J. Korean Math. Soc. **54** (2017), No. 4, pp. 1331–1343
https://doi.org/10.4134/JKMS.j160508
pISSN: 0304-9914 / eISSN: 2234-3008

SLANT HELICES IN THE THREE-DIMENSIONAL SPHERE

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Abstract. A curve γ immersed in the three-dimensional sphere $\mathbb{S}^3$ is said to be a slant helix if there exists a Killing vector field $V(s)$ with constant length along γ and such that the angle between V and the principal normal is constant along γ. In this paper we characterize slant helices in $\mathbb{S}^3$ by means of a differential equation in the curvature κ and the torsion τ of the curve. We define a helix surface in $\mathbb{S}^3$ and give a method to construct any helix surface. This method is based on the Kitagawa representation of flat surfaces in $\mathbb{S}^3$. Finally, we obtain a geometric approach to the problem of solving natural equations for slant helices in the three-dimensional sphere. We prove that the slant helices in $\mathbb{S}^3$ are exactly the geodesics of helix surfaces.

1. Introduction

A curve γ in $\mathbb{R}^3$ with curvature $\kappa > 0$ is called a slant helix if the principal normal lines of γ make a constant angle with a fixed direction, [8]. Observe that the principal normal lines of a general helix (also called a Lancret curve, [1]) is perpendicular to a fixed direction, so that a general helix is also a slant helix. A classical result stated by M. A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 is ([16, p. 34]): a necessary and sufficient condition for a curve γ be a general helix is that the ratio τ/κ of torsion to curvature be constant, where κ and τ stand for the curvature and torsion of γ, respectively. Izumiya and Takeuchi [8] have shown a similar result for slant helices: a necessary and sufficient condition for a curve with $\kappa > 0$ be a slant helix is that the function

$$\sigma = \frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$$

be constant. Observe that $\sigma \equiv 0$ implies that α is a general helix. Somehow, $|\sigma|$ measures how much the curve moves away from being a general helix, and $|\sigma'|$ measures how much the curve moves away from being a slant helix.

The first attempt to extend the Lancret theorem to other ambient spaces is due to M. Barros in [1]. He obtained a theorem of Lancret for general helices

Received July 30, 2016.
2010 *Mathematics Subject Classification.* 53B25, 53B20.
Key words and phrases. slant helix, 3-sphere, helix surface, Killing field, Hopf field.

in a 3-dimensional real space form which gives a relevant difference between hyperbolic and spherical geometries. The first task was to give a consistent definition of general helix in a 3-dimensional real space form and for this goal the author used the concept of Killing vector field along a curve (this concept was introduced by J. Langer and D. A. Singer, [10, 11]). Barros showed that a curve γ in the unit 3-sphere $\mathbb{S}^3$ is a general helix if and only if either (i) $\tau \equiv 0$ and γ is a curve in some unit 2-sphere $\mathbb{S}^2$, or (ii) there exists a constant b such that $\tau = b\kappa \pm 1$. As for curves in the unit hyperbolic space $\mathbb{H}^3$, it is shown that γ is a general helix if and only if either (i) $\tau \equiv 0$ and γ is a curve in some unit hyperbolic plane $\mathbb{H}^2$, or (ii) γ is a helix in $\mathbb{H}^3$. Note that the spherical case is nicely analogous to the Euclidean case, whereas in the hyperbolic case there are only trivial general helices (i.e., plane curves and ordinary helices).

This work is inspired by the papers of M. Barros [1], and S. Izumiya and N. Takeuchi [8]. We use the concept of Killing vector field along a curve γ in the 3-dimensional sphere $\mathbb{S}^3$ to define the concept of slant helix in $\mathbb{S}^3$. Of course this is the natural extension of that for slant helices in the Euclidean 3-space. A curve γ immersed in the three-dimensional sphere $\mathbb{S}^3$ will be called a slant helix if there exists a Killing vector field $V(s)$ with constant length along γ and such that the angle θ between V and the principal normal vector N of γ is constant. V is said to be an axis of the slant helix. Trivial examples of slant helices in $\mathbb{S}^3$ are plane curves (for $\theta = 0$) and general helices (for $\theta = \pi/2$), so that we assume $\theta \in (0, \pi/2)$. We obtain the following characterization of slant helices in $\mathbb{S}^3$ (Theorem 2):

Let γ be a unit speed curve in the 3-sphere $\mathbb{S}^3$, with curvature $\kappa > 0$ and torsion τ. The curve γ is a slant helix if and only if there are two constants λ and θ, with $\lambda^2 = 1$ and $\theta \in (0, \pi/2)$, such that

$$\frac{\kappa^2}{\left(\kappa^2 + (\tau - \lambda)^2\right)^{3/2}} \left(\frac{\tau - \lambda}{\kappa}\right)' = \cot\theta.$$

In this case, an axis of the slant helix is given by

$$V = \frac{\tau - \lambda}{\sqrt{\kappa^2 + (\tau - \lambda)^2}} \sin\theta\, T + \cos\theta\, N + \frac{\kappa}{\sqrt{\kappa^2 + (\tau - \lambda)^2}} \sin\theta\, B.$$

($\{T, N, B\}$ is the Frenet frame of the curve.)

