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One Quadratic Equation, Different Understandings: the 13th Century Interpretations by Li Ye and Later Commentaries in the 18th and 19th Centuries

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The Chinese algebraic method, the *tian yuan shu*, was developed during Song period (960–1279), of which Li Ye's works contain the earliest testimony. Two 18th century editors commentated on his works: the editor of the *Siku quanshu* and Li Rui, the latter responding to the former. Korean scholar Nam Byeong-gil added another response in 1855. Differences can be found in the way these commentators considered mathematical objects and procedures. The conflicting nature of these commentaries shows that the same object, the quadratic equation, can beget different interpretations, either a procedure or an assertion of equality. Textual elements in this paper help modern readers reconstruct different authors' understandings and reconsider the evolution of the definition of the object we now call 'equation'.

Keywords: algebra, quadratic equation, Song China, Qing China, Joseon Korea

MSC: 01A13, 01A25, 01A35, 01A50, 01A55, 12–03

1 Introduction

It is well-known to sinologists that our present knowledge of Song dynasty (960–1279) mathematics derives from reconstructions made by Qing dynasty editors [24]. This time period is known for its evidential studies particularly in the fields of philology, phonology and exegesis, and for the editions of Classics in order to republish ancient Chinese texts. Some editors, convinced of the Chinese origin of algebra, used philological techniques to recover lost materials and restore the roots of 'Chinese mathematics' [15, 30]. The mathematical works of Song-dynasty mathematician Li Ye (李冶, 1192–1279) were republished twice in this context in the 18th cen-

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tury and engendered two commentaries—one attributed to Dai Zhen (戴震, 1724–1777) and another by Li Rui (李銳, 1773–1817). Nam Byeong-gil (南秉吉, 1820–1869) in Korea also gave his opinions on parts of Li Ye's works and earlier commentaries. Not surprisingly, the later commentators appear to be highly critical of their predecessors.

Li Ye produced two mathematical studies: The *Ceyuan haijing* (測圓海鏡, *Sea Mirror of Circle Measurements*, 1248) [1] and the *Yigu yanduan* (益古演段, *Development of Pieces* [*of Areas*] [*according to the Collection*] *Augmenting the Ancient* [*Knowledge*], 1259)1) [2]. As of 1782, both treatises were included in imperial encyclopaedias. They were first included in the Ming dynasty (1368–1644) encyclopaedia, the *Yongle dadian* (永樂大典, *Great Canon of Yongle*), and then were copied in the Qing dynasty encyclopaedia, the *Siku quanshu* (四庫全書, *Complete Library of the Four Treasuries*). They were copied and accompanied by a commentary. According to Li Yan [23], the commentary on the *Ceyuan haijing* was written by Dai Zhen, who was in charge of the re-edition of the mathematics section of the *Yongle dadian* in the *Siku quanshu*. However, we have no evidence that Dai Zhen was also the commentator on the *Yigu yanduan*. In 1798, the mathematician and philologist Li Rui meticulously prepared a new critical edition of Li Ye's treatises for a private edition—the *Zhibuzu zhai congshu* (知不足齋叢書, *Collected Works of the Private Library of Knowing Our Own Insufficiencies*) [30]. He included the commentary found in the *Siku quanshu* and added his own commentary criticising the earlier one. In 1855, Nam Byeong-gil in Korea reproduced several excerpts from both of Li Ye's works in his *Mu-i hae* (無異解, *Solutions of No Differences* [3]. He presented the commentary written by Li Rui and a criticism of it. Each commentator had a different mathematical interpretation depending on how they related the procedure found in Li Ye's works to mathematical discoveries made in their time.

Li Ye's studies contain the earliest evidence of a procedure later called *tian yuan shu* (天元術, literally 'Celestial Source Procedure'), which is used to establish and solve algebraic equations [22].2) The three commentaries mainly relate to Problem One of the

1) Although there are debates about the meaning of the term '*yanduan*', we translate it as 'development (*yan*) of pieces [of areas] (*duan*)', because in the texts we usually see that the author refers to a piece of area as one '*duan*', and one major topic of this text is about a method of manipulating, or developing, pieces of areas to solve problems.

2) The expression *tian yuan shu* was translated as 'technique of the celestial element' in [29] or as 'heavenly element method' in [10]. It is not uncommon to translate the character *shu* 術, as the technical mathematical term of 'procedure', or to use its synonym 'algorithm'. It is an effective method expressed as an ordered and finite list of well-defined instructions for calculating researched values starting from given values [9, pp. 21 and 986]. The prefix *tian*—literally 'sky' or 'celestial'— indicates that it is the first unknown or the only one. The character *yuan* is sometimes translated as 'element' and represents the unknown. Joseph Needham pointed out the disadvantage of translating *yuan* by 'element', because of the confusion with the elements of chemistry [29]. Jock Hoe notices that in philosophical texts, *yuan* is 'the source from which all the matter in the universe stems'. *Yuan* means 'the origin' or 'the beginning'. However, he chose to translate it technically as 'unknown' [14]. Our

Yigu yanduan, which is the simplest and clearest problem illustrating the procedure used to set up quadratic equations. Moreover, this basic problem is a perfect pretext to a discussion on the fundamentals of the procedure. None of the commentators agree on the meaning of the expression Li Ye obtains at the end of the procedure. This will be the starting point of our study.

The final expression of the *tian yuan shu* found in the *Yigu yanduan* can be represented in modern transcription as $ax^2 + bx + c = 0$. This expression can be considered as a procedure or an assertion of equality. None of the four authors named above share the same interpretation concerning the *tian yuan shu* and its resulting expression. Historians know that the same mathematical object, procedure or concept can have several interpretations over different time periods. Indeed, we will not describe the evolution of a procedure but how an expression resulting from the procedure was conceptualised at different times. The four authors discussed the same aforementioned procedure, its resulting objects and transcriptions, but the way they commented and transcribed the mathematical components attests to different ways of working. Therefore, our research questions are as follows: what are the available textual elements in the various re-editions that can help us consider expressions of 'equation' found in Li Ye's text? How did the commentators understand the equations and polynomials found in Li Ye's works? How did they understand how he worked with polynomials? Are there any textual components revealing clues to the difference between 'polynomial' and 'equation' for Li Ye? Is the distinction between the two objects the same in Song and Qing dynasties?

2 One quadratic equation with *tian yuan shu*

Much literature is dedicated to the presentation of the *tian yuan shu* [e.g., 19, 23, 26, 32]. To answer the questions raised above, we shall first present the context and brief history of *tian yuan shu*, and then use Problem One of the *Yigu yanduan* as an example to discuss the general elements of the procedure and its interpretations.

