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ON SPLIT LEIBNIZ TRIPLE SYSTEMS

Yan Cao and Liangyun Chen

Abstract. In order to study the structure of arbitrary split Leibniz triple systems, we introduce the class of split Leibniz triple systems as the natural extension of the class of split Lie triple systems and split Leibniz algebras. By developing techniques of connections of roots for this kind of triple systems, we show that any of such Leibniz triple systems T with a symmetric root system is of the form $T = U + \sum_{[j]\in\Lambda^1/\sim} I_{[j]}$ with U a subspace of T_0 and any $I_{[j]}$ a well described ideal of T, satisfying $\{I_{[j]}, T, I_{[k]}\} = \{I_{[j]}, I_{[k]}, T\} = \{T, I_{[j]}, I_{[k]}\} = 0$ if $[j] \neq [k]$.

1. Introduction

The notion of Leibniz algebras was introduced by Loday [14], which is a "nonantisymmetric" generalization of Lie algebras. So far, many results of this kind of algebras have been considered in the frameworks of low dimensional algebras, nilpotence and related problems [1, 2, 3, 4, 5]. Leibniz triple systems were introduced by Bremner and Sánchez-Ortega [6]. Leibniz triple systems were defined in a functorial manner using the Kolesnikov-Pozhidaev algorithm, which took the defining identities for a variety of algebras and produced the defining identities for the corresponding variety of dialgebras [13]. In [6], Leibniz triple systems were obtained by applying the Kolesnikov-Pozhidaev algorithm to Lie triple systems. In [15], Levi's theorem for Leibniz triple systems is determined. Furthermore, Leibniz triple systems are related to Leibniz algebras in the same way that Lie triple systems related to Lie algebras. So it is natural to prove analogs of results from the theory of Lie triple systems to Leibniz triple systems.

Recently, in [7, 8, 11, 10, 12], the structures of arbitrary split Lie algebras, arbitrary split Leibniz algebras, arbitrary split Lie triple systems and graded Leibniz triple systems have been determined by the techniques of connections of

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roots. Our work is essentially motivated by the work on split Leibniz algebras and split Lie triple systems [11, 10].

Throughout this paper, Leibniz triple systems T are considered of arbitrary dimension and over an arbitrary field $\mathbb{K}$. It is worth to mention that, unless otherwise stated, there is not any restriction on $\dim T_\alpha$ or $\{k \in \mathbb{K}\colon\ k\alpha \in \Lambda^1$ for a fixed $\alpha \in \Lambda^1\}$, where T_α denotes the root space associated to the root α, and Λ^1 the set of nonzero roots of T. This paper proceeds as follows. In Section 2, we establish the preliminaries on split Leibniz triple systems theory. In Section 3, we show that such an arbitrary Leibniz triple system with a symmetric root system is of the form $T = U + \sum_{[j]\in\Lambda^1/\sim} I_{[j]}$ with U a subspace of T_0 and any $I_{[j]}$ a well described ideal of T, satisfying $\{I_{[j]}, T, I_{[k]}\} = \{I_{[j]}, I_{[k]}, T\} = \{T, I_{[j]}, I_{[k]}\} = 0$ if $[j] \neq [k]$.

2. Preliminaries

Definition 2.1 ([10])**.** A *right Leibniz algebra* L is a vector space over a field $\mathbb{K}$ endowed with a bilinear product $[\cdot,\cdot]$ satisfying the Leibniz identity

$$[[y, z], x] = [[y, x], z] + [y, [z, x]]$$

for all $x, y, z \in L$.

Definition 2.2 ([6])**.** A *Leibniz triple system* is a vector space T endowed with a trilinear operation $\{\cdot,\cdot,\cdot\} : T \times T \times T \to T$ satisfying

$$\begin{aligned} \{a, \{b, c, d\}, e\} = {} & \{\{a, b, c\}, d, e\} - \{\{a, c, b\}, d, e\} - \{\{a, d, b\}, c, e\} \\ & + \{\{a, d, c\}, b, e\}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \{a, b, \{c, d, e\}\} = {} & \{\{a, b, c\}, d, e\} - \{\{a, b, d\}, c, e\} - \{\{a, b, e\}, c, d\} \\ & + \{\{a, b, e\}, d, c\} \end{aligned} \tag{2.2}$$

for all $a, b, c, d, e \in T$.

Example 2.3. A Lie triple system gives a Leibniz triple system with the same ternary product. If L is a Leibniz algebra with product $[\cdot,\cdot]$, then L becomes a Leibniz triple system by putting $\{x, y, z\} = [[x, y], z]$. More examples refer to [6].

