $V = \{d\} \in \tilde{\lambda}$, then there is no $U \in \tilde{\lambda}$ such that $U \subset V$ and $U \cap f(B) = \emptyset$. Therefore, $f(B)$ is not a λ-strongly nowhere dense set in X_4.

(d) Consider the set X_4 with two generalized topologies μ, λ where $\mu = \{\emptyset, \{a\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$ and $\lambda = \{\emptyset, \{d\}, \{c\}, \{a,d\}, \{c,d\}, \{a,c,d\}\}$. Define a function $f : (X_4, \mu) \to (X_4, \lambda)$ by $f(a) = c, f(b) = a, f(c) = d, f(d) = b$. Then f is an injective and feebly (μ, λ)-open mapping. If $A = \{d\} \in \tilde{\lambda}$, then $i_\mu(f^{-1}(A)) = \emptyset$. Therefore, f is not feebly (μ, λ)-continuous. Let $B = \{a\} \subset X_4$. Then B is a λ-strongly nowhere dense set in X_4. If $V = \{b, c\} \in \tilde{\mu}$, then there is no $U \in \tilde{\mu}$ such that $U \subset V$ and $U \cap f^{-1}(B) = \emptyset$. Therefore, $f^{-1}(B)$ is not a μ-strongly nowhere dense set in X_4.

(e) Consider the set X_4 with two generalized topologies μ, λ where $\mu = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$ and $\lambda = \{\emptyset, \{a\}, \{a,d\}, \{c,d\}, \{a,c,d\}\}$. Define a function $f : (X_4, \mu) \to (X_4, \lambda)$ by $f(a) = c, f(b) = a, f(c) = d, f(d) = b$. Then f is an injective and feebly (μ, λ)-continuous mapping. If $A = \{a\} \in \tilde{\mu}$, then $i_\lambda(f(A)) = \emptyset$. Therefore, f is not feebly (μ, λ)-open. Let $B = \{c, d\} \subset X_4$. Then B is a μ-strongly nowhere dense set in X_4. If $V = \{c, d\} \in \tilde{\lambda}$, then there is no $U \in \tilde{\lambda}$ such that $U \subset V$ and $U \cap f(B) = \emptyset$. Therefore, $f(B)$ is not a λ-strongly nowhere dense set in X_4.

(f) Consider the set X_4 with two generalized topologies μ, λ where $\mu = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$ and $\lambda = \{\emptyset, \{c\}, \{c,d\}, \{b,c,d\}\}$. Define a function $f : (X_4, \mu) \to (X_4, \lambda)$ by $f(a) = c, f(b) = d, f(c) = b, f(d) = a$. Then f is an injective and feebly (μ, λ)-continuous mapping. If $A = \{b\} \in \tilde{\mu}$, then $i_\lambda(f(A)) = \emptyset$. Therefore, f is not feebly (μ, λ)-open. Let $B = \{d\} \subset X_4$. Then B is a λ-strongly nowhere dense set in X_4. If $V = \{b\} \in \tilde{\mu}$, then there is no $U \in \tilde{\mu}$ such that $U \subset V$ and $U \cap f^{-1}(B) = \emptyset$. Therefore, $f^{-1}(B)$ is not a μ-strongly nowhere dense set in X_4.

Corollary 4.10. *Let (X, μ) and (Y, λ) be two GTSs and $f : (X, \mu) \to (Y, \lambda)$ be a bijective, (μ, λ)-continuous and (μ, λ)-open. Then the following hold.*

(a) *$A \subset X$ is of μ-s-meager in X if and only if $f(A)$ is of λ-s-meager in Y.*

(b) *$B \subset X$ is μ-s-residual in X if and only if $f(B)$ is λ-s-residual in Y.*

(c) *$C \subset Y$ is of λ-s-II category in Y if and only if $f^{-1}(C)$ is of μ-s-II category in X.*

5. Weak Baire Spaces

In this section, we discuss the properties of weak Baire spaces in generalized topological spaces. Also, we characterize weak Baire spaces in terms of dense and codense subsets. The following Theorem 5.1 characterizes weak Baire spaces in GTSs.

