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2-Absorbing δ-primary Ideals in Commutative Rings

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Abstract. In this paper we investigate 2-absorbing δ-primary ideals which unify 2-absorbing ideals and 2-absorbing primary ideals. A number of results about 2-absorbing ideals and 2-absorbing primary ideals are extended into this general framework.

1. Introduction

We assume throughout this paper that all rings are commutative with $1 \neq 0$ and all ring homomorphisms preserve the identity. Let R be a commutative ring. We use $Id(R)$ to denote the set of all ideals of the ring R.

The notion of 2-absorbing (resp.2-absorbing primary) ideal, which is a generalization of prime (primary) ideal, was introduced by Badawi (Badawi and all) in [3] (resp.[4]) and further studied by several authors (see for instance, [1], [2], [5] and [6]). In [7], Zhao introduced and investigated the notions of expansion ideal and δ-primary ideal (for more detail see [7]). In this short paper we introduce the notion of 2-absorbing δ-primary ideal where δ is a mapping that assigns to each ideal I an ideal $\delta(I)$ of the same ring. Such 2-absorbing δ-primary ideals unify the 2-absorbing and 2-absorbing primary ideals under one frame. This approach clearly reveals how similar the two structures are and how they are related to each other. The paper is organized as follows. In the second section, we define and introduce

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2-absorbing δ-primary ideals with respect to an expansion ideal function δ. In the third section, we state more results on 2-absorbing δ-primary ideals.

2. Properties of 2-Absorbing δ-primary Ideals

Recall that a proper ideal I of R is called a 2-*absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. For more properties of 2-absorbing ideal see [3]. A proper ideal I of R is called a 2-*absorbing primary ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. See [4] for more properties of 2-absorbing primary ideal.

An expansion of ideals, or briefly an ideal expansion, is a function δ which assigns to each ideal I of a ring R to another ideal $\delta(I)$ of the same ring, such that the following conditions are satisfied:

$$I \subseteq \delta(I), \text{ and } I \subseteq J \text{ implies } \delta(I) \subseteq \delta(J).$$

For more details on expansions of ideals see [7].

Example 2.1.

(1) The identity function δ_0, where $\delta_0(I) = I$ for every $I \in Id(R)$, is an expansion of ideals [7].

(2) For each ideal I define $\delta_1(I) = \sqrt{I}$, the radical I. Then δ_1 is an expansion of ideals [7].

(3) For each ideal P, let $\delta_M(P)$ be the intersection of all maximal ideals containing P when P is a proper ideal of the ring R, and $\delta_M(R) = R$. Then δ_M is an expansion of ideals [7].

(4) Let Ψ be a collection of ideals of R. For each ideal P of R, define

$$\delta_\Psi(P) = \bigcap\{J \in \Psi : P \subseteq J\}.$$

Then δ_Ψ is an expansion of ideals [7].

Definition 2.2. Given an expansion δ of ideals, a proper ideal I of R is called a 2-*absorbing δ-primary ideal* of R if whenever $a, b, c \in R$, $abc \in I$, $ab \notin I$ and $ac \notin \delta(I)$, then $bc \in \delta(I)$.

Remark 2.3. Let I be an ideal of R.

(1) I is 2-absorbing δ_0-primary if and only if it is 2-absorbing.

(2) I is 2-absorbing δ_1-primary if and only if it is 2-absorbing primary.

(3) If δ and γ are two ideal expansions, and $\delta(I) \subseteq \gamma(I)$ for each ideal I, then every 2-absorbing δ-primary ideal is also 2-absorbing γ-primary. In particular, a 2-absorbing ideal is 2-absorbing δ-primary for every δ.