2.1 Context and brief history of *tian yuan shu*

In the *Yigu yanduan*, the name *tian yuan shu* is found only in the postface by the commentator and editor, Li Rui. Li Ye did not give it a name. The name is derived from the expression Li Ye uses to start each procedure: *li tian yuan yi*(立天元一), or 'to set up one *tian yuan*'. It was thereafter qualified as *shu* 術, or 'procedure' by the com-

preference is to follow the ancient philosophers and Jock Hoe's observation and to translate *yuan* as 'source'. However, as this article discusses the technical interpretations of mathematical terms, we decided to remain neutral and refer to it using Hanyu Pinyin.

mentator. What was a procedure for the Qing dynasty commentator had first been a mathematical object for the Song dynasty author, as we will see later. This important distinction, which is related to the history of understanding of the nature of 'procedure' and of 'equation', will make sense later.

In the context of Chinese mathematics, we know three early occurrences of the terms *tian yuan*. The term first appeared in Qin Jiushao (秦九韶)'s *Shushu jiuzhang* (數書九章, *Mathematical Treatise in Nine Chapters*, 1247), but its usage was related to *da yan* (大衍), which stands for 'indeterminate analysis', and is thus different from that of Li Ye [25, pp. 345–6; 26]. Qin Jiushao and Li Ye were contemporaries. However, most modern scholars admit that there is no evidence that the two mathematicians ever met. They worked independently, living far apart in rival kingdoms. If the procedure had been widespread at this time in Northern China, one would expect to find earlier sources testifying its elaboration. Strangely, nothing tangible on the *tian yuan shu* before Li Ye has survived in the printed form. Li Ye's studies remain the earliest available testimony of this procedure. The procedure was later generalised independently by Zhu Shijie (朱世傑) in the *Siyuan yujian* (四元玉鑒, *Precious Mirror of Four Sources*, 1303) to the 'procedure of four sources', *si yuan shu*(四元術), which was generalised from a solution to an equation with one unknown to that of equations with at most four unknowns. These are the three testimonies with which we could reconstruct the history of the procedure [12]. Consequently, for editors who are responsible of reconstructing and re-editing ancient mathematics, the studies of Li Ye, being the oldest remaining evidence, possess inestimable value, and thus is the reason for him being the centre of the debate.

In contrast, there are other testimonies of the existence of this procedure prior to Li Ye's studies. In the preface by Zu Yi(祖頤) to Zhu Shijie's *Siyuan yujian*, we read several references concerning the term *tian yuan* [14]. Although there is no access to the content of the books referred to, and it is difficult to guess their content, it is possible to deduce that the *tian yuan shu* was probably not a completely new invention at the end of 13th century. Li Yan and Du Shiran dated the origin of the procedure back to the beginning of the 13th century or earlier [23, p. 139]. Needham believed that the procedure could be pushed 'well back into the 12th century' [34, p.41], as did Qian Baocong [32, pp. 190–191]. Martzloff, who underlined that there could be different ways of doing the procedure, wrote that 'in fact, the set of *tian yuan* procedures, which has been preserved, seems to have been invented in Northern China towards the 11th century' [27, p. 259]. Qian Baocong also argued that mathematicians in the Song-Yuan period could comfortably add, subtract, multiply, and divide (by integral powers of the unknown) polynomials with one unknown [33, pp. 187–190]. It is impossible to be more accurate about its origin; the *tian yuan shu* as it appears to us in 13th-century texts has the form of a finished and mature product, already fully developed. The algebraic structure of *tian yuan shu* is also the topic of many studies, and the reader may refer to, for instance, [15].

After its appearance in the Song dynasty and before its re-discovery by 18th century editors, the procedure was forgotten in China for several centuries. The procedure was re-discovered in the early Qing dynasty by mathematician Mei Juecheng (梅瑴成, 1681–1763), who was first acquainted with a newly imported Western algebraic method called *jie gen fang* (借根方)—'Borrowing the Root and Powers'. This method uses abbreviations in Chinese characters to represent different powers of the unknown. For instance, *zhen shu* (真數) is used for the constant, *gen* (根) represents the root, *ping fang* (平方) the square, and li fang (立方) the cube [18]. Anyone using this method has to remember the names of the powers to set up equations. An equation such as $x^2 - 3x + 2 = 5$ would be rendered as '一平方 $-$ 三根 $\perp$ 二 $=$ 五', and the signs for 'plus' ($\perp$) and 'minus' ($-$) are read as '*duo*' (多, more) and '*shao*' (少, less) instead of '*jia*' (加, to add) and '*jian*' (減, to subtract) as in traditional Chinese mathematics. As can be seen, it is tacitly assumed that the powers in the equation belong to the same unknown, and no other unknown is supposed to be in the equation. All commentators of that time were familiar with this kind of notation. Yet, Mei Juecheng was the first to understand the forgotten ancient Chinese algebra *tian yuan shu* [14]. Other mathematicians of the Qianlong–Jiaqing period (The Qianlong emperor 1736–1795 and his successor Jiaqing 1796–1820) struggled to prove the anteriority and greater efficiency of Chinese or Western procedures [16, 17]. The *tian yuan shu* was considered an ambassador of the so-called 'Chinese traditional' algebra [14, 25, 35]. Although in the late 17th and early 18th century, Qing emperor Kangxi made it an important policy for his court to learn and digest Western science and mathematics, certain incidents late in Kangxi's reign caused distrust between the Qing court and the Jesuits, so a current to build a system of scientific knowledge independent of the West grew in Kangxi's court [18]. The academic trends later in Qianlong-Jiaqing period strengthened the sense of independence. Many scholars made it their life-long calling to revive ancient Chinese learning, including Chinese mathematics, and the theory of the 'Chinese origin of Western learning' that was discussed in the 17th century was brought up again. However, there are also scholars, such as Li Rui's friend Wang Lai (汪萊, 1768–1813), who still believed that Western mathematical methods had their value and wrote his research on the basis of Western knowledge. There were fierce debates among these scholars, and also some government officials, on mathematical procedures from China and Europe, and the *tian yuan shu* was one of the methods that were used to show the efficiency and the anteriority of algebra from China [16, 17]. Li Rui, belonging to this camp, strongly believed in the Chinese roots of algebra and commented on the works of Li Ye accordingly [31]. While modern scholars believe Li Rui to be very familiar with traditional Chinese mathematics, several elements of his commentary seem to reveal alternative understanding. This current paper shall present the differences between the interpretations by Li Rui and Li Ye concerning the same expression of quadratic equations constructed with the *tian yuan shu*. And then we will show how the three commentators—Dai Zhen, Li Rui and Nam Byeong-gil—disagreed on how to interpret the procedure and how they understood the polynomials expressed with the *tian yuan shu* and how textual components reveal their understanding.

Before discussing their different understandings, we present Problem One of the *Yigu yanduan* as an example of the procedure *tian yuan shu*.