In the case of curves of the unit hyperbolic space $\mathbb{H}^3$, we show that the only slant curves are the trivial ones (i.e., plane curves and helices). On the other hand, M. Barros also obtained a geometric approach to the problem of solving natural equations for general helices in the 3-dimensional sphere. He proved that a fully immersed curve in $\mathbb{S}^3$ is a general helix if and only if it is a geodesic in some Hopf cylinder in $\mathbb{S}^3$. To extend this result to slant helices in the 3-sphere we need to work with helix surfaces. To define helix surfaces in $\mathbb{S}^3$ we use the model of quaternions, in which $\mathbb{S}^3$ is identified with the set of unit quaternions. In this model, the Hopf fibration $h : \mathbb{S}^3 \to \mathbb{S}^2$ is given by

$h(p) = p\mathbf{i}\bar{p}$ (see Section 2 for details). A surface $M^2 \subset \mathbb{S}^3$, with unit normal vector field η, will be called a helix surface if there is a Hopf vector field V such that the angle between η and V is constant along M. As an example of helix surface we have any Hopf cylinder, for which the angle between η and V is $\pi/2$. The first property of this family of surfaces is stated in Section 4: *A helix surface M_ϕ in the 3-dimensional sphere $\mathbb{S}^3$ is flat.* At this point, the construction of flat surfaces given by Kitagawa [9] (see also [6, 7]) allow us to describe the family of helix surfaces (Theorem 7):

Let $M^2 \subset \mathbb{S}^3$ be an orientable surface. Then M^2 is a helix surface if and only if M^2 is a flat surface and, according to Kitagawa's representation, $c_1(u)$ is a general helix in $\mathbb{S}^2 \subset \mathbb{R}^3$.

The construction of flat surfaces in $\mathbb{S}^3$ is recalled in Theorem 6.

Finally, in Section 5 we study the problem of solving natural equations for slant helices in the 3-sphere. We prove the following result (Theorem 9):

Let $\gamma(s)$ be a unit speed curve in $\mathbb{S}^3$, with $\kappa > 0$. Then γ is a slant helix of constant angle θ if and only if γ is locally congruent to a geodesic of a helix surface M_θ.

2. Preliminaries

Let $\mathbb{S}^3$ denote the 3-dimensional unit sphere in $\mathbb{R}^4$, with constant curvature $c = 1$. Let us consider $\gamma = \gamma(t) : I \subset \mathbb{R} \to \mathbb{S}^3$ an immersed curve in $\mathbb{S}^3$ with speed $v(t) = |\gamma'(t)|$. If $\{T, N, B\}$ is the Frenet frame along γ and $\overline{\nabla}$ denotes the Levi-Civita connection of $\mathbb{S}^3$, then one can write the Frenet equations of γ as follows

$$\overline{\nabla}_T T = \kappa N, \quad \overline{\nabla}_T N = -\kappa T + \tau B, \quad \overline{\nabla}_T B = -\tau N, \tag{1}$$

where κ and τ denote the curvature and torsion of γ, respectively. If ∇^0 stands for the Levi-Civita connection of $\mathbb{R}^4$, then the Gauss formula gives

$$\nabla^0_T X = \overline{\nabla}_T X - c\,\langle X, T\rangle\,\gamma$$

for any tangent vector field $X \in \mathfrak{X}(\gamma)$. In particular, we have

$$\nabla^0_T T = \kappa N - c\gamma, \quad \nabla^0_T N = -\kappa T + \tau B, \quad \nabla^0_T B = -\tau N. \tag{2}$$

We consider variations $\Gamma = \Gamma(t, z) : I \times (-\varepsilon, \varepsilon) \to \mathbb{S}^3$, with $\Gamma(t, 0) = \gamma(t)$, whose variational vector field along γ is given by $V(t) = \frac{\partial \Gamma}{\partial z}(t, 0)$. We will use the notation $V = V(t, z)$, $T = T(t, z)$, $v = v(t, z)$, etc. with the obvious meaning. Finally we use s to denote the arclength parameter of the t-curves in the variation Γ and write $v(s, z)$, $\kappa(s, z)$, $\tau(s, z)$, etc. for the corresponding reparametrizations. According to [10], a vector field $V(s)$ along $\gamma(s)$ is said to be a Killing vector field along γ if

$$\left.\frac{\partial v}{\partial z}\right|_{z=0} = \left.\frac{\partial \kappa^2}{\partial z}\right|_{z=0} = \left.\frac{\partial \tau^2}{\partial z}\right|_{z=0} = 0. \tag{3}$$