Theorem 2.4. *If I is a 2-absorbing δ-primary ideal of R such that $\sqrt{\delta(I)} = \delta(\sqrt{I})$, then $\sqrt{I}$ is a 2-absorbing δ-primary ideal of R.*

Proof. Let $a, b, c \in R$ such that $abc \in \sqrt{I}$, $ac \notin \sqrt{I}$ and $bc \notin \delta(\sqrt{I})$. Since $abc \in \sqrt{I}$, there exists a positive integer n such that $(abc)^n \in I$. Now $ac \notin \sqrt{I}$ and $bc \notin \delta(\sqrt{I}) = \sqrt{\delta(I)}$ imply $a^n c^n \notin I$ and $b^n c^n \notin \delta(I)$, hence $a^n b^n \in \delta(I)$, that is $ab \in \sqrt{\delta(I)} = \delta(\sqrt{I})$. Thus $\sqrt{I}$ is a 2-absorbing δ-primary ideal of R. □

Proposition 2.5. *Let $\{J_i : i \in D\}$ be a directed collection of 2-absorbing δ-primary ideals of R. Then the ideal $J = \bigcup_{i \in D} J_i$ is a 2-absorbing δ-primary ideal of R.*

Proof. Let $abc \in J$ and $ab \notin J$ and $ac \notin \delta(J)$. Then there is a J_i with $abc \in J_i$. In addition, $ab \notin J_i$ and $ac \notin \delta(J_i)$. So $ab \in \delta(J_i) \subseteq \delta(J)$, implying $ab \in \delta(J)$. Hence J is a 2-absorbing δ-primary ideal of R. □

Definition 2.6. Given an expansion δ of ideals, a proper ideal I of R is called a *strongly* 2*-absorbing δ-primary ideal* of R if whenever I_1, I_2, I_3 are ideals of R, $I_1 I_2 I_3 \subseteq I$, $I_1 I_3 \nsubseteq I$ and $I_2 I_3 \nsubseteq \delta(I)$, then $I_1 I_2 \subseteq \delta(I)$.

In the following result, we show that an ideal I is 2-absorbing δ-primary if and only if I is strongly 2-absorbing δ-primary ideal of R, it is an extension of [4, Theorem 2.19]. But first we have the following lemma, it is an extension of [4, Lemma 2.18]. Its proof is similar to that of [4, Lemma 2.18].

Lemma 2.7. *Let I be a 2-absorbing δ-primary ideal of a ring R and suppose that $abJ \subseteq I$ for some elements $a, b \in R$ and some ideal J of R. If $ab \notin I$, then $aJ \subseteq \delta(I)$ or $bJ \subseteq \delta(I)$.*

Theorem 2.8. *An ideal I is 2-absorbing δ-primary if and only if I is strongly 2-absorbing δ-primary ideal of R.*

Proof. The proof follows from Lemma 2.7 and a similar argument of [4, Theorem 2.19]. □

Remark 2.9. A concept more general than 2-absorbing δ-primary (resp., strongly 2-absorbing δ-primary) ideal of R is the concept of n-absorbing δ-primary (resp., strongly n-absorbing δ-primary) ideal of R, where n is a positive integer. Here we shall just state the definition of n-absorbing δ-primary (resp. strongly n-absorbing δ-primary) ideal of R.

Definition 2.10. Given an expansion δ of ideals and n is a positive integer, a proper ideal I of R is called a *n-absorbing δ-primary* (resp., *strongly n-absorbing δ-primary*) *ideal* of R if whenever $a_1, a_2, \cdots, a_{n+1} \in R$ and $a_1 a_2 \cdots a_{n+1} \in I$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ for ideal $I_1, \cdots, I_{n+1}$ of R), then there are n of the a_i's (resp. n of the I_i's) where product is in I.

The strongly n-absorbing δ-primary ideals are defined in a similar manner as strongly 2-absorbing δ-primary ideals. Based on Theorem 2.8 and Definition 2.10, we propose the following conjecture, which is similar to the [1, Conjecture 1].

Conjecture 2.11. *Let n be a positive integer. Then a proper ideal I of a ring R is n-absorbing δ-primary if and only if I is strongly n-absorbing δ-primary.*

Recall that for two ideals P and Q of a ring R, the residual division of P and Q is defined to be the ideal $(P : Q) = \{x \in R \mid xQ \subseteq P\}$.