2.2 Problem One of the *Yigu yanduan*

In the *Yigu yanduan*, *tian yuan shu* is used to elaborate quadratic or sometimes linear equations with one unknown. There are no cases containing several unknowns or a higher degree. The *Yigu yanduan* is composed of 64 problems solved first by *tian yuan shu*, then by a second procedure called '*tiao duan*' (條段, Section of Pieces [of Areas]), which is not the topic of debate by the commentators [30]. Each of the problems proposes a question relating to field surveying as a pretext for working on a quadratic equation. The first solution to the problems in the *Yigu yanduan* is composed of three steps made up of the nine repetitive sentences composing the mathematical discourse.[3] These sentences give a list of operations rhetorically leading to the construction of mathematical expressions and imply manipulations of rods. These manipulations are not explicitly described in the text, and the reader is assumed to be acquainted with them. As an illustration, we present here the original contents, a translation and the mathematical transcriptions of Problem One in the *Yigu yanduan*. Figure 1 is the original illustration to the problem.

> 【第一問】今有方田一段, 內有圓池. 水占之外, 計地一十 三畝七分半. 竝不記內圓外方. 只云 從外田楞至內池楞, 四邊各二十步. 問內圓外方各多少?
>
> [Problem One] Let us suppose there is one piece of square field, inside which there is a circular pond. Outside the [area] occupied by water, one counts thirteen *mu* seven *fen* and a half of land. Moreover, there is no record of the [dimensions] of the inner circle and the outer square. It is only said that [the distance] from the edge of the outer field reaching the edge of the inside pond on [all] four sides is twenty *bu*. One asks

3) We refer to 'mathematical discourse' the lines written with characters and mathematical expressions by Li Ye. Other parts of the text are composed of diagrams and commentaries.

how long are [the diameter of] the inner circle and [the side of] the outer square.

荅曰: 外田方六十步, 內池徑二十步.

The answer says: The side of the outer field is sixty *bu*. The diameter of the inside pond is twenty *bu*.

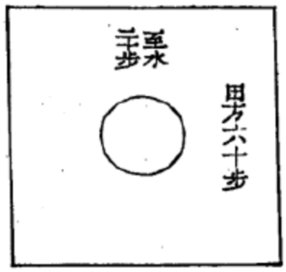
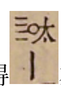
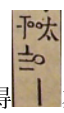


Figure 1. Original illustration of Problem One in the *Yigu yanduan* [13, Vol.1, p.876]. Caption on the top: 至水二十步 (Distance from the pond to the side: 20 *bu*.) Caption on the right: 田方六十步 (The side of the square: 60 *bu*.)

法曰:

The method says:

[Step 1][4)]

立天元一為內池徑. 加倍至步得為田方面. 以自增 乘得為方積, 於頭.

Set up one *tian yuan* as the diameter of the inside pond. Adding twice the reaching *bu* yields as the side of the field. Augmenting this by self-multiplication yields as the area of the square, which is sent to the top position.

Mathematical transcription:

Refer to Figure 2. Let a be the distance from the midpoint of the side of the square to the pond, 20 *bu*; let A be the area of the square field (S) minus the area of the circular pond (C), 13 *mu* 7.5 *fen*, or 3300 *bu*. Let x (*tian yuan* 天元) be the diameter of the pond.

4) We have separated the solution to the problem into three steps for the convenience of the mathematical transcriptions. The original text has only a long paragraph for the solution without differentiation of steps.

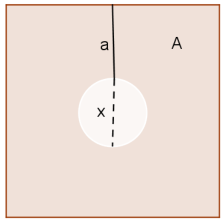
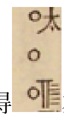


Figure 2. Pictorial configuration for Problem One.

Side of the square $= 2a + x = 40 + x$.

$S = (2a + x)^2 = 1600 + 80x + x^2$.

[Step 2] 再立天元一為內池徑. 以自之, 又三因, 四而一得為池積.

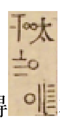
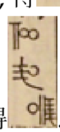
Again, set up one *tian yuan* as the diameter of the inside pond. This timed by itself, and multiplied further by three and then divided by four, yields

0	*tai*
0	
0.75	

as the area of the pond.

Mathematical transcription:

$C = \frac{3}{4}x^2 = 0.75x^2$, with $\pi = 3$.

[Step 3] 以減頭位, 得為一段虛積, 寄左. 然後列真積. 以畝法通之, 得三千三百步. 與左相消得.[5] 開平方, 得二十步, 為圓池徑也. 倍至步, 加池徑, 即外方面也.

Subtracting (減, jian) this from the top position yields

1600	*tai*
80	
0.25	

as a piece of the empty area, which is sent to the left. Next, place the real area. With the divisor of mu, communicating[6] this yields three thou-

5) The rod expression for 1700 in the picture and in Figure 5 is not clear due to the quality of the printing in [13].

6) *Tong* 通 is translated as "to communicate". See [9, pp. 994–8]. Here $1mu = 240bu$. The operation of communication applies to integers $a, b, c \cdots$ and a fraction e/f to transform them into numbers expressed by the same unit (here *bu*). They become $fa, fb, fc \cdots$. Technical vocabulary related to fractions is not scarce in the *Yigu yanduan*. There is clearly an underlying fractional conception of numbers. Secondary literature has often considered numbers in the above tabular setting as decimal,

sand, three hundred *bu*. With what is on the left, eliminating from one another (相消, xiang xiao) yields $\begin{array}{c}1700\\-80\\-0.25\end{array}$. Opening the square yields 20 bu, which is the diameter of the circular pond. Double the reaching *bu* and add the diameter of the pond, and the answer is the side of the outer square.

Mathematical transcription:
$A = 3300bu = S - C = 1600 + 80x + x^2 - 0.75x^2 = 1600 + 80x + 0.25x^2$.
$3300bu = 1600 + 80x + 0.25x^2$.
Hence the equation: $1700 - 80x - 0.25x^2 = 0$.

From this problem, we outline the first three steps for the general procedure:

(1) A first mathematical expression corresponding to the area of one of the figures named in the statement of the problem is computed. We understand this as an expression of the first area as a polynomial.

(2) A second mathematical expression corresponding to the other figure is then computed. We understand this as giving a second polynomial. The second expression is subtracted from the first, or, infrequently, added (pb.21; 23 to 30; 38; 43; 46; 63 of the *Yigu yanduan*).

(3) The expression resulting from this operation is equal to the area given in the statement. The difference or sum of the two polynomials is equal to a constant, and they cancel each other out to give the final expression, which we understand as a quadratic equation.