By using standard arguments (see [10] for details) we deduce that Killing vector fields along γ are characterized by the following equations:

$$\left.\frac{\partial v}{\partial z}\right|_{z=0} = \left\langle \overline{\nabla}_T V, T \right\rangle v = 0, \tag{4}$$

$$\left.\frac{\partial \kappa^2}{\partial z}\right|_{z=0} = 2\kappa \left\langle \overline{\nabla}_T^2 V, N \right\rangle - 4\kappa^2 \left\langle \overline{\nabla}_T V, T \right\rangle + 2c\kappa \left\langle V, N \right\rangle = 0, \tag{5}$$

$$\begin{aligned} \left.\frac{\partial \tau^2}{\partial z}\right|_{z=0} &= \frac{2\tau}{\kappa} \left\langle \overline{\nabla}_T^3 V, B \right\rangle - \frac{2\kappa'\tau}{\kappa^2} \left\langle \overline{\nabla}_T^2 V + cV, B \right\rangle \\ &\quad + \frac{2\tau(c+\kappa^2)}{\kappa} \left\langle \overline{\nabla}_T V, B \right\rangle - 2\tau^2 \left\langle \overline{\nabla}_T V, T \right\rangle = 0, \end{aligned} \tag{6}$$

where $\langle , \rangle$ denotes the standard metric of the 3-sphere and $\kappa' = \frac{\partial \kappa}{\partial s}(s, 0)$. An interesting property (see [10, 11]) is that a vector field V along γ is a Killing vector field along γ if and only if it extends to a Killing field on $\mathbb{S}^3$. This property is also true for immersed curves in a complete, simply connected real space form.

Inspired by the definition of general helix in $\mathbb{S}^3$ given by M. Barros [1], we present here the following definition.

Definition 1. A curve γ immersed in the three-dimensional sphere $\mathbb{S}^3$ will be called a *slant helix* if there exists a Killing vector field $V(s)$ along γ with constant length, whose extension to $\mathbb{S}^3$ is also of constant length, and such that the angle θ between V and the normal vector N is a constant along γ. We will say that V is an axis of the slant helix γ.

Without loss of generality, we may assume that V is a unit Killing vector field. Trivial examples of slant helices in $\mathbb{S}^3$ are plane curves (i.e., curves with $\tau \equiv 0$) for $\theta = 0$, and general helices for $\theta = \pi/2$. Without loss of generality, from now on we assume that $\theta \in (0, \pi/2)$.

A useful model for the unit 3-sphere is to regard $\mathbb{R}^4$ as the set of quaternions $\mathbb{H}$ and $\mathbb{S}^3$ as the set of unit quaternions:

$$\mathbb{S}^3 = \{q = q_1\mathbf{1} + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k} \in \mathbb{H} : |q|^2 = q\bar{q} = 1\}.$$

Here, $\bar{q} = q_1\mathbf{1} - q_2\mathbf{i} - q_3\mathbf{j} - q_4\mathbf{k}$ is the conjugate of q. In this setting, the 2-sphere is identified with the set of unit pure quaternions. One of the advantages of this model is that $\mathbb{S}^3$ is endowed with a Lie group structure. The usual metric of $\mathbb{S}^3$ is bi-invariant with respect to this structure (i.e., the left and right translations turn out to be isometries), and then

$$\langle p, q\rangle = \langle pa, qa\rangle = \langle ap, aq\rangle \quad \text{for any } p, q, a \in \mathbb{S}^3.$$

The conjugation is an orientation reversing isometry of $\mathbb{S}^3$. Moreover, $\bar{q} = q^{-1}$ whenever $q \in \mathbb{S}^3$, and in addition $\bar{q} = -q$ if $q \in \mathbb{S}^2 \subset \mathbb{S}^3$.

The classic Hopf fibration has a nice description via the quaternions. Let us define $\mathrm{Ad}(q)p := qp\bar{q}$, where $p, q \in \mathbb{S}^3$, then the Hopf fibration $h : \mathbb{S}^3 \to \mathbb{S}^2$

is given by $h(q) = \mathrm{Ad}(q)\mathbf{i} = q\mathbf{i}\bar{q}$. Associated to this fibration we have the canonical Hopf vector field in $\mathbb{S}^3$, defined by $V_1(q) = \mathbf{i}q$, $q \in \mathbb{S}^3$. V_1 is a unit Killing vector field tangent to the fibers of h. In general, a Hopf vector field is any vector field V in $\mathbb{S}^3$ congruent to V_1. It is a well known result that the unit Killing vector fields in $\mathbb{S}^3$ are exactly the Hopf vector fields, [17, Theorem 1]. V_1 is a vertical vector field and can be completed to a basis of $\mathbb{S}^3$ with two horizontal unit vector fields V_2 and V_3.

The Hopf fibration is a particular case of Killing submersion $\pi : \mathcal{M}^3 \to \mathbb{M}^2$, [5], whose fibers are the trajectories of a unit Killing vector field. By using Proposition 2.6 and Lemma 2.8 of [5] we get

$$\overline{\nabla}_X V_1 = \lambda\, X \times V_1 \quad \text{for any } X \in \mathfrak{X}(\mathbb{S}^3), \tag{7}$$

where λ is a constant such that $\lambda^2 = c$.

