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ON $\mathcal{S}$-CLOSED SUBMODULES

Yilmaz Durğun and Salahattin Özdemir

Abstract. A submodule N of a module M is called $\mathcal{S}$-closed (in M) if M/N is nonsingular. It is well-known that the class $\mathcal{C}losed$ of short exact sequences determined by closed submodules is a proper class in the sense of Buchsbaum. However, the class $\mathcal{S}-\mathcal{C}losed$ of short exact sequences determined by $\mathcal{S}$-closed submodules need not be a proper class. In the first part of the paper, we describe the smallest proper class $\langle\mathcal{S}-\mathcal{C}losed\rangle$ containing $\mathcal{S}-\mathcal{C}losed$ in terms of $\mathcal{S}$-closed submodules. We show that this class coincides with the proper classes projectively generated by Goldie torsion modules and coprojectively generated by nonsingular modules. Moreover, for a right nonsingular ring R, it coincides with the proper class generated by neat submodules if and only if R is a right SI-ring. In abelian groups, the elements of this class are exactly torsion-splitting. In the second part, coprojective modules of this class which we call *ec-flat* modules are also investigated. We prove that injective modules are ec-flat if and only if each injective hull of a Goldie torsion module is projective if and only if every Goldie torsion module embeds in a projective module. For a left Noetherian right nonsingular ring R of which the identity element is a sum of orthogonal primitive idempotents, we prove that the class $\langle\mathcal{S}-\mathcal{C}losed\rangle$ coincides with the class of pure-exact sequences of modules if and only if R is a two-sided hereditary, two-sided CS-ring and every singular right module is a direct sum of finitely presented modules.

1. Introduction

Closed submodules have offered rich topics of research, especially in the last 20 years, due to their important role played in ring and module theory and relative homological algebra. In parallel, several generalizations of closed submodules have been considered. For instance, neat submodules (see [20]), coneat submodules (see [10]) and $\mathcal{S}$-closed submodules (see [11]) are some of these generalizations. The purpose of the present paper is to study $\mathcal{S}$-closed submodules which have been studied recently in [1, 27, 28].

All rings considered in this paper will be associative with an identity element. Unless otherwise stated R denotes an arbitrary ring and all modules

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will be *right* unitary R-modules. Let R be a ring and M an R-module. Denote by $N \leq M$ that N is a submodule of M or M is an extension of N. The injective hull of M will be denoted by $E(M)$. By M^+ we shall denote the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M. A submodule N of M is *essential* (or *large*) in M if for every nonzero submodule $K \leq M$, we have $N \cap K \neq 0$. N is said to be *closed in* M if N has no proper essential extension in M. We also say in this case that N is a closed submodule. The notion of nonsingularity was introduced in [11]. The *singular submodule* of M is $Z(M) = \{x \in M \mid xI = 0$ for some essential right ideal I of $R\}$; this takes the place of the torsion submodule in general setting. The module M is called *nonsingular* if $Z(M) = 0$, and *singular* if $M = Z(M)$, while the *right singular ideal* of R is $Z_r(R) = Z(R_R)$. The ring R is said to be *right nonsingular* if it is nonsingular as a right R-module. The *second singular* (or *Goldie torsion*) submodule $Z_2(M)$ of M is defined by the equality $Z_2(M)/Z(M) = Z(M/Z(M))$. A module M is called *Goldie torsion* if $Z_2(M) = M$.

As a generalization of closed submodules, Goodearl introduced $\mathcal{S}$-closed submodules in [11]. A submodule $N \leq M$ is called *$\mathcal{S}$-closed* if M/N is nonsingular. Every $\mathcal{S}$-closed submodule is closed, and every closed submodule of a nonsingular module is $\mathcal{S}$-closed by [23, Lemma 2.3]. It is well-known that the class $\mathcal{C}losed$ of short exact sequences determined by closed submodules is a proper class in the sense of Buchsbaum (see [5, 10.5]). However, the class $\mathcal{S}-\mathcal{C}losed$ of short exact sequences determined by $\mathcal{S}$-closed submodules need not be a proper class (see Example 3.1). So we consider the smallest proper class $\langle \mathcal{S}-\mathcal{C}losed \rangle$ containing $\mathcal{S}-\mathcal{C}losed$. In Section 3, we describe the class $\langle \mathcal{S}-\mathcal{C}losed \rangle$ in terms of $\mathcal{S}$-closed submodules. We show that the class $\langle \mathcal{S}-\mathcal{C}losed \rangle$ coincides with the proper classes projectively generated by Goldie torsion modules and coprojectively generated by nonsingular modules. Moreover, for a right nonsingular ring R, we prove that $\langle \mathcal{S}-\mathcal{C}losed \rangle$ coincides with the proper class generated by neat submodules if and only if R is a right SI-ring (i.e., every singular right R-module is injective). In abelian groups, the elements of this class are exactly torsion-splitting. In Section 4, we investigate $\langle \mathcal{S}-\mathcal{C}losed \rangle$-coprojective modules which we shall call *ec-flat* modules. We prove that injective modules are ec-flat if and only if each injective hull of a Goldie torsion module is projective if and only if every Goldie torsion module embeds in a projective module. We also prove that R is a right nonsingular ring if and only if every principal right ideal of R is ec-flat if and only if every ec-flat R-module is nonsingular if and only if every submodule of an ec-flat R-module is ec-flat. For a right perfect ring R, we show that every element in the class $\langle \mathcal{S}-\mathcal{C}losed \rangle$ splits if and only if every finitely generated nonsingular R-module is projective. We prove that a commutative artinian ring R is a QF-ring if and only if every injective R-module is ec-flat, where R is called a *quasi-Frobenius* ring (or *QF-ring* for short) if it is right Noetherian and right self-injective. We show that every short exact sequence in $\langle \mathcal{S}-\mathcal{C}losed \rangle$ is pure if and only if every ec-flat module is flat. Finally, for a left Noetherian, right nonsingular ring R of which the identity element is a sum of orthogonal primitive idempotents, we prove that the class $\langle \mathcal{S} - \mathcal{C}losed \rangle$ and the class $\mathcal{P}ure$ of all pure-exact sequences coincide if and only if R is a two-sided hereditary, two-sided CS-ring and every singular right module is a direct sum of finitely presented modules. We refer the reader to [5, 12, 24] for the undefined notions used in the text.