Theorem 5.1. *Let (X, μ) be a GTS. Then the following are equivalent.*

(a) *X is wBS.*

(b) *If A is μ-s-residual in X, then $A \in \mathcal{D}(\mu)$.*

(c) *If B is μ-s-meager in X, then B is μ-codense.*

Proof. (a) $\Rightarrow$ (b). Let A be a μ-s-residual set in X. Then $X - A$ is a μ-s-meager set in X and so $X - A = \bigcup\limits_{n\in\mathbb{N}} A_n$ where each A_n is a μ-strongly nowhere dense set. Let $G \in \tilde{\mu}$. Suppose $A \cap G = \emptyset$. Then $G \subset X - A$ and so G is a μ-s-meager set in X, by Theorem 3.1 which is a contradiction to X is wBS. Therefore, $A \cap G \neq \emptyset$. Hence $A \in \mathcal{D}(\mu)$.
(b) $\Rightarrow$ (c). Suppose B is μ-s-meager in X. Then $X - B$ is a μ-s-residual set in X and so $X - B \in \mathcal{D}(\mu)$, by (b). Therefore, B is μ-codense.
(c) $\Rightarrow$ (a). Suppose that there exists a nonempty μ-open subset U which is a μ-s-meager set in X. By (c), U is a μ-codense set in X. Then $X - U$ is μ-dense which implies that $(X - U) \cap U \neq \emptyset$ which is not possible. Therefore, U is of μ-s-II category. Hence X is a wBS. $\square$

Theorem 5.2. *Let (X, μ) be a wBS. Then the following hold and also they are equivalent.*

(a) *If $\bigcap\limits_{n\in\mathbb{N}} G_n \neq \emptyset$ is μ-s-residual set where $\{G_n : n \in \mathbb{N}\}$ is a family of subsets of X, then $G_n \in \mathcal{D}(\mu)$ for each $n \in \mathbb{N}$.*

(b) *If $\bigcup\limits_{n\in\mathbb{N}} F_n \neq X$ is μ-s-meager set where $\{F_n : n \in \mathbb{N}\}$ is a family of subsets of X, then F_n is μ-codense for each $n \in \mathbb{N}$.*

Proof. (a). Suppose $\bigcap\limits_{n\in\mathbb{N}} G_n \neq \emptyset$ is μ-s-residual set where $\{G_n : n \in \mathbb{N}\}$ is a family of subsets of X. Then by Corollary 3.2 (b), each G_n is a μ-s-residual set. By Theorem 5.1, $G_n \in \mathcal{D}(\mu)$ for each $n \in \mathbb{N}$.
(b). Suppose $\bigcup\limits_{n\in\mathbb{N}} F_n \neq X$ is μ-s-meager set where $\{F_n : n \in \mathbb{N}\}$ is a family of subsets of X. By Theorem 3.1, each F_n is a μ-s-meager set. Then $X - F_n$ is a μ-s-residual set for each $n \in \mathbb{N}$. By Theorem 5.1, $X - F_n \in \mathcal{D}(\mu)$ for each $n \in \mathbb{N}$. Therefore, F_n is μ-codense for each $n \in \mathbb{N}$.
(a) $\Rightarrow$ (b). Suppose $\bigcup\limits_{n\in\mathbb{N}} F_n \neq X$ is μ-s-meager set where $\{F_n : n \in \mathbb{N}\}$ is a family of subsets of X. Then $X - \bigcup\limits_{n\in\mathbb{N}} F_n \neq \emptyset$ and so $\bigcap\limits_{n\in\mathbb{N}} (X - F_n) \neq \emptyset$. Since each F_n

is a μ-s-meager set, $X - F_n$ is a μ-s-residual set for each $n \in \mathbb{N}$. By Theorem3.3, $\bigcap_{n\in\mathbb{N}} (X - F_n)$ is a μ-s-residual set. By (a), $X - F_n \in \mathcal{D}(\mu)$ for each $n \in \mathbb{N}$. Therefore, F_n is μ-codense for each $n \in \mathbb{N}$.
(b) $\Rightarrow$ (a). Suppose $\bigcap_{n\in\mathbb{N}} G_n \neq \emptyset$ is μ-s-residual set where $\{G_n : n \in \mathbb{N}\}$ is a family of subsets of X. Then $X - \bigcap_{n\in\mathbb{N}} G_n \neq X$ and so $\bigcup_{n\in\mathbb{N}} (X - G_n) \neq X$. Since each G_n is a μ-s-residual set, $X - G_n$ is a μ-s-meager set for each $n \in \mathbb{N}$. By Theorem 3.3, $\bigcup_{n\in\mathbb{N}} (X - G_n) \neq X$ is a μ-s-meager set. By (b), each $X - G_n$ is μ-codense and so $G_n \in \mathcal{D}(\mu)$ for each $n \in \mathbb{N}$. □

Theorem 5.3. *Let (X, μ) be a wBS and (Y, λ) be a GTS. Let $f : (X, \mu) \to (Y, \lambda)$ be a feebly (μ, λ)-continuous, injective mapping. Then the following hold.*

(a) *If A is a μ-s-meager set in X, then $f(A)$ is a λ-codense set in Y.*

(b) *If f is a surjective mapping and B is a μ-s-residual set in X, then $f(B)$ is a λ-dense set in Y.*