A surface M of the Euclidean space $\mathbb{R}^3$ is said to be a *helix surface* if there is a fixed direction $\mathbf{u} \in \mathbb{R}^3$ such that the tangent space of the surface makes a constant angle with $\mathbf{u}$, [4]. This definition is equivalent to the fact that the angle between the unit normal vector η and $\mathbf{u}$ is a constant function along M, i.e., M is a constant angle surface (see e.g. [3, 13] and references therein). Since a fixed direction in the Euclidean space can be regarded as a Killing vector field of constant length (we may assume of unit length), the following definition is natural, [12].

Definition 2. A surface M immersed in the three-dimensional sphere $\mathbb{S}^3$, with unit normal vector field η, will be called a *helix surface* if there is a unit Killing vector field (i.e., a Hopf vector field) V such that the angle ϕ between η and V (i.e., $\langle V, \eta\rangle = \cos\phi$) is a constant along M.

A helix surface $M \subset \mathbb{S}^3$ of angle ϕ will be denoted by M_ϕ. Without loss of generality, we can assume $\phi \in [0, \pi/2]$ and that V is the canonical Hopf vector field V_1. The cases $\phi = 0, \pi/2$ can be discarded, [12]. If $\phi = 0$, then the vector fields V_2 and V_3 would be tangent to the surface M, but the horizontal distribution of the Hopf fibration is not integrable; this is a contradiction. On the other hand, if $\phi = \pi/2$, then V_1 is always tangent to M and hence M is a Hopf tube, [14]. Therefore, from now on we assume that the constant angle $\phi \in (0, \pi/2)$.

3. Slant helices

Let $\gamma(s)$ be a slant helix with axis $V(s)$, and let us denote by $\theta \in (0, \pi/2)$ the constant angle between V and N. Then

$$V(s) = \sin\theta\, \sin\varphi(s)\, T(s) + \cos\theta\, N(s) + \sin\theta\, \cos\varphi(s)\, B(s) \tag{8}$$

for a certain differentiable function $\varphi(s)$. We can assume that $V(s)$ is the restriction to $\gamma(s)$ of the Hopf vector field $V_1 \in \mathfrak{X}(\mathbb{S}^3)$.

From (8), and by using the Frenet equations of γ, we obtain

$$\overline{\nabla}_T V = (\varphi' \sin\theta \cos\varphi - \kappa\cos\theta)T + \sin\theta(\kappa\sin\varphi - \tau\cos\varphi)N$$

$$+ (\tau \cos\theta - \varphi' \sin\theta \sin\varphi)B,$$

and then (4) leads to

$$\kappa = \tan\theta\, \varphi' \cos\varphi, \tag{9}$$

$$\overline{\nabla}_T V = g\,(-\sin\theta\, \cos\varphi\, N + \cos\theta\, B), \tag{10}$$

where $g = \tau - \varphi' \tan\theta \sin\varphi$. Bearing (9) in mind, we easily obtain the following equation

$$\frac{\kappa^2}{\left(\kappa^2 + (\tau - g)^2\right)^{3/2}} \left(\frac{\tau - g}{\kappa}\right)' = \cot\theta. \tag{11}$$

A direct computation from (8), and bearing (10) in mind, yields

$$\overline{\nabla}_T V = g\, T \times V. \tag{12}$$

From here and (7) we deduce that the function g is constant, with $g^2 = c$. Therefore, bearing (11) in mind, we have shown the following result.

Proposition 1. *Let γ be a unit speed curve in the 3-sphere $\mathbb{S}^3$, with curvature $\kappa > 0$ and torsion τ. If γ is a slant helix, then the following equation holds*:

$$\frac{\kappa^2}{\left(\kappa^2 + (\tau - \lambda)^2\right)^{3/2}} \left(\frac{\tau - \lambda}{\kappa}\right)' = \cot\theta, \tag{13}$$

where λ and θ are constant, with $\lambda^2 = c$ and $\theta \in (0, \pi/2)$.

We will now prove that equation (13) characterize the slant curves in $\mathbb{S}^3$. Suppose that γ is a unit speed curve satisfying (13). By taking derivative there we obtain

$$\cos\theta = \frac{\tau'\kappa - (\tau - \lambda)\kappa'}{\left(\kappa^2 + (\tau - \lambda)^2\right)^{3/2}} \sin\theta. \tag{14}$$

Define the following vector field along γ:

$$V = \frac{\tau - \lambda}{\sqrt{\kappa^2 + (\tau - \lambda)^2}} \sin\theta\, T + \cos\theta\, N + \frac{\kappa}{\sqrt{\kappa^2 + (\tau - \lambda)^2}} \sin\theta\, B. \tag{15}$$

