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Using Survival Pairs to Characterize Rings of Algebraic Integers

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Abstract. Let R be a domain with quotient field K and prime subring A. Then R is integral over each of its subrings having quotient field K if and only if (A, R) is a survival pair. This shows the redundancy of a condition involving going-down pairs in a earlier characterization of such rings. In characteristic 0, the domains being characterized are the rings R that are isomorphic to subrings of the ring of all algebraic integers. In positive (prime) characteristic, the domains R being characterized are of two kinds: either $R = K$ is an algebraic field extension of A or precisely one valuation domain of K does not contain R.

1. Introduction

All rings considered in this note are commutative with identity. Any inclusion of rings of the form $R_1 \subseteq R_2$ will be interpreted to mean that R_1 is a unital subring of R_2. As usual, $\mathrm{Spec}(R)$ will denote the set of prime ideals of a ring R. If $R \subseteq T$ is a ring extension, it is convenient to let $[R, T]$ denote the set of intermediate rings (that is, the set of rings S such that $R \subseteq S \subseteq T$). If $\mathcal{P}$ is a property of (some) ring extensions and $R \subseteq T$ are rings, then (R, T) is called a $\mathcal{P}$-*pair* if $A \subseteq B$ satisfies $\mathcal{P}$ for all $A, B \in [R, T]$ such that $A \subseteq B$. Among the best known fruits of the "pair" methodology is the Folklore Theorem (cf. [8, page 454]) stating that a ring extension $R \subseteq T$ is integral if and only if (R, T) is both an INC-pair and an LO-pair. (Following [13, page 28], we let INC, LO, GU and GD respectively denote the incomparable, lying-over, going-up and going-down properties of ring extensions; tweaking the usage in [13, page 35], we say that a ring extension $R \subseteq T$ satisfies the *survival* property if $PT \neq T$ for all $P \in \mathrm{Spec}(R)$.) The INC-pairs first appeared (without the terminology) in [7] and were explicitly considered for (commutative integral) domains in [2]. The LO-pairs and survival pairs were introduced in [8] and studied further in [5] and [6]. The GD-pairs appeared explicitly in [10], with subsequent

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variants appearing in [4] and [1]. Recall that if $R \subseteq T$ are domains, then T is called an *overring* of R (equivalently, R is called an *underring* of T) if R and T have the same quotient field. Overrings have often played a role in "pair" characterizations of important classes of domains, for instance, in Richman's result [14, Theorem 4] that if R is a domain with quotient field K, then R is a Prüfer domain if and only if (R, K) is a flat pair.

It is well known that GU $\Rightarrow$ LO $\Rightarrow$ survival (for extensions) and neither of these implications is reversible. Thus, GU-pair $\Rightarrow$ LO-pair $\Rightarrow$ survival-pair. Though it is not at all obvious, both of these implications are reversible. The fact that every LO-pair is a GU-pair was shown in [8, Corollary 3.2]. The proof that every survival pair is an LO-pair was given as part of the proofs of [5, Lemma 2.1] and [5, Theorem 2.2]. It may be fair to say that GD-pairs have been studied less than survival pairs (if one takes into account that survival pairs are the same as LO-pairs). However, both of these types of pairs figure prominently in our starting point, which is the principal result of [10]: if R is a domain with quotient field K, then R is integral over each of its underrings if and only if (exactly) one of the following three conditions holds: (i) R is (isomorphic to) a subring of the ring of all algebraic integers; (ii) $R = K$ has characteristic $p > 0$ and is an algebraic field extension of $\mathbb{F}_p$; (iii) R has positive characteristic and precisely one valuation domain of K does not contain R. The domains R being characterized in this result received some additional characterizations in [9, Corollary 2.3], including the following: if A is the prime (sub)ring of R, then (A, R) is a GD-pair and an LO-pair. Our main purpose here is to answer the question of whether there is any redundancy in this condition that involves both types of pairs.

The answer is "Yes." More precisely, our main result (Theorem 2.2) is that if one considers only domains of characteristic 0, the above condition on GD-pairs is redundant. In other words, Theorem 2.2 establishes that if R is a domain whose prime ring A is isomorphic to $\mathbb{Z}$, then R is isomorphic to a subring of the ring of all algebraic integers if and only if (A, R) is a survival pair (equivalently, an LO-pair). Proposition 2.3 establishes a similar redundancy for the above condition on GD-pairs in positive characteristic. However, the examples in Remark 2.5 show that the above condition on LO-pairs is not redundant: see part (a) for such an example in characteristic 0 and part (b) for such an example in any positive (prime) characteristic.

As usual, if A is a ring, it will be convenient to denote the (Krull) dimension of A by $\dim(A)$; $\mathbb{F}_q$ denotes the finite field of cardinality q; and $\subset$ denotes proper inclusion. Any unexplained material is standard, as in [12, 13].

2. Results

We begin with a result of some independent interest.