Definition 2.4 ([6])**.** Let I be a subspace of a Leibniz triple system T. Then I is called a *subsystem* of T, if $\{I, I, I\} \subseteq I$; I is called an *ideal* of T, if $\{I, T, T\} + \{T, I, T\} + \{T, T, I\} \subseteq I$.

Proposition 2.5 ([15])**.** *Let T be a Leibniz triple system. Then the following assertions hold.*

(1) *J is generated by $\{\{a, b, c\} - \{a, c, b\} + \{b, c, a\} : a, b, c \in T\}$, then J is an ideal of T satisfying $\{T, T, J\} = \{T, J, T\} = 0$.*
(2) *J is generated by $\{\{a, b, c\} - \{a, c, b\} + \{b, c, a\} : a, b, c \in T\}$, then T is a Lie triple system if and only if $J = 0$.*

(3) $\{\{c, d, e\}, b, a\} - \{\{c, d, e\}, a, b\} - \{\{c, b, a\}, d, e\} + \{\{c, a, b\}, d, e\}$
$- \{c, \{a, b, d\}, e\} - \{c, d, \{a, b, e\}\} = 0$ *for all* $a, b, c, d, e \in T$.

Definition 2.6 ([6])**.** The *standard embedding* of a Leibniz triple system T is the two-graded right Leibniz algebra $L = L^0 \oplus L^1$, L^0 being the $\mathbb{K}$-span of $\{x \otimes y,\ x, y \in T\}$, $L^1 := T$ and where the product is given by

$$[(x \otimes y, z), (u \otimes v, w)] := (\{x, y, u\} \otimes v - \{x, y, v\} \otimes u + z \otimes w, \{x, y, w\} + \{z, u, v\} - \{z, v, u\}).$$

Let us observe that L^0 with the product induced by the one in $L = L^0 \oplus L^1$ becomes a right Leibniz algebra.

Definition 2.7. Let T be a Leibniz triple system, $L = L^0 \oplus L^1$ be its standard embedding, and H^0 be a maximal abelian subalgebra (MASA) of L^0. For a linear functional $\alpha \in (H^0)^*$, we define the root space of T (with respect to H^0) associated to α as the subspace $T_\alpha := \{t_\alpha \in T : [t_\alpha, h] = \alpha(h)t_\alpha$ for any h $\in H^0\}$. The elements $\alpha \in (H^0)^*$ satisfying $T_\alpha \neq 0$ are called roots of T with respect to H^0 and we denote $\Lambda^1 := \{\alpha \in (H^0)^* \setminus \{0\} : T_\alpha \neq 0\}$.

Let us observe that $T_0 = \{t_0 \in T : [t_0, h] = 0$ for any h $\in H^0\}$. In the following, we shall denote by Λ^0 the set of all nonzero $\alpha \in (H^0)^*$ such that $L^0_\alpha := \{v^0_\alpha \in L^0 : [v^0_\alpha, h] = \alpha(h)v^0_\alpha$ for any h $\in H^0\} \neq 0$.

Lemma 2.8. *Let T be a Leibniz triple system, $L = L^0 \oplus L^1$ be its standard embedding, and H^0 be a* MASA *of L^0. For $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}$ and $\delta \in \Lambda^0 \cup \{0\}$, the following assertions hold.*

(1) *If $[T_\alpha, T_\beta] \neq 0$, then $\alpha + \beta \in \Lambda^0 \cup \{0\}$ and $[T_\alpha, T_\beta] \subseteq L^0_{\alpha+\beta}$.*
(2) *If $[L^0_\delta, T_\alpha] \neq 0$, then $\delta + \alpha \in \Lambda^1 \cup \{0\}$ and $[L^0_\delta, T_\alpha] \subseteq T_{\delta+\alpha}$.*
(3) *If $[T_\alpha, L^0_\delta] \neq 0$, then $\alpha + \delta \in \Lambda^1 \cup \{0\}$ and $[T_\alpha, L^0_\delta] \subseteq T_{\alpha+\delta}$.*
(4) *If $[L^0_\delta, L^0_\gamma] \neq 0$, then $\delta + \gamma \in \Lambda^0 \cup \{0\}$ and $[L^0_\delta, L^0_\gamma] \subseteq L^0_{\delta+\gamma}$.*
(5) *If $\{T_\alpha, T_\beta, T_\gamma\} \neq 0$, then $\alpha+\beta+\gamma \in \Lambda^1 \cup \{0\}$ and $\{T_\alpha, T_\beta, T_\gamma\} \subseteq T_{\alpha+\beta+\gamma}$.*

Proof. (1) For any $x \in T_\alpha$, $y \in T_\beta$ and $h \in H^0$, by Leibniz identity, one has $[[x, y], h] = [x, [y, h]] + [[x, h], y] = [x, \beta(h)y] + [\alpha(h)x, y] = (\alpha + \beta)(h)[x, y]$.