Note that $|V| = 1$ and $\langle V, N\rangle = \cos\theta$ is constant. Let us see now that V is a Killing vector field along γ. A straightforward computation taking into account the above equations yields:

$$\overline{\nabla}_T V = -\frac{\lambda\kappa \sin\theta}{\sqrt{\kappa^2 + (\tau - \lambda)^2}} N + \lambda \cos\theta\, B, \tag{16}$$

$$\overline{\nabla}^2_T V = \frac{\lambda\kappa^2 \sin\theta}{\sqrt{\kappa^2 + (\tau - \lambda)^2}} T - \lambda^2 \cos\theta\, N - \frac{\lambda\kappa\tau \sin\theta}{\sqrt{\kappa^2 + (\tau - \lambda)^2}} B, \tag{17}$$

$$\overline{\nabla}^3_T V = \left(\frac{\lambda\kappa\kappa' \sin\theta}{\sqrt{\kappa^2 + (\tau - \lambda)^2}} + \lambda\kappa(2\lambda - \tau)\cos\theta\right) T + \frac{\lambda\kappa(\kappa^2 + \tau^2) \sin\theta}{\sqrt{\kappa^2 + (\tau - \lambda)^2}} N$$

$$+\left(\lambda\tau(\tau-2\lambda)\cos\theta-\frac{\lambda\kappa\tau'\sin\theta}{\sqrt{\kappa^2+(\tau-\lambda)^2}}\right)B. \tag{18}$$

It is not difficult to see that V satisfies the equations (4)–(6). Therefore, γ is a slant helix in $\mathbb{S}^3$ with axis V. This result and Proposition 1 can be put together as follows.

Theorem 2. *Let γ be a unit speed curve in the 3-sphere $\mathbb{S}^3$, with curvature $\kappa > 0$ and torsion τ. The curve γ is a slant helix if and only if there are two constants λ and θ, with $\lambda^2 = c$ and $\theta \in (0, \pi/2)$, such that*

$$\frac{\kappa^2}{\left(\kappa^2+(\tau-\lambda)^2\right)^{3/2}}\left(\frac{\tau-\lambda}{\kappa}\right)'=\cot\theta. \tag{19}$$

In this case, an axis of the slant helix is given by

$$V=\frac{\tau-\lambda}{\sqrt{\kappa^2+(\tau-\lambda)^2}}\sin\theta\, T+\cos\theta\, N+\frac{\kappa}{\sqrt{\kappa^2+(\tau-\lambda)^2}}\sin\theta\, B.$$

Remark 1. In the Euclidean space $\mathbb{R}^3$, the constant λ vanishes, and therefore the equation (19) is exactly the same that appears in [8].

3.1. Slant helices in the 3-dimensional hyperbolic space

Let $\mathbb{H}^3$ denote the 3-dimensional hyperbolic space of constant curvature $c = -1$. The notion of a slant helix γ in $\mathbb{H}^3$ can be defined similarly as in $\mathbb{S}^3$ (see Definition 1); in this case, a Killing vector field along γ satisfies the equations (4)–(6), with $c = -1$.

The condition $\lambda^2 = c$ in Theorem 2 suggests that probably there are no slant helices in the 3-dimensional hyperbolic space $\mathbb{H}^3$, other than plane curves (i.e., $\tau \equiv 0$) and ordinary helices (i.e., κ and τ are nonzero constants). This is compatible with the results obtained in [1], where the author shows that these curves are the only general helices in the 3-dimensional hyperbolic space $\mathbb{H}^3$. And certainly this is what happens.

In [2, Proposition 3], the authors show that on a Riemannian manifold of nonpositive sectional curvature, the only Killing fields of constant length are the parallel ones. As a consequence, the only unit Killing vector fields along a slant curve γ in $\mathbb{H}^3$ are the parallel ones. Then, if $V(s)$ is given by (8), the equations (4)–(6) yields

$$c\kappa\cos\theta = 0,$$
$$c\kappa'\tau\sin\theta\cos\varphi = 0.$$

Therefore, we have shown the following result.

Theorem 3 (cf. Theorem 1 of [1])**.** *A curve γ in $\mathbb{H}^3$ is a slant helix if and only if either*

(1) *$\tau \equiv 0$ and γ is a curve in some unit hyperbolic plane $\mathbb{H}^2 \subset \mathbb{H}^3$, or*
(2) *γ is a helix in $\mathbb{H}^3$.*

4. Helix surfaces

In this section we will find a useful representation of the helix surfaces in $\mathbb{S}^3$.

Let M_ϕ be an oriented helix surface in $\mathbb{S}^3$ of constant angle ϕ, with unit normal vector field η. Then we have

$$\langle V_1, \eta\rangle = \cos\phi.$$

Decomposing V_1 in its tangential and normal components, we have

$$V_1 = \sin\phi\, T_V + \cos\phi\, \eta, \tag{20}$$

where T_V is a unit tangent vector field, $T_V \in \mathfrak{X}(M_\phi)$. From Gauss and Weingarten formulae we have

$$\overline{\nabla}_X V_1 = (\sin\phi\, \nabla_X T_V - \cos\phi\, A_\eta X) + \sin\phi\, \langle A_\eta T_V, X\rangle\, \eta, \quad X \in \mathfrak{X}(M_\phi), \tag{21}$$

where ∇ stands for the induced Levi-Civita connection on M_ϕ and A_η represents the shape operator of the surface in the 3-sphere.