Proof. (a) Follows from Theorem 4.4 and Theorem 5.1.
(b) Suppose B is a μ-s-residual set in X. Then by (a), $f(X - B)$ is a λ-codense set in Y. This implies $Y - f(X - B)$ is a λ-dense set in Y which implies $f(B)$ is a λ-dense set in Y. □

Theorem 5.4. *Let (X, μ) and (Y, λ) be two GTSs and $f : X \to Y$ be an feebly (μ, λ)-open mapping. If Y is a wBS, then the following hold.*

(a) *If A is λ-s-meager in Y, then $f^{-1}(A)$ is μ-codense in X.*

(b) *If A is λ-s-residual in Y, then $f^{-1}(A) \in \mathcal{D}(\mu)$.*

Proof. (a) Suppose A is λ-s-meager in Y. By Theorem 5.1, A is λ-codense. By Theorem 4.6, $f^{-1}(A)$ is μ-codense in X.
(b) Suppose A is λ-s-residual in Y. Then $Y - A$ is λ-s-meager in Y. By (a), $f^{-1}(Y - A)$ is μ-codense in X. Now $f^{-1}(Y - A) = f^{-1}(Y) - f^{-1}(A) = X - f^{-1}(A)$. Therefore, $f^{-1}(A)$ is dense in X. □

Theorem 5.5. *Let (X, μ) be a GTS. If every μ-dense subset of X is a wBS, then X is a wBS.*

Proof. Let A be a μ-dense subset of X. Suppose that there exists a nonempty μ-open subset U is a μ-s-meager set in X. This implies $U \cap A$ is a μ-s-meager set in X. Then by Theorem 3.6, $U \cap A$ is a μ_A-s-meager set in A. Since A is μ-dense and $U \in \tilde{\mu}$, $U \cap A \in \tilde{\mu}_A$. Thus, $U \cap A$ is a nonempty μ_A-open, μ_A-s-meager set in A and so A is not a wBS, which is a contradiction to our hypothesis. Therefore, U is of μ-s-II category set. Hence X is a wBS. □

Theorem 5.6. *Let (X, μ) be the GT-sum of $\{(X_\alpha, \mu_\alpha) : \alpha \in \Gamma\}$ and X be a wBS. If A is μ-s-residual, then $A \cap X_\alpha \in \mathcal{D}(\mu_\alpha)$ for any $\alpha \in \Gamma$.*

Proof. Suppose A is a μ-s-residual set in X. By Theorem 5.1, A is a μ-dense set in X. By hypothesis and Lemma 2.5, $A \cap X_\alpha \in \mathcal{D}(\mu_\alpha)$ for any $\alpha \in \Gamma$. □

Theorem 5.7. *Let (X, μ) and (Y, λ) be two GTSs and $f : X \to Y$ be an feebly (μ, λ)-open, feebly (μ, λ)-continuous, injective mapping. Then (X, μ) is a wBS if and only if (Y, λ) is a wBS.*

Proof. Suppose X is a wBS. Let $V \in \tilde{\lambda}$. Suppose V is a λ-s-meager set. Then $V = \bigcup_{n\in\mathbb{N}} V_n$ where each V_n is a λ-strongly nowhere dense set. By hypothesis and Theorem 4.8, $f^{-1}(V_n)$ is a μ-strongly nowhere dense set in X for each $n \in \mathbb{N}$. Therefore, $f^{-1}(V)$ is a μ-s-meager set which implies $i_\mu f^{-1}(V)$ is a nonempty μ-open, μ-s-meager set in X, since f is feebly (μ, λ)-continuous. Therefore, X is not a wBS, which is not possible. Hence Y is a wBS. Conversely, assume that Y is a wBS. Let $U \in \tilde{\mu}$. Suppose U is a μ-s-meager set in X. Then $U = \bigcup_{n\in\mathbb{N}} U_n$ where each U_n is a μ-strongly nowhere dense set in X. By hypothesis and Theorem 4.8, $f(U_n)$ is a λ-strongly nowhere dense set in Y for each $n \in \mathbb{N}$. Therefore, $f(V)$ is a λ-s-meager set which implies $i_\lambda f(V)$ is a nonempty λ-open, λ-s-meager set in Y, since f is feebly (μ, λ)-open. Therefore, Y is not a wBS, which is not possible. Hence X is a wBS. □

The following Corollary 5.8 follows from Theorem 5.5 and Theorem 5.7.

Corollary 5.8. *Let (X, μ) and (Y, λ) be two GTSs and $f : X \to Y$ be an feebly (μ, λ)-open, feebly (μ, λ)-continuous, injective mapping. Then the following hold.*

(a) *If every μ-dense subset of X is a wBS, then Y is a wBS.*

(b) *If every λ-dense subset of Y is a wBS, then X is a wBS.*

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