Lemma 2.1. *Let A be a Noetherian domain and B an overring of A such that A is integrally closed in B. Then $B = A$ if (and only if) (A, B) is a survival pair.*

Proof. The parenthetical assertion holds since (R, R) is a survival pair for any ring R. We turn to the converse. Since A is a domain with B an overring of A, the hypothesis that (A, B) is a survival pair is equivalent, by [6, Theorem 2.17], to the condition that $A \subseteq B$ is a strongly 1-almost integral extension. Since A is integrally closed in B, this condition implies that if $b \in B$, then for each nonzero element a belonging to the conductor $I := (A :_A b)$, one has that $ab^n \in A$ for each positive integer n. Fix $b \in B$. Our task is to show that $b \in A$. As A is integrally closed in B, we need only show that b is integral over A. Since b is an element of the quotient field of A, the ideal I is nonzero. Choose any nonzero element $a \in I$. Then $ab^n \in A$ for each positive integer n. By [12, Theorem 13.1 (3)], this means that b is almost integral over A. Since A is Noetherian, this means that b is integral over A (cf. [12, page 133]), as desired. □

We can now sharpen the above-mentioned part of [9, Corollary 2.3] for domains of characteristic 0.

Theorem 2.2. *Let R be a domain of characteristic 0 with prime (sub)ring A ($\cong \mathbb{Z}$). Then R is isomorphic to a subring of the ring of all algebraic integers if and only if (A, R) is a survival pair.*

Proof. By the Lying-over Theorem ([12, Theorem 11.5], [13, Theorem 44]), every integral ring extension satisfies LO and, hence, satisfies the survival property. This fact establishes the "only if" assertion. For the converse, suppose that (A, R) is a survival pair. So, by [5], (A, R) is an LO-pair. The same conclusion must then hold for (A, T) for any ring $T \in [A, R]$. As $\dim(A) = 1 \neq 0$, it therefore follows from [8, Proposition 2.13] that no such T can be of the form $T = A[X]$ with X being transcendental over A. Hence, R is algebraic over A. Let E denote the integral closure of A in R. It will suffice to prove that if $r \in R$, then $r \in E$.

Consider $S := A[r]$. Then $E_1 := E \cap S$ is the integral closure of A in S. Since S is algebraic over A, clearing denominators (as in, for instance, the proof of [15, Theorem 7, page 264]) shows that S is an overring of E_1. Let K ($\cong \mathbb{Q}$) denote the quotient field of A, and let L denote the quotient field of S. Then $L = K(r)$ and, since r is algebraic over A, we have $[L : K] < \infty$. Next, since A ($\cong \mathbb{Z}$) is a one-dimensional Noetherian domain, it follows from a general form of the Krull-Akizuki Theorem ([3, page 500], cf. also [11]) that E_1 is a Noetherian ring. Moreover, E_1 is integrally closed in S and (E_1, S) inherits the "survival pair" property from (A, R). Therefore, by Lemma 2.1, $S = E_1$. Thus, $r \in S = E_1 \subseteq E$. The proof is complete. □

We next sharpen the above-mentioned part of [9, Corollary 2.3] for domains of positive characteristic.

Proposition 2.3. *Let p be a prime number and R a domain of characteristic p with prime (sub)ring A ($\cong \mathbb{F}_p$) and quotient field K. Then the following conditions are equivalent:*

(1) *Either $R = K$ is an algebraic field extension of A or precisely one valuation domain of K does not contain R;*

(2) *(A, R) is a survival pair.*

Proof. By [9, Corollary 2.3], it suffices to prove that if (A, R) is a survival pair, then (A, R) is a GD-pair. Suppose that (A, R) is a survival pair. It suffices to prove that $\dim(B) \leq 1$ for all $B \in [A, R]$. This, in turn, follows from the fact that (A, B) is an LO-pair, for [8, Proposition 3.7] ensures that $\dim(B) \leq \dim(A) + 1$ $(= 1)$. □

Combining Theorem 2.2 and Proposition 2.3, we can now give the following characteristic-free formulation of a sharpening of part of [9, Corollary 2.3].

Corollary 2.4. *Let R be a domain with prime (sub)ring A. Then R is integral over each of its underrings if and only if (A, R) is a survival pair.*

The next remark gives two examples showing that Theorem 2.2 and Proposition 2.3 found the only redundancies in the above-mentioned part of [9, Corollary 2.3].

Remark 2.5.

(a) We will show that the "LO-pair" condition in [9, Corollary 2.3] is not redundant in characteristic 0 by finding an example of a GD-pair $(\mathbb{Z}, R)$ which is not an LO-pair. Let p be any prime number, and consider $R := \mathbb{Z}_{p\mathbb{Z}}$. Any subring of R is an overring of $\mathbb{Z}$ and, hence by [12, Theorem 26.1 (1)], must be a Prüfer domain. Thus, as mentioned in the Introduction, it follows from [14] that $(\mathbb{Z}, R)$ is a flat pair. Since any flat ring extension satisfies GD (cf. [13, Exercise 37, page 44]), $(\mathbb{Z}, R)$ is a GD-pair. (This could also be established by showing that each ring in $[\mathbb{Z}, R]$ is one-dimensional.) However, $(\mathbb{Z}, R)$ is not an LO-pair since any nonzero prime ideal of $\mathbb{Z}$ other than $p\mathbb{Z}$ does not survive in R.

(b) Let p be any prime number. Put $F := \mathbb{F}_p$. We will show that the "LO-pair" condition in [9, Corollary 2.3] is not redundant in characteristic p by finding an example of a GD-pair (F, R) which is not an LO-pair. Let X be a transcendental element over F, and consider $R := F[X]_{XF[X]}$. It is known that if $A \in [F, R]$, then $\dim(A) \leq 1$ (cf. [12, Theorem 30.11 (a)]). Thus, (F, R) is a GD-pair. However, (F, R) is not an LO-pair. Indeed, the extension $F[X] \subset R$ does not have the survival property since $(X + 1)F[X]$ does not survive in R.

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