2. Proper class

Throughout this section, let $\mathcal{P}$ be a class of short exact sequences of modules and module homomorphisms. If a short exact sequence

$$E : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

belongs to $\mathcal{P}$, then f is said to be a $\mathcal{P}$-*monomorphism* and g is said to be a $\mathcal{P}$-*epimorphism*. A short exact sequence E is determined by each of the monomorphisms f and the epimorphisms g uniquely up to isomorphism.

Definition 2.1. The class $\mathcal{P}$ is said to be *proper* (in the sense of Buchsbaum) if it satisfies the following conditions (see, for example, [5]):

P-1) If a short exact sequence E is in $\mathcal{P}$, then $\mathcal{P}$ contains every short exact sequence isomorphic to E.

P-2) $\mathcal{P}$ contains all splitting short exact sequences.

P-3) The composite of two $\mathcal{P}$-monomorphisms (respectively $\mathcal{P}$-epimorphisms) is a $\mathcal{P}$-monomorphism (respectively $\mathcal{P}$-epimorphism) if this composite is defined.

P-4) If g and f are monomorphisms and $g \circ f$ is a $\mathcal{P}$-monomorphism, then f is a $\mathcal{P}$-monomorphism. If g and f are epimorphisms and $g \circ f$ is a $\mathcal{P}$-epimorphism, then g is a $\mathcal{P}$-epimorphism.

The class $\mathcal{S}plit$ of all splitting short exact sequences of modules is the smallest proper class and the class $\mathcal{A}bs$ of all short exact sequences of modules is the largest proper class. Another important example is $\mathcal{P}ure$, the class of all pure short exact sequences in the sense of Cohn [6], that is, the class of all short exact sequences E such that $E \otimes M$ is exact for every left R-module M. We will identify a class of isomorphic short exact sequences with any of its elements.

The intersection of all proper classes containing the class $\mathcal{P}$ is clearly a proper class, denoted by $\langle \mathcal{P} \rangle$. The class $\langle \mathcal{P} \rangle$ is the smallest proper class containing $\mathcal{P}$, called the proper class *generated* by $\mathcal{P}$. A module M is called $\mathcal{P}$-*projective* if it is projective with respect to all short exact sequences in $\mathcal{P}$, that is, $\mathrm{Hom}(M, E)$ is exact for every E in $\mathcal{P}$. Notice that the proper class $\langle \mathcal{P} \rangle$ has the same projective modules as $\mathcal{P}$ (see [18]). A module M is called $\mathcal{P}$-*coprojective* if every short exact sequence of the form $0 \to A \to B \to M \to 0$ is in $\mathcal{P}$. For a given class $\mathcal{M}$ of modules, denote by $\overline{k}(\mathcal{M})$ the smallest proper class for which each $M \in \mathcal{M}$ is $\overline{k}(\mathcal{M})$-coprojective; it is called the proper class *coprojectively generated* by $\mathcal{M}$. The largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is $\mathcal{P}$-projective is called the proper class *projectively generated* by $\mathcal{M}$. For a homomorphism $f : A \to B$

and a module C, the induced homomorphisms $\mathrm{Ext}^1_R(1_C, f) : \mathrm{Ext}^1_R(C, A) \to \mathrm{Ext}^1_R(C, B)$ and $\mathrm{Ext}^1_R(f, 1_C) : \mathrm{Ext}^1_R(B, C) \to \mathrm{Ext}^1_R(A, C)$ will denoted by f_* and f^*, respectively.

Throughout by a short exact sequence we mean a short exact sequence of modules and module homomorphisms. See [24] and [14] for further details on proper classes.

3. Proper classes relative to Goldie torsion theory

The torsion theory generated by all singular modules is called the *Goldie torsion theory*. Its torsion class consists of the Goldie torsion modules and its torsion free class consists of the nonsingular modules. If R is right nonsingular, then the Goldie torsion modules and the singular modules coincide. The torsion class of the Goldie torsion theory is closed under homomorphic images, direct sums, extensions, submodules and injective hulls, while the torsion free class is closed under submodules, direct products, extensions and injective hulls.

It is known that $\mathcal{S}$-closed submodules are always closed. However, closed submodules need not be $\mathcal{S}$-closed in general. For example, the zero submodule 0 is closed in any module M, but it is $\mathcal{S}$-closed in M only if M is nonsingular. Denote by $\mathcal{S} - \mathcal{C}losed$ the class of all short exact sequences $0 \to A \xrightarrow{f} B \to C \to 0$ such that $\mathrm{Im}\, f$ is $\mathcal{S}$-closed in B. In contrast to the class $\mathcal{C}losed$ of short exact sequences determined by closed submodules, the class $\mathcal{S} - \mathcal{C}losed$ need not be a proper class in general as the following example shows.

Example 3.1. Let M be a module which is not nonsingular. Then the short exact sequence $E : 0 \to 0 \to M \to M \to 0$ splits, but E is not $\mathcal{S}$-closed exact sequence since $M/0 \cong M$ is not nonsingular. Thus, by the condition $P - 2)$ of being a proper class, we see that $\mathcal{S} - \mathcal{C}losed$ is not a proper class.

Note that the class $\mathcal{S} - \mathcal{C}losed$ satisfies all conditions in Definition 2.1 except $P - 2)$ in general. Now we give some conditions for the class $\mathcal{S} - \mathcal{C}losed$ to be proper.

Proposition 3.2. *For a ring R, the following statements are equivalent.*

(1) *$\mathcal{S} - \mathcal{C}losed$ is a proper class.*
(2) *Every module is nonsingular.*
(3) *$\mathcal{A}bs = S - \mathcal{C}losed$*
(4) *R is semisimple.*

Proof. (1) $\Rightarrow$ (2) For a module M, the short exact sequence $0 \to 0 \to M \to M \to 0$ splits, so by assumption it is $\mathcal{S}$-closed exact, that is, the zero submodule 0 is $\mathcal{S}$-closed in M. We conclude that M is nonsingular.

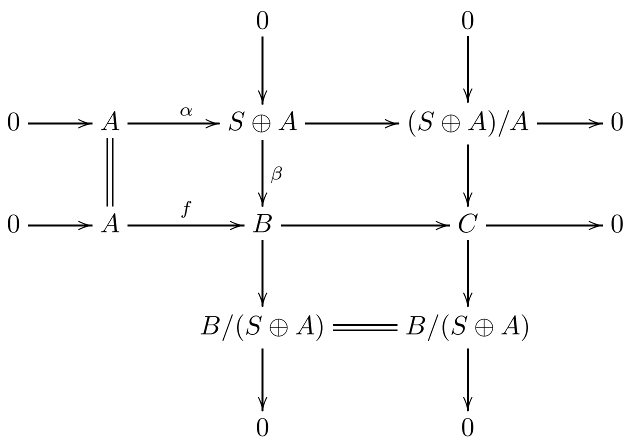
(2) $\Rightarrow$ (3), (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2) are clear.