Bearing (20) in mind, compute $\left\langle \overline{\nabla}_X V_1, \eta\right\rangle$ by using (21) and also by using (7). Then by equating the two equations we deduce

$$A_\eta T_V = \lambda\, T_V \times \eta = -\lambda\, RT_V, \tag{22}$$

where R denotes the positive rotation of angle $\pi/2$ in the tangent space. Then the shape operator A_η can be written, with respect to the basis $\{T_V, RT_V\}$, as

$$A_\eta = \begin{pmatrix} 0 & -\lambda \\ -\lambda & \psi \end{pmatrix}, \tag{23}$$

for a certain function $\psi = \langle A_\eta(RT_V), RT_V\rangle$.

By equating the tangential component of $\overline{\nabla}_{T_V} V_1$ computed by using (7) and (21) we deduce

$$\sin\phi\, \langle \nabla_{T_V} T_V, RT_V\rangle = -2\lambda\cos\phi.$$

This equation leads to the following characterization of T_V.

Proposition 4. *Let M_ϕ be a helix surface in $\mathbb{S}^3$ of angle ϕ. Then the integral curves of the vector field T_V are exactly the curves with constant geodesic curvature $\kappa_g = -2\lambda\cot\phi$.*

Bearing in mind that the Gauss curvature of M_ϕ is given by $K = c+\det(A_\eta)$, the following result is clear.

Proposition 5. *A helix surface M_ϕ in the 3-dimensional sphere $\mathbb{S}^3$ is flat.*

The construction of flat surfaces in $\mathbb{S}^3$ is essentially given by Kitagawa, [9]. We expose the method following [6] (see also [7]). Let us consider $U\mathbb{S}^2$ the unit tangent bundle of $\mathbb{S}^2$, given by

$$U\mathbb{S}^2 = \{(p, q) \in \mathbb{S}^2 \times \mathbb{S}^2 : \langle p, q\rangle = 0\}.$$

Theorem 6. *Let $c_1(u)$, $c_2(v)$ be two regular curves in $\mathbb{S}^2$, with $c_i(0) = \mathbf{i}$, $c_i'(0) = \xi_0$, for some $\xi_0 \in \mathbb{S}^3$ orthogonal to both $\mathbf{1}$ and $\mathbf{i}$, and such that $k_1(u) \neq k_2(v)$ for all u, v, where k_i denotes the geodesic curvature of c_i, $i = 1, 2$. Let $\pi : \mathbb{S}^3 \to U\mathbb{S}^2$ be the double cover given by*

$$\pi(q) = (\mathrm{Ad}(q)\mathbf{i}, \mathrm{Ad}(q)\xi_0).$$

Let us consider $a_1(u)$, $a_2(v)$ two curves in $\mathbb{S}^3$ parametrized by the arclength and satisfying $\pi(a_i) = (c_i, c_i'/||c_i'||)$, and define

$$\begin{aligned}\Phi(u,v) &= a_1(u)\,\overline{a_2}(v),\\ \eta(u,v) &= a_1(u)\,\xi_0\,\overline{a_2}(v),\end{aligned}$$

on a rectangle $\Omega \subset \mathbb{R}^2$. Then $\Phi(\Omega)$ is a simply connected flat surface in $\mathbb{S}^3$ with unit normal η. Conversely, every flat surface in $\mathbb{S}^3$ can be locally constructed in this way for some ξ_0.

Remark 2. If the flat surface is analytic, or complete with bounded mean curvature, then the construction given in Theorem 6 is global, [6].

The next step is to determine which flat surfaces in $\mathbb{S}^3$ are helix surfaces. Take a local parametrization $\Phi(u, v)$ of a flat surface, with unit normal vector field $\eta(u, v)$, for some ξ_0. Let $\mathbf{q} \in \mathbb{S}^2$ and consider a Hopf vector field V given by $V(x) = \mathbf{q}x$, $x \in \mathbb{S}^3$. Then

$$\begin{aligned}\langle V, \eta\rangle (u,v) &= \langle \mathbf{q}\, a_1(u)\,\overline{a_2}(v), a_1(u)\,\xi_0\,\overline{a_2}(v)\rangle\\ &= \langle \mathbf{q}\, a_1(u), a_1(u)\,\xi_0\rangle = \langle \mathbf{q}, a_1(u)\,\xi_0\,\overline{a_1}(u)\rangle\\ &= \langle \mathbf{q}, T_{c_1}(u)\rangle,\end{aligned}$$

where $T_{c_1}(u)$ denotes the unit tangent vector to the curve c_1. This equation yields the following result (cf. [12, Theorem 3.1]).