The same steps are found in all 64 problems. For each problem, first an unknown number is chosen based on the condition given in the statement, then polynomials are constructed and the equation that governs the unknown is found. Second, the problem is solved after one of the roots of the equation, which suits the conditions of the problem, is found. That one root, always positive, is found by means of computations with the polynomials, but the actual procedure to solve the equation is never explicitly given because the reader is assumed to be familiar with it. Li Ye uses no term that we could render as 'equation', nor does he use a term to refer to the tabular settings used to represent mathematical expressions. The tabular setting for all mathematical expressions, polynomials or equations seems to be the same at first sight.

but they could be linked to fractions. 5.87 could be read as $5 + 87/100$. It is possible that what Li Ye worked on with fractional conceptions were seen as decimals by Li Rui. This could be the object of further discussions. See [4].

However, if an 'equation' is an equality containing one or more unknowns whose value(s) can be determined in order to make the equality true, then, transcribed in modern terms, the last mathematical expression looks like an equation.

We saw in the translation of Problem One above that at first sight, this setting looks the same for polynomials and equations, but there are several ways to read expressions depending on the interpretation of tabular layouts. Thus we need to take a closer look at the mathematical expressions of *tian yuan shu*.

2.3 Mathematical expressions of *tian yuan shu* as seen in the *Yigu yanduan*

Mathematical expressions in the *Yigu yanduan* are presented on three rows with horizontal and vertical lines, which represent numbers as counting rods on a surface. For example,  denotes $2700 + 252x + 5.87x^2$. Each row indicates a term of consecutive powers. The upper row contains the constant term, the middle row contains the coefficient of the first power of the indeterminate (x) and the lower row contains the coefficient of the second power of the indeterminate (x^2). The characters 太 (*tai*) or 元 (*yuan*) at the side of the array indicate the significance of the numbers relative to the marked position. The character *tai* on the upper row indicates the constant term. *Yuan* is used when there is no constant term; it indicates the first power. There is no specific character for the second power and there are no cubic equations in the *Yigu yanduan*. The *tai* and *yuan* characters disappear in the last mathematical expression, which appears after the polynomials are 'eliminated from one another' (*xiang xiao* 相消). One of the difficulties of the text is the interpretation of the mathematical expression resulting from the operation of elimination (*xiang xiao*). Li Ye makes strict use of technical vocabulary [8, 30]. The term *jian* (減), or 'to subtract', appears only when two constants or coefficients are subtracted in the construction of polynomials. The term *xiang xiao* is used only when the two polynomials are subtracted from one another to give the last mathematical expression, the one we understand as an equation.

As the equality is never stated and the placing of rods on a tabular setting is the same for all mathematical expressions, some scholars claimed that 'these columnar arrays of numbers do not differentiate between mere algebraic expressions and equations' [20, p. 245] and 'various configurations can be regarded as either equations or polynomials' [24, p. 138]. We argue that the appearance of *tai* and *yuan* and the technical vocabulary contrasts these statements. There *is* a difference. It seems that the question of understanding Song-dynasty *tian yuan shu* in the 18th century is still meaningful in the 20th and 21st centuries.

In problems 38, 44, 48, 56, 59, and 60 of the *Yigu yanduan*, Li Ye refers to different rows as, *xia fa shang shi* (下法上實), or 'the divisor is below and the dividend is above'. These names are also used systematically in the other procedure, the *tiao duan*, in the *Yigu yanduan* [30]. These are names borrowed from the procedure of division. Another expression related to division is used in problem 23: *kai ping fang chu* (開平方除), or 'to divide by the extraction of the square root'. The relation between division, root extraction and tabular settings, well-known to historians of mathematics written in Chinese, is precisely the key to understanding Li Ye's conceptualisation of polynomial and equation, as we shall see in the next section.

3 Different understandings of the equation

Now we are ready to discuss different understandings of the same quadratic equation. We shall first discuss Li Ye in the 13th century, then two Chinese editors in late imperial China, and finally Nam Byeong-gil in the 19th century.

3.1 Li Ye's understanding

The setting for polynomial and equation finds its origin in the procedure for root extraction, which in turn finds its origin in the procedure for division. Several studies have already thoroughly investigated the relation of these operations (e.g., [9, pp. 314–22], [21, ch 3.4], and [24, pp. 17–19]). We summarise a few of the elements based on simple examples provided in the *Sunzi suanjing* (孫子算經, *Classic of Computation of Master Sun*), where the method is described in great detail.7)

Let us first note that the procedure of division is the mirror of that for multiplication, where the multiplier is placed in the upper position and the multiplicand in the lower position. The latter is moved to the left according to the number of digits. The *Sunzi suanjing* prescribes multiplying the number placed below, digit after digit, by the greatest digit of the multiplier and to place the intermediate results in the middle row where they are progressively added. Similar to multiplication, where its operations are based on the position of the multiplier relative to the multiplicand, the division operation is based on the placing of the divisor (*fa*) relative to the dividend (*shi*). The quotient (*shang*) is placed on the top. The initial position of the divisor relative to the dividend determines the position of the first digit (from the left) of the

7) [21, 32]. The first reference was the object of a critical review by [36]. *Classic of Computation of Master Sun* belongs to the *Shibu suanjing* (十部算經, *Ten Books of Mathematical Classic*) used in education during the Sui and Tang dynasties (581–907). Wang Ling suggested that it was composed between 280 and 473 CE [38]. Its earliest surviving edition is dated from the Southern Song dynasty [24, pp. 92–3].

quotient (See Figure 3).

商 *shang*
實 *shi*
法 *fa*

Figure 3. Relative positions of the divisor (*fa*), the dividend (*shi*) and the quotient (*shang*).

The characters *fa* 法 and *shi* 實 are also those used by Li Ye for the last two rows of the expression and in the second geometrical procedure. In problems 38, 44, 48, 56, 59, and 60, *shi* is used for the constant term and *fa* for the first power in linear equations.[8] The procedure of square root extraction has remarkable similarities with the procedures for division. Sunzi explains the method of extracting the square root with two examples in Ch. 2, pb. 19–20 ([19], [24] and [32]). Parallelism appears with the following observations; the dividend is called *shi* and the number whose root is extracted is called *shi*. This is the first number to be placed on the working surface, and its digits set the values of places for other digits on the surface. The divisor termed *fa* is moved from right to left such that its first digit from the left is placed below the first or second digit of the dividend. The quantity added at the place of *fang fa* is used like a divisor, and the quantity placed at the *shi* row is treated like a dividend. The operation returns the extraction of the root to a division.