(2) $\Rightarrow$ (4) For an essential right ideal I of R, R/I is nonsingular by assumption, and so $I = R$ which implies that R is a semisimple ring. □

Kepka introduced in [14] certain conditions on a class of monomorphisms which are needed to study new kind of proper classes. Since in an abelian category any monomorphism induces an epimorphism and a short exact sequence, these conditions lead to the new type of proper classes of such sequences. This study motivates us to introduce extended $\mathcal{S}$-closed submodules. A submodule A of a module B is called *extended $\mathcal{S}$-closed* in B if there is a submodule S in B such that $S \cap A = 0$ and $B/(S \oplus A)$ is nonsingular. $\mathcal{S}$-closed submodules are extended $\mathcal{S}$-closed, but the converse is not true in general (see Example 3.1). Denote by $\mathcal{S} - \overline{\mathcal{C}losed}$ the class of all short exact sequences $0 \to A \xrightarrow{f} B \to C \to 0$ such that $\operatorname{Im} f$ is extended $\mathcal{S}$-closed in B. It is known that the class $\mathcal{S} - \overline{\mathcal{C}losed}$ forms a proper class (see [14, Theorem 2.1]). Now we show that $\mathcal{S} - \overline{\mathcal{C}losed}$ is the smallest proper class generated by the class $\mathcal{S} - \mathcal{C}losed$.

Proposition 3.3. *$\langle \mathcal{S} - \mathcal{C}losed \rangle = \mathcal{S} - \overline{\mathcal{C}losed}$. In particular, $\mathcal{S} - \overline{\mathcal{C}losed} \subseteq \mathcal{C}losed$.*

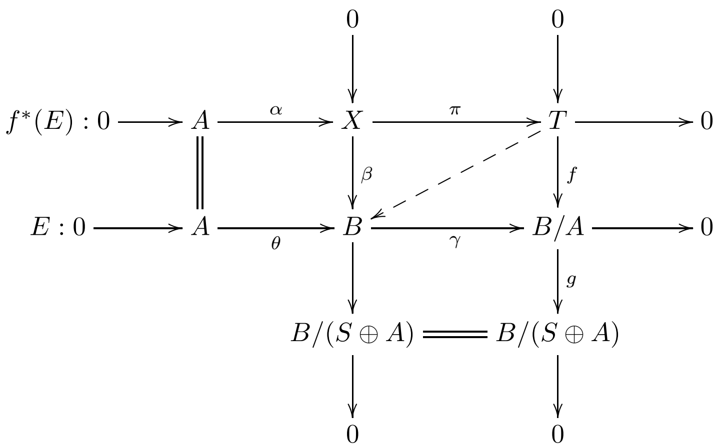
Proof. Since $\langle \mathcal{S} - \mathcal{C}losed \rangle$ is the smallest proper class containing $\mathcal{S} - \mathcal{C}losed$ and $\mathcal{S} - \mathcal{C}losed \subseteq \mathcal{S} - \overline{\mathcal{C}losed}$, we have $\langle \mathcal{S} - \mathcal{C}losed \rangle \subseteq \mathcal{S} - \overline{\mathcal{C}losed}$. For the converse, let $0 \to A \xrightarrow{f} B \to C \to 0$ be an extended $\mathcal{S}$-closed exact sequence. Then there is a submodule S in B such that $S \cap A = 0$ and $B/(S \oplus A)$ is nonsingular. Now, in the following commutative diagram, α is a $\mathcal{S}plit$-monomorphism, and so an $\langle \mathcal{S} - \mathcal{C}losed \rangle$-monomorphism by $P - 2)$ of Definition 2.1. Moreover, β is an $\langle \mathcal{S} - \mathcal{C}losed \rangle$-monomorphism by the nonsingularity of $B/(S \oplus A)$. Thus $f = \beta\alpha$ is also an $\langle \mathcal{S} - \mathcal{C}losed \rangle$-monomorphism by $P - 3)$ of Definition 2.1. For the particular case, since $B/(S \oplus A)$ is nonsingular, we conclude by [23, Lemma 2.3] that β is a $\mathcal{C}losed$-monomorphism. Hence $f = \beta\alpha$ is also a $\mathcal{C}losed$-monomorphism.



□

Proposition 3.4. *A submodule A of a module B is extended $\mathcal{S}$-closed in B if and only if* $\mathrm{Hom}(T, B) \to \mathrm{Hom}(T, B/A) \to 0$ *is exact for each Goldie torsion module T.*

Proof. ($\Rightarrow$) Let $E : 0 \to A \hookrightarrow B \to B/A \to 0$ be an extended $\mathcal{S}$-closed exact sequence and let $f : T \to B/A$ be a homomorphism with T a Goldie torsion module. Then the sequence $f^*(E) : 0 \to A \to X \to T \to 0$ is extended $\mathcal{S}$-closed by the properties of a proper class. So there is a submodule $S \leq X$ such that $S \cap A = 0$ and $X/(S \oplus A)$ is nonsingular. But $X/(S \oplus A)$ is isomorphic to a homomorphic image of a Goldie torsion module T, so $X = S \oplus A$. Now consider the following diagram:



Since $f^*(E)$ splits, there is a homomorphism $\pi^{-1} : T \to X$ such that $\pi\pi^{-1} = id$. So we find a homomorphism $\beta\pi^{-1} : T \to B$ such that $\gamma(\beta\pi^{-1}) = f$ as desired.

($\Leftarrow$) Assume that $Z_2(B/A) = T/A$ for some submodule $T \leq B$. Since T/A is a Goldie torsion module, it is projective relatively to the short exact sequence $E : 0 \to A \hookrightarrow B \to B/A \to 0$ by assumption. So the inclusion map $T/A \hookrightarrow B/A$ can be extended to $h : T/A \to B$ with h necessarily monic. Therefore there exists a submodule $S \cong T/A$ of B such that $S \cap A = 0$, and $B/(S \oplus A) \cong B/T \cong (B/A)/(T/A)$ is nonsingular. Thus A is an extended $\mathcal{S}$-closed submodule of B. □

Proposition 3.5. *Let $\mathcal{F}$ be the class of all nonsingular modules. Then $\overline{k}(\mathcal{F}) = \mathcal{S} - \overline{\mathcal{C}losed}$.*

Proof. Since all nonsingular modules are $\mathcal{S} - \overline{\mathcal{C}losed}$-coprojective, we see that $\overline{k}(\mathcal{F}) \subseteq \mathcal{S} - \overline{\mathcal{C}losed}$. For the converse, let $0 \to A \xrightarrow{f} B \to C \to 0$ be an extended $\mathcal{S}$-closed exact sequence. Then there is a submodule $S \leq B$ such that $S \cap A = 0$ and $B/(S \oplus A)$ is nonsingular. In the diagram in the proof of Proposition 3.3, since $B/(S \oplus A)$ is nonsingular, β is a $\overline{k}(\mathcal{F})$-monomorphism. Thus $f = \beta\alpha$ is also a $\overline{k}(\mathcal{F})$-monomorphism. □

We collect all results obtained so far in the following corollary.