(2) For any $x \in L^0_\delta$, $y \in T_\alpha$ and $h \in H^0$, by Leibniz identity, one has $[[x, y], h] = [x, [y, h]] + [[x, h], y] = [x, \alpha(h)y] + [\delta(h)x, y] = (\delta + \alpha)(h)[x, y]$.

(3) For any $x \in T_\alpha$, $y \in L^0_\delta$, and $h \in H^0$, by Leibniz identity, one has $[[x, y], h] = [x, [y, h]] + [[x, h], y] = [x, \delta(h)y] + [\alpha(h)x, y] = (\alpha + \delta)(h)[x, y]$.

(4) For any $x \in L^0_\delta$, $y \in L^0_\gamma$ and $h \in H^0$, by Leibniz identity, one has $[[x, y], h] = [x, [y, h]] + [[x, h], y] = [x, \gamma(h)y] + [\delta(h)x, y] = (\delta + \gamma)(h)[x, y]$.

(5) It is a consequence of Lemma 2.8(1) and (2). □

Definition 2.9. Let T be a Leibniz triple system, $L = L^0 \oplus L^1$ be its standard embedding, and H^0 be a MASA of L^0. We shall call that T is a *split Leibniz triple system* with a coherent 0-root space (with respect to H^0) if :

(1) $T = T_0 \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha)$,

(2) $\{T_0, T_0, T_0\} = 0$,
(3) $\{T_\alpha, T_{-\alpha}, T_0\} = 0$ for $\alpha \in \Lambda^1$.
We say that Λ^1 is the root system of T.

From now on, all of the split Leibniz triple system T will be considered having a coherent 0-root space.

We also note that the facts $H^0 \subset L^0 = [T, T]$ and $T = T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha)$ imply

$$H^0 = [T_0, T_0] + \sum_{\alpha\in\Lambda^1} [T_\alpha, T_{-\alpha}]. \tag{2.3}$$

Finally, as $[T_0, T_0] \subset L^0_0 = H^0$, we have

$$[T_0, [T_0, T_0]] = 0. \tag{2.4}$$

We finally note that $\alpha \in \Lambda^1$ does not imply $\alpha \in \Lambda^0$.

Definition 2.10. A root system Λ^1 of a split Leibniz triple system T is called *symmetric* if it satisfies that $\alpha \in \Lambda^1$ implies $-\alpha \in \Lambda^1$.

A similar concept applies to the set Λ^0 of nonzero roots of L^0.

In the following, T denotes a split Leibniz triple system with a symmetric root system Λ^1, and $T = T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha)$ the corresponding root decomposition. We begin the study of split Leibniz triple systems by developing the concept of connections of roots.

Definition 2.11. Let α and β be two nonzero roots, we shall say that α and β are *connected* if there exists a family $\{\alpha_1, \alpha_2, \dots, \alpha_{2n}, \alpha_{2n+1}\} \subset \Lambda^1 \cup \{0\}$ of roots of T such that
(1) $\{\alpha_1, \alpha_1+\alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5, \dots, \alpha_1+\cdots+\alpha_{2n}+\alpha_{2n+1}\} \subset \Lambda^1$;
(2) $\{\alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \dots, \alpha_1+\cdots+\alpha_{2n}\} \subset \Lambda^0$;
(3) $\alpha_1 = \alpha$ and $\alpha_1+\cdots+\alpha_{2n}+\alpha_{2n+1} \in \pm\beta$.
We shall also say that $\{\alpha_1, \alpha_2, \dots, \alpha_{2n}, \alpha_{2n+1}\}$ is a connection from α to β.

We denote by

$$\Lambda^1_\alpha := \{\beta \in \Lambda^1 : \alpha \text{ and } \beta \text{ are connected}\},$$

we can easily get that $\{\alpha\}$ is a connection from α to itself and to $-\alpha$. Therefore $\pm\alpha \in \Lambda^1_\alpha$.