The observation of the procedure for root extraction found in various texts shows that there is an evolution in the procedure, with the appearance of a place value notation for the equation associated with these extractions [6, 7]. Chemla interprets the parallelism between the division and the extraction of a square root thus: such terms as 'dividend' and 'divisor' for positions corresponding to the successive steps of computation have two functions. First, it allows the same list of operations to be reproduced in an iterative way. Second, it allows the extraction of the square root to be modeled on the operations for division [9, p. 327]. The same position names are used and the way of using the position during the succession of operation is performed in the same way. Names and the management of positions are the key points for the correlation of the two procedures. In another publication, Chemla shows that there is a practice that consists in the exploration of relations between the operations of root extraction and division; its expression is a revision of the different ways of computing and naming positions. She concludes that the practice she analyses in the *Jiuzhang suanshu* (九章算術, first century CE)[9] has in fact been preserved until the

8) In the geometrical procedure, *shi* also names the constant term. *Cong fa* 從法 is the first power, and *chang fa* 常法 is the second power. These appellations are well known to historians of Chinese mathematics. Their geometrical origins as well as their relations to the square root extraction procedure have been clearly established [9, pp. 322–335] and [19, pp. 251–274].

9) *Jiuzhang suanshu* (九章算術), or the *Nine Chapters on Mathematical Art*, compiled in the beginning of

13th century, i.e. in Li Ye's time [7].

The elaboration of the division procedure led to a general technique that mechanically extracts the square root of a number. This method is used as a procedure and also provides the basis for the further development of procedures solving quadratic equations. The configuration necessary for conveying the meaning of these equations is inextricably expressed in the positions occupied by the rod numerals on the support. This justifies the use of the same divisor terms to name positions on the surface for division, extracting the root and finally the terms of the equation as they are established at the same position. Once the equation is set up on the surface one just has to apply a well-known procedure to solve it. The development of the procedure of root extraction leads to the conception and solution of the equation.

It has been shown by historians that, just as the development of the method of square root extraction was based on the knowledge of division, the concept of the polynomial equation was derived from the procedure for square root extraction. The conception of the equation is procedural; it is a series of operations with two different terms—a dividend and one or two divisors—which is solved by the square root extraction procedure. The equality is expressed by the operation of 'eliminating one another'. However, an important premise is that only when two polynomials are equal can they 'eliminate one another'. Thus, if the question is to determine an unknown which satisfies an equality, then, in that sense, there is an equation. In fact, what we identify as equations in the representation of tabular settings is an opposition between a 'dividend' and other terms. This peculiarity of the concept of an equation is due to the essential role played by the counting surface and the way of assembling different procedures for division and root extraction.

In Li Ye's studies, the last mathematical expression resulting from *xiang xiao* (eliminating from one another) never contains a the character *tai* or *yuan* 元, i.e., the character represented in the polynomials is absent from the equation. This is because once the procedure of setting up the equation is completed, the last expression obtained will not be the object of further operations [5, note (a) in ch. 8.3]. The number of ranks is the only pertinent information for extracting the positive root. Having obtained the equation after cancellation, the marks on the right can be forgotten. Thus, for Li Ye, there is a difference between the mathematical expressions for polynomials and equations. The configuration can produce either an equation or a polynomial, but by adding the character *tai* or *yuan* to the column, Li Ye makes what we call a

the common era, was considered a 'canon' of classical Chinese mathematics, and its format, terminologies, and categorization of problems had been considered standard since the 7th century. It was also the object of several commentaries. Among them, two were transmitted along with the canon, one by Liu Hui (3rd C.) and the other by Li Chunfeng (7th C.). It was the most influential mathematical text in pre-modern China.

'polynomial'. The absence of a sign is precisely the mark of the equation.

In other words, the procedure that had been presented to us and later called *tian yuan shu* by the Qing dynasty commentators is a sequence of operations with an object named *tian yuan* for Li Ye. This sequence gave a shape to a problem in the form of 'an equation', or rather, in the form of a square root extraction applied to polynomials. And this would be the next procedure applied to it in order to solve the problem. For Li Ye, *tian yuan* is the name of a component in a multi-step procedure preceding the solution of a problem. This procedure interprets the geometrical data through polynomials and then transforms the problem into a new object we can now call an equation. However, it is not clear whether the later commentators shared the same understanding. For Li Ye, there were several procedures which merged into a big one ending with an equation which is also an operation. For the later commentators, there is a unique procedure with several sub-steps.

3.2 Two Qing Chinese commentators' understandings

More than four centuries later, two commentaries discussing the comparison between *tian yuan shu* and *jie gen fang* suggest that there are other possible interpretations of the equations in the *Yigu yanduan*.

In Problem One translated above, after subtracting the second polynomial from the first, the reader obtains an expression of the area known in the statement according to the unknown. In Li Ye's discourse, this is expressed in the following way:

> Subtracting this from the top position yields $\begin{array}{rl} 1600 & \textit{tai} \\ 80 & \\ 0.25 & \end{array}$ as a piece of the empty area that is sent to the left. Next, place the real area. With the divisor of mu, communicating this yields three thousand three hundred *bu*. With what is on the left, eliminating from one another yields $\begin{array}{r} 1700 \\ -80 \\ -0.25 \end{array}$.

In modern transcription, the equation obtained at the end of the procedure would be $1700 - 80x - 0.25x^2 = 0$, but none of the Chinese commentators read it that way. To understand both cases, we should first consider that in the handwritten copies of *Yigu yanduan* in the *Siku quanshu*, there are no signs to discriminate positive from negative coefficients [Figure 4], while there are diagonals strokes in Li Rui's edition [Figure 5].

Figure 4: the equation of Problem One in the *Siku quanshu* copy of 1789.

Figure 5: the equation of Problem One in Li Rui's edition of 1798.

The commentator of the *Siku quanshu* interprets the last expression as an equation expressing equality. The 'eliminating from one another' (*xiang xiao*) operation is interpreted in the following way:

> 相消者, 兩邊同減一千六百步. 後凡言相消者, 皆兩邊加減一數也.
>
> 'Eliminating from one another' means subtracting equally one thousand six hundred *bu* from the two sides. Next, each time one says 'eliminating from one another', one will add or subtract a [same] quantity from each of the two sides.

We have the following expression $1600 + 80x + 0.25x^2 = 3300$. According to the commentator of the *Siku quanshu*, the same quantity has to be removed from the two sides, i.e. $1600 + 80x + 0.25x^2 - 1600 = 3300 - 1600$. Thus, the equation becomes $80x + 0.25x^2 = 1700$. The commentator of the *Siku quanshu* clearly states equality between a constant and a polynomial composed of the two terms of different powers:

> 此即一千七百步與八十池徑, 二分半平方等.
>
> This means one thousand seven hundred *bu* equals eighty diameters of the pond and two and a half tenth of the square.