Corollary 3.6. *Let $E : 0 \to A \to B \to M \to 0$ be a short exact sequence. Then the following statements are equivalent.*

(1) *E is extended $\mathcal{S}$-closed.*
(2) *$\mathrm{Hom}_R(T, E)$ is exact for each Goldie torsion module T.*
(3) *$E \in \overline{k}(\mathcal{F})$, where $\mathcal{F}$ is the class of all nonsingular modules.*
(4) *The sequence $0 \to Z_2(A) \to Z_2(B) \to Z_2(M) \to 0$ is splitting.*

Proof. (1) $\Leftrightarrow$ (2) by Proposition 3.4, (1) $\Leftrightarrow$ (3) by Proposition 3.5 and (2) $\Leftrightarrow$ (4) by [26, Theorem 3.4]. $\square$

Remark 3.7. In case $R = \mathbb{Z}$, it is known that an R-module A is singular if and only if it is a torsion group, and that A is nonsingular if and only if it is a torsion-free group. A short exact sequence $E : 0 \to A \to B \to C \to 0$ of abelian groups is called *torsion-splitting* if E splits for the injection $\iota : T \to C$, where $T = T(C)$ is the torsion part of C. A short exact sequence of abelian groups is torsion-splitting exactly if the torsion groups are projective relative to it (see [9, Exercises 58.2]). So, a short exact sequence of abelian groups is torsion-splitting exactly if it is an extended $\mathcal{S}$-closed exact by Corollary 3.6.

Proposition 3.8. *Every nonsingular module is projective if and only if every extended $\mathcal{S}$-closed exact sequence splits, that is, $\mathcal{S}plit = \overline{\mathcal{S} - \mathcal{C}losed}$.*

Proof. ($\Rightarrow$) Let $0 \to A \xrightarrow{f} B \to C \to 0$ be an extended $\mathcal{S}$-closed short exact sequence. Then there is a submodule $S \leq B$ such that $S \cap A = 0$ and $B/(S \oplus A)$ is nonsingular. Now, consider the diagram in the proof of Proposition 3.3. Since $B/(S \oplus A)$ is projective by assumption, α and β are $\mathcal{S}plit$-monomorphisms, and so $f = \beta\alpha$ is also a $\mathcal{S}plit$-monomorphism.

($\Leftarrow$) Let C be a nonsingular module. Let us consider the short exact sequence $E : 0 \to K \to P \to C \to 0$ with P projective. Since $C \cong P/K$ is nonsingular, E is an $\mathcal{S}$-closed, and so an extended $\mathcal{S}$-closed short exact sequence. Then E splits by assumption, and so C is projective as a direct summand of P. $\square$

Neat subgroups of abelian groups were introduced by Honda [13] in order to characterize closed subgroups. A subgroup A of an abelian group B is called *neat* in B if $Ap = A \cap Bp$ for every prime p. It is known that a subgroup A of an abelian group B is closed if and only if it is neat if and only if the sequence $\mathrm{Hom}(S, B) \to \mathrm{Hom}(S, B/A) \to 0$ is exact for each simple abelian group S. Neatness over arbitrary rings are considered by Renault [20]. A submodule A of a module B is called *neat* in B if the sequence $\mathrm{Hom}(S, B) \to \mathrm{Hom}(S, B/A) \to 0$ is exact for each simple module S. Closed submodules are neat, but the converse is true exactly for C-rings, where R is called a *C-ring* if the socle of R/I is nonzero for every proper essential right ideal I of R. So, in particular, extended $\mathcal{S}$-closed submodules are neat. For the converse, we have the following result.

A ring R is a right *SC-ring* if every singular right R-module is semisimple (see [21, Theorem 3.2]).

Lemma 3.9. *Let R be a ring. Neat submodules are extended $\mathcal{S}$-closed if and only if each Goldie torsion module T can be represented as $T = S \oplus P$, where S is a semisimple module and P is a projective module. In particular, R is a right SC-ring.*

Proof. ($\Rightarrow$) Let T be Goldie torsion module, and let $E : 0 \to A \to B \to B/A \to 0$ be a neat exact sequence. Then E is an extended $\mathcal{S}$-closed sequence by assumption. So, the sequence $\mathrm{Hom}(T, B) \to \mathrm{Hom}(T, B/A) \to 0$ is exact by Proposition 3.4, that is, T is neat projective. Thus, T is a direct sum of a projective module and a semisimple module by [10, Theorem 2.6].

($\Leftarrow$) By assumption, we see that every Goldie torsion module T is neat-projective. Then every neat exact sequence is extended $\mathcal{S}$-closed by Proposition 3.4.

For the particular case, let Z be a singular right module. Since Z is a Goldie torsion module, it follows by assumption that $Z = S \oplus P$ with S semisimple and P projective. Moreover, since singular modules are closed under submodules and singular projective modules are zero, we have $P = 0$. Thus Z is semisimple, and hence R is a right SC-ring. □

Since a nonsingular SC-ring is an SI-ring, we have the following result.

Corollary 3.10. *Let R be a right nonsingular ring. Then the following statements are equivalent.*

(1) *Neat submodules are extended $\mathcal{S}$-closed.*
(2) *Closed submodules are extended $\mathcal{S}$-closed and R is right C-ring.*
(3) *R is a right SI-ring.*

4. *ec*-flat modules

A module M is called *flat* if the short exact sequence $0 \to A \to B \to M \to 0$ is pure exact for any modules A and B (see, for example, [22]). Motivated by the relation between flat modules and pure submodules, some classes of modules that are defined via closed submodules and neat submodules have been studied recently (see, for example, [3, 29]). A module M is said to be *weakly-flat* (respectively *neat-flat*) if the kernel of any epimorphism $Y \to M$ is closed (respectively neat) in Y. These concepts lead us to investigate the modules M for which any short exact sequence ending with M is extended $\mathcal{S}$-closed. We call M an *extended $\mathcal{S}$-closed flat* module (or *ec-flat* for short) if the kernel of any epimorphism $Y \to M$ is extended $\mathcal{S}$-closed in Y. By Proposition 3.4, we infer that M is ec-flat if and only if for any epimorphism $Y \to M$, the induced map $\mathrm{Hom}(T, Y) \to \mathrm{Hom}(T, M)$ is surjective for each Goldie torsion module T.