Theorem 2.12. *Let δ be an expansion of ideals of R and I a proper ideal of R. Then the following statements are equivalent:*

(1) *I is 2-absorbing δ-primary.*

(2) *For every elements $x, y \in R$ with $xy \notin \delta(I)$, $(I : xy) \subseteq (\delta(I) : y) \cup (I : x)$.*

Proof. (1) $\Rightarrow$ (2) suppose that $x, y \in R$ with $xy \notin \delta(I)$. Let $a \in (I : xy)$. So $axy \in I$. If $ax \in I$, then $a \in (I : x)$. Assume that $ax \notin I$. Since I is 2-absorbing δ-primary, $ay \in \delta(I)$. So $a \in (\delta(I) : y)$.
(2) $\Rightarrow$ (1) Let $xyz \in I$ such that $xy \notin I$ and $yz \notin \delta(I)$, then $x \in (I : yz)$. Since $(I : yz) \subseteq (\delta(I) : z) \cup (I : y)$, $x \in (\delta(I) : z) \cup (I : y)$. But $xy \notin I$, so $xz \in \delta(I)$. Therefore I is 2-absorbing δ-primary. □

Theorem 2.13. *If δ is an ideal expansion such that $\delta(I) \subseteq \delta_1(I)$ and $\delta(I)$ is a semiprime ideal of R for every ideal I, then for any 2-absorbing δ-primary ideal I, $\delta(I) = \delta_1(I)$.*

Proof. Let $a \in \delta_1(I)$. Then there exists k which is the least positive integer k with $a^k \in I$. If $k = 1$, then $a \in I \subseteq \delta(I)$. If $k > 1$, then $a^{k-2}aa \in I$. But $a^{k-1} \notin I$, so $a^2 \in I$. Since $\delta(I)$ is semiprime, $a \in \delta(I)$. Hence $\delta_1(I) \subseteq \delta(I)$ and $\delta_1(I) = \delta(I)$. □

3. Expansions with Extra Properties and 2-Absorbing δ-primary Ideals

In this section we investigate 2-absorbing δ-primary ideals where δ satisfy additional conditions, and prove more results with respect to such expansions. Recall from [7] that an ideal expansion δ is intersection preserving if it satisfies $\delta(I \cap J) = \delta(I) \cap \delta(J)$ for any $I, J \in Id(R)$.

Proposition 3.1. *Let δ be an intersection preserving ideal expansion. If $Q_1, \cdots, Q_n$ are 2-absorbing δ-primary ideals of R, and $P = \delta(Q_i)$ for all i, then*

$$Q = \bigcap_{i=1}^{i=n} Q_i$$

is 2-absorbing δ-primary.

Proof. If $xyz \in Q$, $xy \notin Q$ and $xz \notin \delta(Q)$, then $xy \notin Q_k$ and $xz \notin \delta(Q_j)$, for some k and j. As $\delta(Q_j) = \delta(Q_k)$, we have that $xy \notin Q_k$ and $xz \notin \delta(Q_k)$, therefore $yz \in \delta(Q_k)$. But $\delta(Q) = \delta(\cap_{i=1}^{i=n} Q_i) = \cap_{i=1}^{i=n} \delta(Q_i) = P = \delta(Q_k)$, $yz \in \delta(Q)$. It follows that Q is 2-absorbing δ-primary. □

Let $f : R \longrightarrow S$ be a ring-homomorphism, γ be an expansion function of ideals of R, and δ be an expansion function of ideals of S. We say that f is a *$\gamma\delta$-ring-homomorphism* if $\gamma(f^{-1}(I)) = f^{-1}(\delta(I))$ for all $I \in Id(S)$. Note that if f is a surjective $\gamma\delta$-ring-homomorphism and $ker(f) \subseteq I$, for some ideal $I \in Id(R)$, then $f(\gamma(I)) = \delta(f(I))$. In particular, if f is an $\gamma\delta$-ring-isomorphism, then $f(\gamma(I)) = \delta(f(I))$ for every ideal I of R.

Definition 3.4. The ring R is called a *δ-2-integral domain* if the zero ideal is 2-absorbing δ-primary.