His terminology clearly states the equality (*yu...deng* 與...等, 'equals') and the second power (*ping fang* 平方, 'square') using the vocabulary of *Jie gen fang*. We cannot interpret the reading of the editor of the *Siku quanshu* as a way to fit with tabular notation. Indeed, the tabular setting shows only positive terms and the commentator reads the equation as made of positive terms. Yet, Problem Two in the *Yigu yanduan* also contains a commentary by the editor of the *Siku quanshu*. The equation contains two negative terms ($-3100 + 160x - 0.25x^2 = 0$) recorded in Li Rui's edition. The tabular setting does not show any specific marks for positive and negative coefficients in the *Siku quanshu*, but contains the following commentary:

> 此即三千一百步與一百六十徑少二分半平方等.
>
> This means three thousand one hundred bu *and* one hundred sixty diameters with two and a half tenths squared less *are equal*.

Positive and negative coefficients are indicated by '*duo*' (more) and '*shao*' (less) in problem two. This is the *jie gen fang* notation, as the reader has seen earlier. While the editor of the *Siku quanshu* reads $1700 = 80x + 0.25x^2$, where all the coefficients are positive, Li Rui reads: $1700 - 80x - 0.25x^2$, where two coefficients are negative.

The two commentators have completely different readings of the last tabular setting. Li Rui, in response to the edition made in the *Siku quanshu*, reproduces the commentary of the *Siku quanshu* and adds his own commentary to it. He immediately indicates that the positive and negative quantities were once differentiated by red and black coloured ink. The coloured notation disappeared from use, and in the *Siku quanshu*, there are no specific marks for one sign or the other. Therefore, Li Rui added a diagonal stroke to signify the negative coefficients. His commentary indicates the type of correction he applied to the text and justifies the origin of the mistake by the commentator of the *Siku quanshu* (Translation of the commentary in [30, pp. 409–10]).

Then, as follows, Li Rui vehemently explains why the interpretation by the commentator of *Siku quanshu* is a mistake:

> 此案非也. 蓋西人借根方,即古立天元一,而借根方兩邊加減與立天元一相消. 其法迥 殊加減法. 如案所云, 若相消法, 則但以寄左數減後數, 或以後數減寄左數, 故曰相 消也.
>
> This commentary is wrong. The '*jie gen fang*' of the Westerners is the old 'set up one *tian yuan*', but the '*jie gen fang*' requires 'adding or subtracting from the two sides' and the method of 'set up one *tian yuan*' requires 'eliminating from one another'. The latter is absolutely different from the method 'to add or subtract [from the two sides]'. According to what the commentary says, if one uses the method 'to eliminate from one another', then one just subtracts the quantity placed on the left from the quantity that follows or the quantity that follows from the quantity that is on the left. That is why it says 'eliminating from one another'.

According to Li Rui, there is a crucial difference between the procedure suggested by the commentator of the *Siku quanshu*, 'to add or subtract [from the two sides]' (*jia jian* 加減) and that used by Li Ye, 'to eliminate from one another' (*xiang xiao* 相消). Li Rui insists on the reciprocity of the second procedure. The quantity on the left (i.e. the polynomial $1600+80x+0.25x^2$) can be subtracted from the quantity that follows (i.e. the area expressed as a constant, 3300 *bu*), or reciprocally, but that is not possible with the 'to add or subtract' procedure. In Problem One presented by Li Rui, the expression $1600 + 80x + 0.25x^2 = 3300$ was transformed into $3300 - (1600 + 80x + 0.25x^2)$ to give $1700 - 80x - 0.25x^2$. It causes the signs of the coefficients to change and this is not a problem for the extraction of the root, which can be performed the same way regardless of the signs.

In fact, his last comment on Problem One reveals that Li Rui puts the emphasis on his disagreement with the commentator of the *Siku quanshu*:

> 兩邊加減法, 既加減後 仍分兩邊, 故案云步與池徑平方等. 若相消之後, 則止有減餘, 更不得云彼與此等矣.
>
> In the method 'to add or to subtract from the two sides', after one adds or subtracts one still distinguishes two sides. That is why the commentary above says that 'the *bu* equals the diameters and the square'. If one eliminates from one another, then, afterwards one only has the remainder of a subtraction, and it cannot be said that one thing is equal to another.

There are two interesting points in this piece of the commentary. First, Li Rui underlines that the expression is not an equality, but the remainder of the subtraction $3300 - (1600 + 80x + 0.25x^2)$. Yet, second, he does not seem to conceive either that it is $1700 - 80x - 0.25x^2 = 0$. In that way, he is close to Li Ye's procedural concept of 'equation'. The last tabular setting is a mathematical expression whose signs can be changed.

Li Rui perfectly understands the procedural concept—an equation is a procedure to be carried out which ends with the remainder from a series of operations. He totally understands the implication of the change of signs and the 'eliminating from one another' operation. Yet, for him the last mathematical expression does not have a different 'status' before and after the 'elimination', as it seems to have for Li Ye. On that point, Li Rui's commentary is ambiguous.

In another commentary, to The *Ceyuan haijing* (ch. 2, folio 13), he assimilates the two operations of 'elimination' (*xiang xiao*) and 'mutual subtraction' (*xiang jian*, 相減):

> 相消即相減, 方程所為直除式也.
>
> 'To eliminate from one another' is 'to mutually subtract' in the procedure of 'elimination of two numbers facing each other' in *fang cheng*.

He refers to the vocabulary from the classic, the *Jiuzhang suanshu*. Chapter 8 of the classic details the *fang cheng* procedure. The *zhi chu* operation is an elimination between two columns having the same number of rows. Corresponding numbers are eliminated from one another. Indeed, it is possible that the operations as performed on the counting surface were the same for *zhi chu*, *xiang jian* or *xiang xiao*. Yet, Li Ye used the term *xiang xiao* only for eliminating two polynomials. They may be the same operations, but do not operate with the same objects.

The distinction between objects is recorded by the marks *tai* and *yuan*. Li Rui carefully reproduced and corrected the materials available to him to prepare his edition of Li Ye's studies. He carefully copied the characters *tai* and *yuan* where they were needed, and preserved the final mathematical expression in each problem without them. This leads one to conclude that he understood Li Ye's use of characters. However, his commentary on Problem One reveals a different interpretation of the absence of the characters:

> 凡算式, 真積曰太極, 旁記太字. 虛數曰天元, 旁記元字. 太之下一層為元. 元之下 一層為元自乘冪. 記太字則不記元字, 記元字則不記太字. 其太, 元俱不記者, 則以 上方一層為太也.
>
> For all mathematical expressions, the real area is called *tai ji* (the Great Ultimate), and on its side one writes down the character *tai*. The empty quantity is called *tian yuan*, and on its side, one writes down the character *yuan*. One rank under the rank of *tai* is the rank of *yuan*, and one rank under the rank of *yuan* is the rank of the square, which is *yuan* self-multiplied. If the character *tai* is written down, the character *yuan* is not written, and if the character *yuan* is written down, the character *tai* is not written. In the case of neither of the [characters] *tai* and *yuan* being written down, then the upper rank becomes the rank of *tai*.