For the following proposition we refer to [16, Propositions 1.12 and 1.13]. The proof is included for completeness.

Proposition 4.1. *The following statements are equivalent for a module M.*

(1) *M is ec-flat.*
(2) *There exists an extended $\mathcal{S}$-closed sequence $0 \to K \to F \to M \to 0$ with F projective.*
(3) *There exists an extended $\mathcal{S}$-closed sequence $0 \to K \to F \to M \to 0$ with F ec-flat.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1) Let $0 \to A \to B \overset{g}{\to} M \to 0$ be a short exact sequence. By assumption, there exists an extended $\mathcal{S}$-closed exact sequence $0 \to K \to F \overset{\alpha}{\to} M \to 0$ with F ec-flat. Considering the pullback of g and α, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B' & \overset{\beta}{\longrightarrow} & F & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \overset{g}{\longrightarrow} & M & \longrightarrow & 0 \end{array}$$

Since F is ec-flat, β is an extended $\mathcal{S}$-closed epimorphism. Then $gf = \alpha\beta$ is also an extended $\mathcal{S}$-closed epimorphism, and so g is an extended $\mathcal{S}$-closed epimorphism by $P-4$) of Definition 2.1. This means that $\operatorname{Ker} g$ is an extended $\mathcal{S}$-closed submodule of B, that is, M is ec-flat. □

Remark 4.2. (1) Nonsingular modules and projective modules are ec-flat.
(2) ec-flat modules are weakly-flat, by Proposition 3.3.
(3) A Goldie torsion module T is ec-flat if and only if it is projective, by Proposition 3.4. So, R is a semisimple ring if and only if every right (or left) R-module is ec-flat.
(4) If R is a right nonsingular ring, then ec-flat (weakly-flat) modules are exactly nonsingular modules by [23, Lemma 2.3(a)].
(5) The class of ec-flat modules is closed under extensions and finite direct sums.

Proposition 4.3. *Let M and N be modules and $f : N \to M$ an epimorphism. If M is an ec-flat module, then any Goldie torsion submodule of M is isomorphic to a Goldie torsion submodule of N. In particular, $Z_2(M)$ embeds in a projective module.*

Proof. Let T be a Goldie torsion submodule of M and $\iota : T \to M$ be an inclusion homomorphism. Since M is ec-flat, the map $\operatorname{Hom}(T, N) \to \operatorname{Hom}(T, M)$ is surjective, so there is a homomorphism $g : T \to N$ such that $fg = \iota$. Now, since g is a monomorphism and Goldie torsion modules are closed under homomorphic images, it follows that $g(T)$ is a Goldie torsion submodule of N. The particular case follows by taking an epimorphism $h : P \to M$ with P projective and $T = Z_2(M)$. □

Theorem 4.4. *An injective module E is ec-flat if and only if $Z_2(E)$ is a projective module.*

Proof. (⇒) Suppose that E is ec-flat. Then by Proposition 4.3, $Z_2(E)$ embeds in a projective module, say P. Since $E/Z_2(E)$ is a nonsingular module, $Z_2(E)$ is an $\mathcal{S}$-closed submodule, and so a closed submodule of E. Thus $Z_2(E)$ is a direct summand of E, so it is an injective module. Hence $Z_2(E)$ is a direct summand of the projective module P, so it is projective.

(⇐) Suppose that $Z_2(E)$ is a projective module. Let $0 \to A \to F \xrightarrow{g} E \to 0$ be a short exact sequence with F projective, and let $f : T \to E$ be a homomorphism with T a Goldie torsion module. We claim that there is a homomorphism $h : T \to F$ such that $gh = f$. Since T is a Goldie torsion module, $f(T)$ embeds in $Z_2(E)$, that is, there is an inclusion $\iota : f(T) \to Z_2(E)$. There is also an inclusion $\iota_1 : f(T) \to E$. Combining these maps, we obtain the following diagram:

$$\begin{array}{ccccccccc} & & & & T & & \\ & & & & \downarrow{\scriptstyle f_1} & & \\ & & & & f(T) & \xrightarrow{\iota} & Z_2(E) \\ & & & & \downarrow{\scriptstyle \iota_1} & & \\ 0 \longrightarrow A & \longrightarrow & F & \xrightarrow{g} & E & \longrightarrow & 0 \end{array}$$

By the injectivity of E, there is a homomorphism $v : Z_2(E) \to E$ such that $v\iota = \iota_1$ and so by the projectivity of $Z_2(E)$, there is a homomorphism $u : Z_2(E) \to F$ such that $gu = v$. Setting $h = u\iota f_1 : T \to F$, we obtain that $gh = f$, as desired. Hence, E is an ec-flat module by Proposition 4.1. □

Recall that R is called a *right Kasch* ring if each simple right R-module embeds in R_R.

Theorem 4.5. *The following statements are equivalent.*

(1) *Injective modules are ec-flat.*
(2) *The injective hull $E(T)$ of each Goldie torsion module T is projective.*
(3) *Every Goldie torsion module embeds in a projective module.*
(4) *Every injective module E can be represented as $E = P \oplus N$, where P is a projective Goldie torsion module and N is a nonsingular module.*
(5) *For every free left R-module F, the character module F^+ is ec-flat.*

In particular, R is a right Kasch ring.

Proof. (1) ⇒ (2) The injective hull $E(T)$ of a Goldie torsion module T is Goldie torsion. Since $E(T)$ is aslo ec-flat by assumption, it follows that $E(T)$ is projective by Remark 4.2(3).

(2) ⇒ (3) is clear.

(3) ⇒ (4) Let E be an injective module. Then $E = Z_2(E) \oplus N$, where $Z_2(E)$ is the Goldie torsion submodule of E and N is a nonsingular module. By assumption, $Z_2(E)$ embeds in a projective module, say in F. But since

$Z_2(E)$ is injective as a direct summand of E, it is also a direct summand of F. Thus $Z_2(E)$ is projective.