Theorem 7. *Let $M^2 \subset \mathbb{S}^3$ be an orientable surface. Then M^2 is a helix surface if and only if M^2 is a flat surface and, according to Kitagawa's representation, $c_1(u)$ is a general helix in $\mathbb{S}^2 \subset \mathbb{R}^3$.*

Example 1 (Spherical general helices)**.** Struik have shown that a spherical helix making an angle θ with his axis projects on a plane perpendicular to its axis in an arc of an epicycloid with fixed radius $a = \cos\theta$ and rolling radius $b = \sin^2(\theta/2)$, [16, p. 35]. Let $c_1(u)$ be a helix in $\mathbb{S}^2$ of angle θ with axis $\mathbf{k}$. Then a parametrization of $c_1(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$ is given as follows (see e.g. [15]),

$$\begin{aligned}x(u) &= \frac{1+\cos\theta}{2}\cos(u+u_0+\alpha) - \frac{1-\cos\theta}{2}\cos\left(\frac{1+\cos\theta}{1-\cos\theta}\,(u+u_0)+\alpha\right),\\ y(u) &= \frac{1+\cos\theta}{2}\sin(u+u_0+\alpha) - \frac{1-\cos\theta}{2}\sin\left(\frac{1+\cos\theta}{1-\cos\theta}\,(u+u_0)+\alpha\right),\\ z(u) &= \sin\theta\cos\left(\frac{\cos\theta}{1-\cos\theta}(u+u_0)\right),\end{aligned}$$

α and u_0 being constants; the angle α represents a rotation about $\mathbf{k}$-axis. These constants can be chosen in such a way that $c_1(0) = \mathbf{i}$:

$$u_0 = \frac{\pi}{2}\frac{1-\cos\theta}{\cos\theta} \qquad \text{and} \qquad \alpha = -\frac{\pi}{2}\frac{1+\cos\theta}{\cos\theta}.$$

The arc parameter s and the radius of curvature $R = 1/\kappa$ of the epicycloid $(x(u), y(u))$ are given by [16, p. 27]

$$s(u) = \frac{4b(a+b)}{a}\cos\left(\frac{au}{2b}\right) \quad \text{and} \quad R(u) = \frac{4b(a+b)}{a+2b}\sin\left(\frac{au}{2b}\right).$$

5. Slant helices and helix surfaces

Let M_ϕ be a helix surface in $\mathbb{S}^3$ of constant angle ϕ, and consider $\gamma(s) = \Phi(u(s), v(s))$ a unit speed geodesic of M_ϕ. Then the unit normal vector field η of M_ϕ in $\mathbb{S}^3$ and the principal normal $N(s)$ of γ are collinear; let us assume that $\eta = N$ along the curve γ. Let $V(s)$ be the restriction of V_1 to the curve γ, which is a Killing vector field along γ. Then we have

$$\langle N, V\rangle(s) = \langle \eta, V_1\rangle(u(s), v(s)) = \cos\phi,$$

showing that γ is a slant helix. In the rest of the section we prove the converse, i.e., that every slant curve is a geodesic of a certain helix surface.

The first task is to compute the ODE system that characterizes the geodesic curves of a helix surface. According to Theorem 6, let $\Phi(u, v) = a_1(u)\,\overline{a_2}(v)$ be a helix surface M_ϕ of angle ϕ, with unit normal vector field given by $\eta(u, v) = a_1(u)\,\xi_0\,\overline{a_2}(v)$, for some $\xi_0 \in \mathbb{S}^3$ orthogonal to both $\mathbf{1}$ and $\mathbf{i}$. The first and second fundamental forms of M_ϕ are given, in (u, v)-coordinates, by the following

$$\begin{aligned} \mathrm{I} &= \mathrm{d}u^2 + 2\cos\omega\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}v^2, \\ \mathrm{II} &= 2\sin\omega\,\mathrm{d}u\,\mathrm{d}v, \end{aligned} \tag{24}$$

for a certain differentiable function $\omega \equiv \omega(u, v)$. For a flat immersion in $\mathbb{S}^3$ the function ω is usually called the angle function of the immersion and the Gauss equation of the surface implies $\omega_{uv} = 0$. Then, locally, we can decompose ω as a sum $\omega(u, v) = \omega_1(u) + \omega_2(v)$, where ω_i are differentiable functions, [6, 9].