Definition 2.12. A subset Ω^1 of a root system Λ^1, associated to a split Leibniz triple system T, is called a *root subsystem* if it is symmetric, and for $\alpha, \beta, \gamma \in \Omega^1 \cup \{0\}$ such that $\alpha + \beta \in \Lambda^0$ and $\alpha+\beta+\gamma \in \Lambda^1$, then $\alpha+\beta+\gamma \in \Omega^1$.

Let Ω^1 be a root subsystem of Λ^1. We define

$$T_{0,\Omega^1} := \mathrm{span}_{\mathbb{K}}\{\{T_\alpha, T_\beta, T_\gamma\} : \alpha+\beta+\gamma = 0;\ \alpha, \beta, \gamma \in \Omega^1 \cup \{0\}\} \subset T_0$$

and $V_{\Omega^1} := \oplus_{\alpha\in\Omega^1} T_\alpha$. Taking into account the fact that $\{T_0, T_0, T_0\} = 0$, it is straightforward to verify that $T_{\Omega^1} := T_{0,\Omega^1} \oplus V_{\Omega^1}$ is a subsystem of T. We will say that T_{Ω^1} is a subsystem associated to the root subsystem Ω^1.

Proposition 2.13. *If Λ^0 is symmetric, then the relation $\sim$ in Λ^1, defined by $\alpha \sim \beta$ if and only if $\beta \in \Lambda^1_\alpha$, is of equivalence.*

Proof. This can be proved completely analogously to [9, Proposition 3.1]. □

Proposition 2.14. *Let α be a nonzero root and suppose Λ^0 is symmetric. Then Λ^1_α is a root subsystem.*

Proof. This can be proved completely analogously to [9, Lemma 3.1]. □

3. Decompositions

In this section, we will show that for a fixed $\alpha_0 \in \Lambda^1$, the subsystem $T_{\Lambda^1_{\alpha_0}}$ associated to the root subsystem $\Lambda^1_{\alpha_0}$ is an ideal of T.

Lemma 3.1. *The following assertions hold.*

(1) *If $\alpha, \beta \in \Lambda^1$ with $[T_\alpha, T_\beta] \neq 0$, then α is connected with β.*

(2) *If $\alpha, \beta \in \Lambda^1$, $\alpha \in \Lambda^0$ and $[L^0_\alpha, T_\beta] \neq 0$, then α is connected with β.*

(3) *If $\alpha, \beta \in \Lambda^1$, $\alpha \in \Lambda^0$ and $[T_\beta, L^0_\alpha] \neq 0$, then α is connected with β.*

(4) *If $\alpha, \beta \in \Lambda^1$, $\alpha, \beta \in \Lambda^0$ and $[L^0_\alpha, L^0_\beta] \neq 0$, then α is connected with β.*

(5) *If $\alpha, \overline{\beta} \in \Lambda^1$ such that α is not connected with $\overline{\beta}$, then $[T_\alpha, T_{\overline{\beta}}] = 0$, $[L^0_\alpha, T_{\overline{\beta}}] = 0$ and $[T_{\overline{\beta}}, L^0_\alpha] = 0$ if furthermore $\alpha \in \Lambda^0$. If $\alpha, \overline{\beta} \in \Lambda^1$ such that α is not connected with $\overline{\beta}$, then $[L^0_\alpha, L^0_{\overline{\beta}}] = 0$ if furthermore $\alpha, \overline{\beta} \in \Lambda^0$.*

Proof. (1) Suppose $[T_\alpha, T_\beta] \neq 0$, by Lemma 2.8(1), one gets $\alpha + \beta \in \Lambda^0 \cup \{0\}$. If $\alpha + \beta = 0$, then $\beta = -\alpha$ and so α is connected with β. Suppose $\alpha + \beta \neq 0$. Since $\alpha + \beta \in \Lambda^0$, one gets $\{\alpha, \beta, -\alpha\}$ is a connection from α to β.

(2) Suppose $[L^0_\alpha, T_\beta] \neq 0$, by Lemma 2.8(2), one gets $\alpha + \beta \in \Lambda^1 \cup \{0\}$. If $\alpha + \beta = 0$, then $\beta = -\alpha$ and so α is connected with β. Suppose $\alpha + \beta \neq 0$. Since $\alpha + \beta \in \Lambda^1$, we obtain $\{\alpha, 0, -\alpha - \beta\}$ is a connection from α to β.