(4) $\Rightarrow$ (5) For every free left R-module F, the character module F^+ is injective by [22, Theorem 3.52]. Then by assumption $F^+ = P \oplus N$, where P is a projective Goldie torsion module and N is a nonsingular module. Since P and N are ec-flat modules by Remark 4.2(1), it follows that F^+ is ec-flat by Remark 4.2(5).

(5) $\Rightarrow$ (1) Let E be an injective module. There is a free left R-module F and an epimorphism $F \to E^+$ from which we obtain an exact sequence $0 \to E^{++} \to F^+$. Since E is injective and $E \leq E^{++}$, E is a direct summand of F^+. Thus E is ec-flat since F^+ is ec-flat by assumption.

Finally, since ec-flat modules are neat-flat, it follows by (1) that injective modules are neat-flat. Thus, R is a right Kasch ring by [2, Theorem 4.9] □

For a ring R, a right R-module is projective if and only if it is injective exactly when R is a QF-ring (see [15, Theorem 15.9]).

Corollary 4.6. *The following statements hold for any ring R.*

(1) *If $Z_2(R_R) = R_R$, then every injective module is ec-flat if and only if R is a QF-ring.*
(2) *If $Z_2(R_R) = 0$, then every injective module is ec-flat if and only if R is semisimple.*

Proof. (1) If $Z_2(R_R) = R_R$, then all nonsingular modules are zero. Assume that every injective module is ec-flat. Then by Theorem 4.5(4), every injective module is projective, that is, R is a QF ring. Conversely, if R is a QF-ring, then all injective modules are projective, and so they are ec-flat.

(2) If $Z_2(R_R) = 0$, then all projective modules are nonsingular. Assume that every injective module is ec-flat. Since nonsingular modules are closed under submodules, every singular module is zero by Theorem 4.5(3). In particular, every simple module is projective, that is, R is semisimple. Conversely, if R is semisimple, then all injective modules are projective, and so they are ec-flat. □

Note that if the injective hull of a finitely generated module is projective, then it is also finitely generated [17, Lemma 3.70].

Corollary 4.7. *Assume that R satisfies one of the equivalent conditions of Theorem 4.5. Then, injective hulls of finitely generated Goldie torsion modules are finitely generated. In particular, the module $E(S)$ is finitely generated for each singular simple module S.*

A ring R is right Artinian if and only if every injective module is a direct sum of injective hulls of simple R-modules [15, Exercises-42,§3].

Corollary 4.8. *Let R be a right Artinian ring whose maximal right ideals are essential, that is, every simple module is singular. Then R is a QF-ring if and only if every injective right R-module is ec-flat.*

Lemma 4.9. *Let R be a commutative Artinian ring. Then, R is a QF-ring if and only if every injective module is ec-flat.*

Proof. ($\Rightarrow$) If R is a QF-ring, then every injective module is projective, and so ec-flat.

($\Leftarrow$) Assume that every injective module is ec-flat. Note that if R is commutative and E is an injective cogenerator, then $\mathrm{Hom}(S, E) \cong S$. Let E be any injective module. Then $E = \bigoplus_{i \in I} E(S_i)$ for some index set I, where $E(S_i)$ is the injective hull of simple module S_i for each i. It is well known that every simple module is either projective or singular. In the former case, since $S_i \cong S_i^+$, S_i is injective by [22, Theorem 3.52], and so $E(S_i) = S_i$ is projective. In the later case, $E(S_i)$ is projective by Theorem 4.5(2). Thus in both cases, $E(S_i)$ is projective for each $i \in I$. Thus E is projective as a direct sum of projective modules. Hence R is a QF-ring. $\square$

Theorem 4.10 ([4, Theorem 4.2])**.** *For a right nonsingular ring R, every nonsingular R-module is projective if and only if R is Artinian hereditary serial.*

By Proposition 3.8 and Theorem 4.10, we obtain the following result.

Corollary 4.11. *Let R be a right nonsingular ring. Then every ec-flat module is projective if and only if R is an Artinian hereditary serial ring.*

Proposition 4.12. *Every cyclic nonsingular module is projective if and only if every cyclic ec-flat module is projective.*

Proof. ($\Rightarrow$) Let M be a cyclic ec-flat module. Then $M \cong R/I$ for some right ideal I of R, and so I is an extended $\mathcal{S}$-closed ideal of R. Therefore, there is a right ideal J of R such that $J \cap I = 0$ and $R/(J \oplus I)$ is nonsingular. Now, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \xrightarrow{\alpha} & J \oplus I & \longrightarrow & (J \oplus I)/I & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow & & \\
0 & \longrightarrow & I & \xrightarrow{\theta} & R & \longrightarrow & R/I & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow{\scriptstyle g} & & \\
 & & & & R/(J \oplus I) & = & R/(J \oplus I) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $R/(J \oplus I)$ is nonsingular, it is projective by assumption. So β is a $\mathcal{S}plit$-monomorphism. Moreover, since α is a $\mathcal{S}plit$-monomorphism, $\theta = \beta\alpha$ is also a $\mathcal{S}plit$-monomorphism. Thus R/I is projective as a direct summand of R.

($\Leftarrow$) It follows by the fact that every nonsingular module is ec-flat. □

The following implications will be useful in the sequel:

$$\textbf{nonsingular} \Rightarrow \textbf{ec-flat} \Rightarrow \textbf{weakly flat} \Rightarrow \textbf{neat-flat}$$

Proposition 4.13. *The following statements are equivalent for a ring R.*

(1) *Every principal right ideal is ec-flat.*
(2) *Every principal right ideal is weakly-flat.*
(3) *R is a right nonsingular ring.*

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are clear.

(2) $\Rightarrow$ (3) Let m be a nonzero element of R. Consider the short exact sequence $0 \to Ann_r(m) \to R \to mR \to 0$ of R-modules, where the map $R \to mR$ is a left multiplication by m. Since $(mR)_R$ is weakly-flat by assumption, the sequence is closed exact. Thus $Ann_r(m)$ is a proper closed submodule of R, so it cannot be essential in R. Hence $Z(R_R) = 0$, that is, R a right nonsingular ring. □

Proposition 4.14. *The following statements are equivalent for a ring R.*

(1) *Every weakly-flat module is nonsingular.*
(2) *Every ec-flat module is nonsingular.*
(3) *Every submodule of an ec-flat module is ec-flat.*
(4) *R is right nonsingular ring.*

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (4) The singular submodule $Z(R_R)$ of R is ec-flat by assumption. But $Z(R_R)$ is projective relative to the extended $\mathcal{S}$-closed short exact sequences by Proposition 3.4. Then $Z(R_R)$ is projective, and so it is zero. That is, R is a right nonsingular ring.