Li Rui accurately described where to write the characters. The character *tai*, borrowed from Daoist terminology, has to be written next to the term expressing the known area ('real area') or constant term. The terms written below the constant term correspond to the area expressed according to the indeterminate (empty quantity).[10)] The row of the term of the first power is marked by the character *yuan*. The last row, which is the row of the second power, is not marked. The two characters are never marked together to avoid redundancy. Nevertheless, the last sentence of this commentary is surprising. Although Li Rui does not write either *tai* or *yuan* in the last mathematical expression resulting from *xiang xiao*, it seems that he considers that

10) The reader may refer to [30] for the translation of 'real' and 'empty'. *Xu* is often translated as 'negative' and considered a synonym of *fu* 負. However, in the *Yigu yanduan*, the character xu is firstly used as a geometrical concept. It is used to refer to a type of area for a negative (負) coefficient set on a counting surface. Therefore, its meaning is different from 'negative' as there cannot be negative area. This character is applied only to areas which are expressed with the unknown, because areas expressed by the constant term must be positive. In Problems One and Two of the *Yigu yanduan*, *xu* was applied to polynomials. The term is later generalised by Li Rui to name all polynomials as 'empty quantities'. The meaning of the character *xu* is not always the same and its synonymy with 'negative' is problematic. The evolution of its meaning will be detailed in a forthcoming publication, *The Empty and the Full: Li Ye and the Way of Mathematics* by Pollet.

there should not be a blank space. The reader has to supplement the missing character, i.e. the last expression is to be understood just as any other expression. Li Rui simply negates the distinction between the expressions before and after *xiang xiao*, or between polynomials and equations as modern mathematicians call them.

The absence of *tai* and *yuan* after the *xiang xiao* operations is also clearly acknowledged by Li Rui in the the *Ceyuan haijing*:

> 古人文簡, 不立此法. 既相消後, 即不論天元, 太極等位, [...]. 故相消所得算旁更不記元, 太等字.
>
> The words of the ancients are simple [so they] did not set up this method. Once the operation of *xiang xiao* is performed, [they] do not discuss the positions of *tian yuan* or *tai ji* [...]. Hence, on the sides of the expression after *xiang xiao*, there is no need to record the character *yuan* or *tai*.

We notice that Li Rui and Li Ye make different uses of technical vocabulary. In Li Ye's discourse, *zhen ji* ('real area') refers to a quantity expressing the area in a constant term. This quantity is already known from the statement of the problem. *Xu ji* ('empty area') or in other problems *ru ji* (如積, 'equal area') are used to refer to a polynomial composed of a constant term and two terms of different powers. This polynomial is equal to the area expressed by *zhen ji*. The expression *xu shu* ('empty quantity') never appears in Li Ye's discourse; only Li Rui uses it. Yet, Li Rui applies it to the two terms containing the unknown in the polynomial. *Zhen ji* is used for the constant of the same polynomial. Li Rui and Li Ye do not share the same vocabulary and the same concepts for 'terms' and 'coefficients'. This difference in naming objects shows how the two authors see things differently. It seems evident that Li Rui understands the operations clearly, but he operates with a different conception of mathematical objects. He does not explain the change before and after the 'elimination' operation. Either he does not see the difference between the two mathematical objects before and after, or he prefers not to recognise the change. Recognizing the change would be to admit that there is equality and thus an equation in the sense of the commentator of the *Siku quanshu*. This would lead to a validation of the interpretation made by the other commentator.

Li Rui's commentary did not settle the dispute in East Asia. In fact, it probably provoked more discussions. In the next section we will present the reaction of a Korean scholar, Nam Byeong-gil.

3.3 Nam Byeong-gil's understanding

Nam Byeong-gil was a Confucian astronomer and mathematician in 19th century Joseon Korea. He was not only interested in mathematics *per se*, but also in the application of the study and its popularisation among the noble class *yangban*, the elite of the literati [39]. That is one of the reasons that he also tried to elaborate on the algebraic methods discussed above and criticised previous commentaries. For this paper, we will present only Nam's commentary concerning Li Rui's commentary on Problem One in the *Yigu yanduan*.

Nam Byeong-gil's criticism can be found in his *Mu-i hae*, finished by 1855. Literally 'Solutions of No Differences', the title of this text declares that Nam believed, at least when he was writing it, that there was no difference between *tian yuan shu* and *jie gen fang*. In this text, he brings together a total of only seven problems from the *Yigu yanduan* and the *Ceyuan haijing* along with the commentaries by the editor of *Siku quanshu* and by Li Rui. He gives his comments regarding each problem. His interpretations for Problem One are no less provocative than those of Li Rui.

In the *Mu-i hae*, Nam Byeong-gil has no real argument in constructing and manipulating counting-rod expressions or polynomials before *xian xiao*. The most interesting piece of Nam's commentary on Li Rui's study is on the similarities and differences between 'eliminating from one another' and 'adding or subtracting from the two sides'. Li Rui stresses that after eliminating from one another, there is only one 'remainder', not something equal to another. For this, Nam Byeong-gil says that:

> 凡寄左數與後數, 原是等數, 故相消與加減之後, 皆為開方式也. 相消法使實, 從, 廉, 隅, 務歸一行, 而不辨其肯綮; 加減法使真數與根, 方數, 仍分兩邊, 而昭晰其法理. 縱有詳畧之不同, 若以一行分之, 則為兩邊, 而彼此相等; 以兩邊併之, 則為一行, 而正負相當. 此即一而二, 二而一也. 以等減等, 所餘必等. 何可謂止有減餘, 更不得云彼與此等耶?
>
> The quantity on the left and the quantity that follows are equal, so, after elimination from one another or after addition or subtraction, they are both root extraction expressions. The method of elimination placed the dividend, the edge and the corner on one column without differentiating the key elements. The method of addition or subtraction placed the real number, the root and the powers on two sides, so the principles of the operations are clear. Although there are differences in the detailed or brief [operations], if [one] divides one column, then the two sides are equal, and if [one] combines the two sides, then it is in one column of corresponding positive and negative [terms].[11] This is that one becomes

11) By 'combines the two sides', it seems Nam means that one moves all items from one side of the equality to the other and subtract items of the former from the latter, because there are 'corresponding positive and negative [terms]' (正負相當) that appear after the subtraction. Nam also says 'subtract equals from equals' two sentences later that shows his 'combination' means moving items and subtracting.

> two and two becomes one. Subtract equals from equals, the remainders must be equal, so how can [Li Rui] said that one only has the remainder of a subtraction, and it cannot be said that one thing is equal to another?