Let $\gamma(s) = \Phi(u(s), v(s))$ be a unit speed geodesic in M_ϕ, then we have

$$T(s) = u'(s)\Phi_u(u(s), v(s)) + v'(s)\Phi_v(u(s), v(s)), \tag{25}$$

and therefore

$$u'(s)^2 + v'(s)^2 + 2u'(s)v'(s)\cos\omega(u(s), v(s)) = 1. \tag{26}$$

By taking derivative in (25), and using the Frenet equations of γ, we obtain

$$\begin{aligned} -\gamma(s) + \kappa(s)N(s) = {} & u''(s)\Phi_u(u(s), v(s)) + v''(s)\Phi_v(u(s), v(s)) \\ & + u'(s)^2\,\Phi_{uu}(u(s), v(s)) + v'(s)^2\,\Phi_{vv}(u(s), v(s)) \\ & + 2u'(s)v'(s)\Phi_{uv}(u(s), v(s)), \end{aligned} \tag{27}$$

where κ denotes the curvature of γ. On the other hand, bearing (24) in mind, it is not difficult to see that

$$\Phi_{uu}(u,v) = -\kappa_1(u)\cot\omega(u,v)\,\Phi_u(u,v) + \frac{\kappa_1(u)}{\sin\omega(u,v)}\,\Phi_v(u,v) - \Phi(u,v), \tag{28}$$

$$\Phi_{vv}(u,v) = \frac{\kappa_2(v)}{\sin\omega(u,v)}\,\Phi_u(u,v) - \kappa_2(v)\cot\omega(u,v)\,\Phi_v(u,v) - \Phi(u,v), \tag{29}$$

$$\Phi_{uv}(u,v) = \sin\omega(u,v)\;\eta(u,v) - \cos\omega(u,v)\;\Phi(u,v), \tag{30}$$

where κ_i is the curvature of a_i, $i = 1,2$. Taking into account the above equations, (27) is equivalent to the following ODE system

$$u''(s) + v''(s)\,\cos\omega(u(s),v(s)) + \kappa_2(v(s))\,v'(s)^2\sin\omega(u(s),v(s)) = 0, \tag{31}$$

$$v''(s) + u''(s)\,\cos\omega(u(s),v(s)) + \kappa_1(u(s))\,u'(s)^2\sin\omega(u(s),v(s)) = 0, \tag{32}$$

$$2u'(s)v'(s)\sin\omega(u(s),v(s)) = \kappa(s). \tag{33}$$

A straightforward computation shows that the binormal vector field $B(s)$ of γ is given by

$$\begin{aligned} B(s) = &- \frac{v'(s) + u'(s)\cos\omega(u(s),v(s))}{\sin\omega(u(s),v(s))}\Phi_u(u(s),v(s)) \\ &+ \frac{u'(s) + v'(s)\cos\omega(u(s),v(s))}{\sin\omega(u(s),v(s))}\Phi_v(u(s),v(s)). \end{aligned}$$

By taking derivative here, and using the Frenet equations, we get

$$v'(s)^2 - u'(s)^2 = \tau(s), \tag{34}$$

where τ denotes the torsion of γ. In other words, we have shown the following result.

Proposition 8. *Let M_ϕ be a helix surface in $\mathbb{S}^3$, locally parametrized by $\Phi(u,v)$. Then a curve $\gamma(s) = \Phi(u(s),v(s))$ is a unit speed geodesic of M_ϕ if and only if the equations* (26), (31), (32), (33) *and* (34) *are satisfied.*

Let $\alpha(s)$ be a slant helix in $\mathbb{S}^3$ with curvature κ_α and torsion τ_α. Then there exists a Killing vector field V along α and a constant $\theta \in (0,\pi/2)$ such that $\langle V, N_\alpha\rangle = \cos\theta$, where N_α is the principal normal vector field of α. Let $c_1(u)$ be a spherical helix in $\mathbb{S}^2$ of angle θ, and consider M_θ a helix surface of angle θ corresponding to c_1 and another regular curve c_2, that will be determined later. Is there a geodesic $\gamma(s) = \Phi(u(s),v(s))$ in M_θ congruent to α (i.e., with curvature κ_α and torsion τ_α)? The answer is yes.

From (26) we get

$$2u'(s)v'(s)\cos\omega(u(s),v(s)) = 1 - (u'(s)^2 + v'(s)^2). \tag{35}$$

Squaring (33) and (35), and adding the resulting equations, leads to

$$(u'(s)^2 - v'(s)^2)^2 - 2(u'(s)^2 + v'(s)^2) + \kappa_\alpha(s)^2 + 1 = 0. \tag{36}$$

This equation, jointly with (34), yields

$$u'(s)^2 = \frac{1}{4}(\kappa_\alpha(s)^2 + (\tau_\alpha(s) - 1)^2), \tag{37}$$

which determine, up to a constant, the function $u(s)$. Then $v(s)$ can be computed, up to a constant, from (34). Finally, the geodesic curvature k_2 of c_2 can be determined from (31), (32) and (26). In conclusion, we have shown the following result.

Theorem 9. *Let $\gamma(s)$ be a unit speed curve in $\mathbb{S}^3$, with $\kappa > 0$. Then γ is a slant helix of constant angle θ if and only if γ is locally congruent to a geodesic of a helix surface M_θ.*

Acknowledgements. This work has been partially supported by MINECO (Ministerio de Economía y Competitividad) and FEDER project MTM2015-65430-P, and by Fundación Séneca (Región de Murcia) project 19901/GERM/15, Spain.

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