(4) $\Rightarrow$ (1) Let M be a weakly-flat module. Then there is a closed exact sequence $0 \to A \to P \to M \to 0$ with P projective. Moreover, every projective module must be nonsingular by assumption. Thus M is nonsingular by [23, Lemma 2.3]. □

Proposition 4.15. *The following statements are equivalent.*

(1) *Every extended $\mathcal{S}$-closed submodule is pure.*
(2) *Every ec-flat module is flat.*
(3) *Every nonsingular module is flat.*
(4) *Every finitely generated nonsingular module is flat.*

Proof. The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (1) Firstly, we recall that if every finitely generated submodule of a module is flat, then it is flat by [22, Corollary 3.49]. Hence, every nonsingular

module is flat by assumption. Let $0 \to A \xrightarrow{f} B \to C \to 0$ be an extended $\mathcal{S}$-closed exact sequence. Then there is a submodule $S \leq B$ such that $S \cap A = 0$ and $B/(S \oplus A)$ is nonsingular. Now, consider the diagram in the proof of Proposition 3.3. Since $B/(S \oplus A)$ is flat, β is a $Pure$-monomorphism, and thus $f = \beta\alpha$ is also a $\mathcal{P}ure$-monomorphism. □

It is well known that a ring R is right perfect if and only if every flat module is projective. So, by Propositions 3.8 and 4.15, we have the following result.

Corollary 4.16. *Let R be a right perfect ring. Then $\mathcal{S}plit = \mathcal{S} - \overline{\mathcal{C}losed}$ if and only if every finitely generated nonsingular module is projective.*

For a commutative nonsingular ring R, every nonsingular module is flat if and only if R is semi-hereditary [11, Proposition 2.3]. So, we get the following corollary.

Corollary 4.17. *Let R be a commutative nonsingular ring. Then every extended $\mathcal{S}$-closed submodule is pure if and only if R is semi-hereditary.*

Remark 4.18. Let R be a ring and e be a central idempotent in R. Then for a module M, we have $M = Me \oplus M(1-e)$. So, it can be easily seen that M is an ec-flat (flat) R-module if and only if Me is an ec-flat (flat) eR-module and $M(1-e)$ is an ec-flat (flat) $(1-e)R$-module.

A module M is said to be *extending* or CS if every closed submodule of M is a direct summand. A ring R is called a right *CS-ring* if R is CS as a right module, (see [7]). It is clear that every cyclic weakly-flat R-module is projective if and only if R is right CS-ring.

Proposition 4.19. *Let R be a commutative ring. Then the following are equivalent.*

(1) *Every cyclic ec-flat (nonsingular) module is projective.*
(2) *$R \cong A \times B$, where A is a ring with $Z_2(A) = A$ and B is a CS-ring.*

Proof. (1) ⇒ (2) Since $R/Z_2(R)$ is ec-flat, it is projective by assumption. So, the short exact sequence $0 \to Z_2(R) \hookrightarrow R \to R/Z_2(R) \to 0$ splits, that is, $Z_2(R)$ is direct summand of R. Thus $R \cong A \times B$, where $A = Z_2(R)$ and B is nonsingular. By Remark 4.18, we can assume that either R is a ring with $Z_2(R) = R$ or R is nonsingular. For the later case, R is a CS-ring by [23, Lemma 2.3] and by assumption.

(2) ⇒ (1) Assume that $R \cong A \times B$, where A is a ring with $Z_2(A) = A$ and B is a CS-ring. Let M be a cyclic ec-flat module. Since $M = MA \oplus MB$, it follows that MA is a cyclic ec-flat A-module and MB is a cyclic ec-flat B-module by Remark 4.18. So, MA is a projective A-module since $Z_2(A) = A$, and MB is a projective B-module since B is a CS-ring. Thus M is a projective R-module. □

Proposition 4.20. *Let R be a commutative ring. Assume that every cyclic nonsingular module is projective. Then the following statements hold.*

(1) *A module M is ec-flat if and only if $M = P \oplus N$, where P is a projective module with $Z_2(P) = P$ and N is nonsingular.*
(2) *ec-flat modules are closed under extended $\mathcal{S}$-closed submodules.*

Proof. (1) Let M be an ec-flat R-module. Then by Proposition 4.19, we have $M = MA \oplus MB$, where A is a ring with $Z_2(A) = A$ and B is a CS-ring. So, MA is an ec-flat A-module and MB is an ec-flat B-module, by Remark 4.18. Since $Z_2(A) = A$, every A-module is a Goldie torsion module, and so every extended $\mathcal{S}$-closed exact sequence of A-modules splits by Proposition 3.4. In particular, MA is a projective A-module by Proposition 4.1. Moreover, it can be observed by the proof of (1) $\Rightarrow$ (2) of Proposition 4.19 that B is nonsingular. So, MB is a nonsingular B-module by Proposition 4.14 as it is an ec-flat B-module. Therefore $M = P \oplus N$ where P is a projective module with $Z_2(P) = P$ and N is a nonsingular module. The converse is clear.

(2) Let M be an ec-flat module, and let K be an extended $\mathcal{S}$-closed submodule of M. Then the short exact sequence $0 \to Z_2(K) \to Z_2(M) \to Z_2(M/K) \to 0$ is splitting by Corollary 3.6. This implies that $Z_2(M) \cong Z_2(K) \oplus Z_2(M/K)$. Since M is ec-flat, it follows by (1) that $M = P \oplus N$ where P is a projective module with $Z_2(P) = P$ and N is a nonsingular module. Therefore, $Z_2(M) = Z_2(P) = P$ is projective, and so $Z_2(K)$ is also projective as a direct summand of $Z_2(M)$. Thus, $Z_2(K)$ is ec-flat. Finally, since $Z_2(K)$ and $K/Z_2(K)$ are ec-flat modules in the short exact sequence $0 \to Z_2(K) \hookrightarrow K \to K/Z_2(K) \to 0$, we conclude that K is an ec-flat module by Remark 4.2-(5). □

As a proper generalization of CS-modules, Tercan introduced the concept of CLS-modules in [25]. A module M is called a *CLS-module* if every $\mathcal{S}$-closed submodule of M is a direct summand of M. Recently, CLS-modules have been studied in [27].

Proposition 4.21. *A module B is a CLS-module if and only if every extended $\mathcal{S}$-closed submodule of B is a direct summand of B.*

Proof. ($\Leftarrow$) It is clear, because every $\mathcal{S}$-closed submodule is extended $\mathcal{S}$-closed.