(3) Suppose $[T_\beta, L^0_\alpha] \neq 0$, by Lemma 2.8(3), one gets $\beta + \alpha \in \Lambda^1 \cup \{0\}$. If $\beta + \alpha = 0$, then $\beta = -\alpha$ and it is clear that α is connected with β. Suppose $\beta + \alpha \neq 0$. Since $\beta + \alpha \in \Lambda^1$, one gets $\{\beta, -\alpha - \beta, 0\}$ is a connection from β to α. By the symmetry, we get α is connected with β.

(4) Suppose $[L^0_\alpha, L^0_\beta] \neq 0$, by Lemma 2.8(4), one has $\alpha + \beta \in \Lambda^0 \cup \{0\}$. If $\alpha + \beta = 0$, then $\beta = -\alpha$ and so α is connected with β. Suppose $\alpha + \beta \neq 0$. Since $\alpha + \beta \in \Lambda^0$, one gets $\{\alpha, \beta, -\alpha\}$ is a connection from α to β.

(5) It is a consequence of Lemma 3.1(1), (2), (3) and (4). □

Lemma 3.2. *If $\alpha, \overline{\beta} \in \Lambda^1$ are not connected, then $\{T_\alpha, T_{-\alpha}, T_{\overline{\beta}}\} = 0$.*

Proof. If $[T_\alpha, T_{-\alpha}] = 0$, it is clear. One may suppose that $[T_\alpha, T_{-\alpha}] \neq 0$ and $\{T_\alpha, T_{-\alpha}, T_{\overline{\beta}}\} \neq 0$. By Leibniz identity, one gets

$$\{T_\alpha, T_{-\alpha}, T_{\overline{\beta}}\} = [[T_\alpha, T_{-\alpha}], T_{\overline{\beta}}] \subset [T_\alpha, [T_{-\alpha}, T_{\overline{\beta}}]] + [[T_\alpha, T_{\overline{\beta}}], T_{-\alpha}].$$

So either $[T_\alpha, [T_{-\alpha}, T_{\overline{\beta}}]] \neq 0$ or $[[T_\alpha, T_{\overline{\beta}}], T_{-\alpha}] \neq 0$, contradicting Lemma 3.1(5). Hence, $\{T_\alpha, T_{-\alpha}, T_{\overline{\beta}}\} = 0$. □

Lemma 3.3. *Fix* $\alpha_0 \in \Lambda^1$ *and suppose* Λ^0 *is symmetric. For* $\alpha \in \Lambda^1_{\alpha_0}$ *and* $\beta, \gamma \in \Lambda^1 \cup \{0\}$, *the following assertions hold.*

(1) *If* $\{T_\alpha, T_\beta, T_\gamma\} \neq 0$, *then* β, γ, $\alpha + \beta + \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$.

(2) *If* $\{T_\beta, T_\alpha, T_\gamma\} \neq 0$, *then* β, γ, $\beta + \alpha + \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$.

(3) *If* $\{T_\beta, T_\gamma, T_\alpha\} \neq 0$, *then* β, γ, $\beta + \gamma + \alpha \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Proof. (1) It is easy to see that $[T_\alpha, T_\beta] \neq 0$ for $\alpha \in \Lambda^1_{\alpha_0}$ and $\beta \in \Lambda^1 \cup \{0\}$. By Lemma 3.1(1), one gets $\alpha \sim \beta$ in the case $\beta \neq 0$. From here, $\beta \in \Lambda^1_{\alpha_0} \cup \{0\}$. In order to complete the proof, we will show γ, $\alpha + \beta + \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$. We distinguish two cases.

Case 1. Suppose $\alpha + \beta + \gamma = 0$. It is clear that $\alpha + \beta + \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$. The fact that $\{T_0, T_0, T_0\} = 0$ and $\{T_\alpha, T_{-\alpha}, T_0\} = 0$ for $\alpha \in \Lambda^1$ gives us $\gamma \neq 0$. By Lemma 2.8(1), one gets $\alpha + \beta \in \Lambda^0$. As $\alpha + \beta = -\gamma$, $\{\alpha, \beta, 0\}$ would be a connection from α to γ and we conclude $\gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Case 2. Suppose $\alpha + \beta + \gamma \neq 0$. We treat separately two cases.