Nam Byeong-gil does not devote much space to discussing the uses and omissions of the *yuan* and *tai* characters. On this part, he has no difference with Li Rui, either. However, while Li Rui stresses that the reciprocity only exists in 'elimination from one another', and the remainder of elimination cannot be seen as one thing equal to another, Nam believes that the final steps of the two methods are both 'root extraction expressions' (*gae bang sig* 開方式). There are only *apparent* differences between the two methods, but the *nature* of the two kinds of expressions—the remainder after elimination and the results after addition and subtraction—are the same. This is the key issue with which Nam contests Li Rui's interpretation and is also the main argument in his *Mu-i hae* to show that the two methods have no differences. In fact, Nam makes his disagreement with Li Rui clearly in the preface to his work:

> 天元一術, 即今之借根方法也. 嘉慶間元和李鋭 […] 云: 借根方出於天元術. 其加減乘除之法並同. 惟此相消法與借根方兩邊加減則有異. 此說甚惑矣! 蓋立天元術則相消後歸之一行, 而正負相當借根方法則加減後仍分兩邊, 而彼此相等. 此特殊一行與兩邊也. 且彼此正負因主客而變, 互相往來. 方程篇所謂: '此正則彼負, 彼正則此負' 是也. […] 愚於益古演段及測圓海鏡, 遂卞李鋭不同之案, 名曰: 無異解.
>
> The method of setting up one *tian yuan* in ancient times is just the same as *jie gen fang*. During Jiaqing's reign, Li Rui […] says that *jie gen fang* came from *tian yuan shu* because their ways of doing four basic operations are the same, but only the method of eliminating from one another is different from the addition and subtraction on the two sides in *jie gen fang*. This statement is confusing! To set up *tian yuan*, the terms are placed on one column with positive and negative terms corresponding to each other, and in *jie gen fang* the terms are still separated on the two sides after addition and subtraction with the terms on the two sides being equal to each other. This is a special case with one column and two sides, and the positive and negative signs change with the position of the host and the guest. Terms in one side can go to the other, and terms in the other side can come to this side. This is what [Mei Wending's] *Article of Fang cheng* says—positive on this [side] is negative on that [side] and positive on that [side] is negative on this [side]. […] Therefore, I contest Li Rui's commentary of 'difference' in the *Ceyuan haijing* and the *Yigu yanduan* and call my study the *Mu-i hae* (*Solutions of No Differences*).

Mathematically speaking, what Nam believes about the two algebraic methods is not very different from what a modern mathematician might believe; the two methods have only apparent differences in their procedures and their nature is the same. Where Nam disagrees with Li Rui is not exactly in his interpretation of the two methods *per se*, but his claim that there are *differences* between them. Of course, one can blame Li Rui, working within the atmosphere of Qing evidential studies as the reader has seen, for his political motivation to show their differences so he can say the ancient Chinese method is somehow 'better' for learning mathematics.[12] However, one cannot really say that Nam concerned himself with only mathematics when he wrote his mathematical works. It is because in many other of his studies, Nam tries to use knowledge from the *Shuli jingyun* (數理精蘊, *Essentials of Mathematical Principles*, 1723), the contemporary mathematical canon in Qing China and Joseon Korea in order to explain the republished ancient texts such as the *Jiuzhang suanshu*.[13] Moreover, the reader can see from his final comments on Problem One in the *Yigu yanduan* that Nam solves the whole problem again with *jie gen fang*, the algebraic method used in the *Shuli jingyun* and does it for every other problem in the *Mu-i hae*. If the methods were really identical, he would not need to solve them again. Still, in the *Mu-i hae*, Nam implicitly tries to elevate the epistemological status of *jie gen fang* by writing the final words for each problem with that 'Western' method. The most probable reason for this is that Nam believes contemporary mathematical knowledge should be the standard method of problem-solving, although mathematically the ancient method is 'not different'. Nam Byeong-gil did modify his view on the two algebraic methods later in his life after he studied *si yuan shu*, a method generalised from *tian yuan shu*. Without openly criticising *jie gen fang* in the imperial canon, he used *tian yuan shu* to solve a broader range of problems and believed that it bore a distinctive mark of Korean tradition [40]. Nevertheless, his criticism to Li Rui stands. As far as the 'construction of quadratic equations with one unknown' is concerned, Nam believed that Li Rui had been wrong and the *tian yuan shu* and *jie gen fang* essentially had no difference.

12) For the evidential scholarship in the Qing dynasty, refer to [11].

13) An analysis of Nam's commentary on the *Jiuzhang suanshu* can be found in [39]. The *Shuli jingyun* is the mathematical part of the *Lüli yuanyuan* 律曆淵源 (*Source of Pitch-pipes and the Calendar*, 1723), a great compendium of mathematics, astronomy, and music. Since this compendium was 'imperially composed' (*yuzhi* 御製), the knowledge in it was considered official by the Qing government and many contemporaneous scholars. See [13, Vol.3]. Nam tried to promote the knowledge in the *Shuli jingyun* also because his country was a tributary state of Qing so it had to use the imperial calendar, which was constructed with the mathematics in the *Lüli yuanyuan*. In order for Korean scholars in the 19th century to understand the principles behind the imperial calendar, they must possess the mathematical knowledge in the *Shuli jingyun*. This can also be seen as a political motive for Nam's studies.

4 Concluding remarks

A mathematical object can have several interpretations. Here, the object recorded in modern terms by $ax^2 + bx + c = 0$ was first considered a procedure by Li Ye in the 13th century. In the 18th century, commentators who were also responsible for the re-publication of Li Ye's studies thought differently. The editor of the *Siku quanshu* saw in this object an equation as an equality of polynomials. This interpretation was rejected by another editor, Li Rui, who was famous for his efforts to revitalise and 'sinicize' the classics. According to him, the mathematical expressions used by Li Ye are to be understood in the frame of a procedure: they are the remainder of a series of operations involved in the extraction of square roots. Li Rui seemed to be closer to Li Ye's interpretation, and yet, he did not see or he did not want to admit he saw the change of 'status' of the expression at the end of the procedure. This contrasts with Li Ye's interpretation, which makes a conceptual difference between polynomials and equations. While Li Rui understands the procedural concept of 'equation', he has another conception of object involved in the operations.

A few decades later, the Korean mathematician Nam Byeong-gil published another commentary on the same problem. Neglecting the status of mathematical objects, he focuses on the procedural aspects. He reinterprets the ancient problem with his contemporaneous terms and shows that they are not two different procedures. There are no differences between the procedure promoted by the editor of the *Siku quanshu* and that favoured by Li Rui. The interpretation of procedures and the understanding of mathematical objects vary according to contemporary considerations. None of the editors of Li Ye's study are immune to discussions related to historical contexts. Their conceptions of the procedure are reflected naturally in their commentaries, and also in the way they transcribe mathematical components in equations and polynomials.

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