($\Rightarrow$) Suppose that B is a CLS-module and that A is an extended $\mathcal{S}$-closed submodule of B. Then there is a submodule $S \leq B$ such that $S \cap A = 0$ and $B/(S \oplus A)$ is nonsingular. Now, consider the diagram in the proof of Proposition 3.3. Since $S \oplus A$ is $\mathcal{S}$-closed in B, it is a direct summand of B by assumption. Therefore, β is a $\mathcal{S}plit$-monomorphism, and so is $f = \beta\alpha$. Thus, A is a direct summand of B. □

Example 4.22. The $\mathbb{Z}$-module $\mathbb{Q} \oplus Z_{p^n}$ with p prime and $n \geq 2$ is a CLS-module, but not a CS-module by [27, Example 2.5]. So, by this example, we observe that there are closed submodules which are not extended $\mathcal{S}$-closed by Proposition 4.21.

Proposition 4.23. *The following statements are equivalent.*

(1) *Every nonsingular module is projective.*

(2) *Every ec-flat module is projective.*
(3) *Every projective module is a CLS-module.*

Proof. (1) $\Rightarrow$ (2) Let C be an ec-flat module. Then there is an extended $\mathcal{S}$-closed exact sequence $0 \to A \xrightarrow{f} B \to C \to 0$ with B projective. So, there is a submodule $S \leq B$ such that $S \cap A = 0$ and $B/(S \oplus A)$ is nonsingular. Now, consider the diagram in the proof of Proposition 3.3. Since $B/(S \oplus A)$ is projective by assumption, β is a splitting monomorphism, and so $f = \beta\alpha$ is also a splitting monomorphism. Thus C is projective as a direct summand of B.

(2) $\Rightarrow$ (3) Let B be a projective module and A an $\mathcal{S}$-closed submodule of B. Since we have an extended $\mathcal{S}$-closed exact sequence $E : 0 \to A \to B \to C \to 0$ with B projective, it follows by Proposition 4.1 that C is ec-flat. Then C is projective by assumption, and so the sequence E splits. Thus A is direct summand of B.

(3) $\Rightarrow$ (1) Let C be a nonsingular module. Then there is an $\mathcal{S}$-closed sequence $0 \to A \to B \to C \to 0$ with B projective. Sice B is a CLS-module by assumption, A must be a direct summand of B. In fact, $B \cong A \oplus C$ which implies that C is projective. □

The classes of flat and nonsingular right R-modules coincide if and only if R is a left semi-hereditary ring without an infinite set of orthogonal idempotents such that $\mathcal{Q}^r$ is flat as a right R-module, where $\mathcal{Q}^r$ denotes the maximal right ring of quotients of R (see [1, Theorem 5.2]). Motivated by this result, we investigate the structure of a ring over which the proper classes $\mathcal{S}-\overline{\mathcal{C}losed}$ and $\mathcal{P}ure$ coincide.

An R-module M is said to be *FP-injective* or *absolutely pure* if it is pure in every extension. Every FP-injective right R-module is injective if and only if R is a right Noetherian ring (see [8, Exercises 6.2.3]).

Theorem 4.24. *Let R be a left Noetherian right nonsingular ring. Assume that the identity element of R is a sum of orthogonal primitive idempotents. Then the following are equivalent.*

(1) *A submodule of a right R-module M is extended $\mathcal{S}$-closed in M if and only if it is pure in M. That is, $\mathcal{S}-\overline{\mathcal{C}losed} = \mathcal{P}ure$.*
(2) *R is a two-sided hereditary, two-sided CS-ring and every singular right R-module is a direct sum of finitely presented modules.*

Proof. (1) $\Rightarrow$ (2) By assumption, every pure submodule is extended $\mathcal{S}$-closed, and so it is closed by Proposition 3.3. Then FP-injective modules are injective which implies that R is a right Noetherian ring. Since R is a right nonsingular ring, ec-flat modules are nonsingular, and so they coincide with flat modules by assumption. It follows that every finitely generated nonsingular module is projective by [22, Corollary 3.58]. Thus, R is a two-sided hereditary, two-sided CS-ring by [4, Theorem 4.1]. Now, since R is a two-sided Noetherian hereditary ring and every singular module M is $\mathcal{P}ure$-projective by (1), M is a direct sum of finitely presented right R-modules by [19, Corollary 6.5].

(2) $\Rightarrow$ (1) If R is a two-sided hereditary, two-sided CS-ring, then every finitely generated nonsingular right R-module is projective by [4, Theorem 4.1]. So, every extended $\mathcal{S}$-closed submodule is pure by Proposition 4.15. Conversely, if every singular module is direct sum of finitely presented modules, then every singular module is $\mathcal{P}ure$-projective. This implies that every pure submodule is extended $\mathcal{S}$-closed. □

Proposition 4.25. *Let R be a commutative nonsingular ring. Then $\mathcal{S}-\overline{\mathcal{C}losed} = \mathcal{P}ure$ if and only if R is a semi-hereditary ring and every singular module is a direct sum of finitely presented modules.*

Proof. ($\Rightarrow$) By assumption, every pure submodule is extended $\mathcal{S}$-closed, and so closed by Proposition 3.3. Then FP-injective modules are injective which implies that R is a Noetherian ring. Moreover, R is a semi-hereditary ring by Corollary 4.17, and so it is a hereditary ring. Now, since every singular module M is $\mathcal{P}ure$-projective by assumption, it is a direct sum of finitely presented modules by [19, Corollary 6.5].

($\Leftarrow$) It follows by Corollary 4.17 that $\mathcal{S}-\overline{\mathcal{C}losed} \subseteq \mathcal{P}ure$. Since every singular module M is a direct sum of finitely presented modules by assumption, M is $\mathcal{P}ure$-projective. This implies that $\mathcal{P}ure \subseteq \mathcal{S}-\overline{\mathcal{C}losed}$. □

Remark 4.26. A right R-module M is called *torsion-free* if $\mathrm{Tor}_R^1(M, R/Rr) = 0$ for all $r \in R$. In [1], the authors investigated the structure of a ring over which nonsingular and torsion-free right R-modules coincide. This ring is in fact a nonsingular ring over which ec-flat and torsion-free right R-modules coincide, by Remark 4.2-(4).

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Yilmaz Durğun
Department of Mathematics
Bitlis Eren University
Bitlis, Turkey
E-mail address: ydurgun@beu.edu.tr

Salahattin Özdemir
Department of Mathematics
Dokuz Eylul University
Izmir, Turkey
E-mail address: salahattin.ozdemir@deu.edu.tr