Theorem 3.5. *Let δ be an expansion such $q : R \longrightarrow R/I$ be the natural quotient homomorphism is $\delta\delta$-ring-homomorphism . An ideal I of R is 2-absorbing δ-primary if and only if the quotient ring R/I is a δ-2-integral domain.*

Proof. Let I be a 2-absorbing δ-primary ideal. Assume that $\tilde{x}\tilde{y}\tilde{z} \in \{0_{R/I}\}$, $\tilde{x}\tilde{y} \notin \{0_{R/I}\}$ and $\tilde{x}\tilde{z} \notin \delta(0_{R/I})$. Let $q : R \longrightarrow R/I$ be the natural quotient homomorphism such that q is $\delta\delta$-ring-homomorphism. We have

$$\delta(I) = \delta(q^{-1}(\{0_{R/I}\})) = q^{-1}(\delta(\{0_{R/I}\})).$$

Since q is onto, so $\delta(I)/I = q(\delta(I)) = \delta(\{0_{R/I}\})$. Then $xy \notin I$ and $xz \notin \delta(I)$, so $yz \in \delta(I)$, implying $\tilde{y}\tilde{z} \in \delta(0_{R/I})$. Hence $\{0_{R/I}\}$ is 2-absorbing δ-primary.

Conversely assume that $\{0_{R/I}\}$ is 2-absorbing δ-primary. Let $xyz \in I$, $xy \notin I$ and $xz \notin \delta(I)$. Then $\tilde{x}\tilde{y} \notin \{0_{R/I}\}$ and $\tilde{x}\tilde{z} \notin \delta(I)/I$. Since $\delta(I)/I = q(\delta(I)) = \delta(\{0_{R/I}\})$.

Hence $\tilde{x}\tilde{y} \notin \{0_{R/I}\}$ and $\tilde{x}\tilde{z} \notin \delta(\{0_{R/I}\})$. So $\tilde{y}\tilde{z} \in \delta(\{0_{R/I}\}) = \delta(I)/I$. Hence $yz \in \delta(I)$. □

Lemma 3.2. *If f is a $\gamma\delta$-ring-homomorphism, then for any 2-absorbing δ-primary ideal I of S, $f^{-1}(I)$ is a 2-absorbing γ-primary of R.*

Proof. Let $a, b, c \in R$ with $abc \in f^{-1}(I)$, $ab \notin f^{-1}(I)$ and $ac \notin \gamma(f^{-1}(I)) = f^{-1}(\delta(I))$. So $f(a)f(b) \notin I$ and $f(a)f(c) \notin \delta(I)$. So $f(b)f(c) \in \delta(I)$. So $bc \in f^{-1}(\delta(I)) = \gamma(f^{-1}(I))$. □

Theorem 3.3. *Let $f : R \longrightarrow S$ be an surjective $\delta\gamma$-ring-homomorphism. Then an ideal I of R that contains $ker(f)$ is 2-absorbing δ-primary if and only if $f(I)$ is 2-absorbing γ-primary ideal of S.*

Proof. If $f(I)$ is 2-absorbing γ-primary, then by $f^{-1}(f(I)) = I$ and the Lemma 3.2, I is 2-absorbing δ-primary. Now suppose I is 2-absorbing δ-primary. If a, b and $c \in S$ such that $abc \in f(I)$, $ab \notin f(I)$ and $ac \notin \gamma(f(I))$. Then there are x, y and z such that $a = f(x), b = f(y)$ and $c = f(z)$. Thus $xyz \in f^{-1}(f(I)) = I$, $xy \notin I, xz \notin f^{-1}(\gamma(f(I)))$. Since $f^{-1}(\gamma(f(I))) = \delta(f^{-1}(f(I))) = \delta(I)$ and f is surjective. So $f(\delta(I)) = \gamma(f(I))$. Hence $yz \in \delta(I) = f^{-1}(\gamma(f(I))$, implying $bc \in \gamma(f(I))$. □

Corollary 3.6. *Let I be any ideal of R and J be an ideal of R containing I. Then J/I is a 2-absorbing δ-primary ideal of R/I if and only if J is a 2-absorbing δ-primary ideal of R.*

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