Suppose $\alpha + \beta \neq 0$. By Lemma 2.8(1), one gets $\alpha + \beta \in \Lambda^0$ and so $\{\alpha, \beta, \gamma\}$ is a connection from α to $\alpha + \beta + \gamma$. Hence $\alpha + \beta + \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$. In the case $\gamma \neq 0$, $\{\alpha, \beta, -\alpha - \beta - \gamma\}$ is a connection from α to γ. So $\gamma \in \Lambda^1_{\alpha_0}$. Hence $\gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Suppose $\alpha + \beta = 0$. Then necessarily $\gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$. Indeed, if $\gamma \neq 0$ and α is not connected with γ, by Lemma 3.2, $\{T_\alpha, T_\beta, T_\gamma\} = \{T_\alpha, T_{-\alpha}, T_\gamma\} = 0$, a contradiction. We also have $\alpha + \beta + \gamma = \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$.

(2) The fact that $[T_\beta, T_\alpha] \neq 0$ implies by Lemma 3.1(1) that $\alpha \sim \beta$ in the case $\beta \neq 0$. From here, $\beta \in \Lambda^1_{\alpha_0} \cup \{0\}$. In order to complete the proof, we will show γ, $\beta + \alpha + \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$. We distinguish two cases.

Case 1. Suppose $\beta + \alpha + \gamma = 0$. It is clear that $\beta + \alpha + \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$. The fact that $\{T_0, T_0, T_0\} = 0$ and $\{T_\alpha, T_{-\alpha}, T_0\} = 0$ for $\alpha \in \Lambda^1$ gives us $\gamma \neq 0$. By Lemma 2.8(1), one has $\beta + \alpha \in \Lambda^0$. As $\beta + \alpha = -\gamma$, $\{\alpha, \beta, 0\}$ would be a connection from α to γ and we conclude $\gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Case 2. Suppose $\beta + \alpha + \gamma \neq 0$. We treat separately two cases.

Suppose $\beta + \alpha \neq 0$. By Lemma 2.8(1), one gets $\beta + \alpha \in \Lambda^0$ and so $\{\alpha, \beta, \gamma\}$ is a connection from α to $\beta + \alpha + \gamma$. Hence $\beta + \alpha + \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$. In the case $\gamma \neq 0$, we have $\{\alpha, \beta, -\alpha - \beta - \gamma\}$ is a connection from α to γ. So $\gamma \in \Lambda^1_{\alpha_0}$. Hence $\gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Suppose $\beta + \alpha = 0$. Then necessarily $\gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$. Indeed, if $\gamma \neq 0$ and α is not connected with γ, by Lemma 3.2, $\{T_\alpha, T_\beta, T_\gamma\} = \{T_\alpha, T_{-\alpha}, T_\gamma\} = 0$, a contradiction. We also have $\beta + \alpha + \gamma = \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$.

(3) By Leibniz identity,

$$\begin{aligned} \{T_\beta, T_\gamma, T_\alpha\} &= [[T_\beta, T_\gamma], T_\alpha] \\ &\subset [T_\beta, [T_\gamma, T_\alpha]] + [[T_\beta, T_\alpha], T_\gamma]. \end{aligned}$$

From $\{T_\beta, T_\gamma, T_\alpha\} \neq 0$, we obtain either $[T_\beta, [T_\gamma, T_\alpha]] \neq 0$ or $[[T_\beta, T_\alpha], T_\gamma] \neq 0$. We treat separately two cases.

Csse 1. Suppose $[T_\beta,[T_\gamma,T_\alpha]]\neq 0$, we will show β, γ, $\beta+\gamma+\alpha\in\Lambda^1_{\alpha_0}\cup\{0\}$. First to show $\gamma\in\Lambda^1_{\alpha_0}\cup\{0\}$. The fact that $[T_\gamma,T_\alpha]\neq 0$ implies by Lemma 3.1(1) that $\gamma\sim\alpha$ in the case $\gamma\neq 0$. From here, $\gamma\in\Lambda^1_{\alpha_0}\cup\{0\}$.

Next to show $\beta\in\Lambda^1_{\alpha_0}\cup\{0\}$. Indeed, if $\beta\neq 0$ and suppose α is not connected with β, then β is not connected with γ in the case $\gamma\neq 0$. By Lemma 3.1(1), $[T_\beta,T_\gamma]=0$ whenever $\gamma\neq 0$, contradicting $\{T_\beta,T_\gamma,T_\alpha\}\neq 0$. Next to show if $\beta\neq 0$ and in the case $\gamma=0$, we also get $\beta\in\Lambda^1_{\alpha_0}$. Indeed, suppose α is not connected with β, in the case $\gamma=0$, one has $\{T_\beta,T_\gamma,T_\alpha\}=\{T_\beta,T_0,T_\alpha\}=[[T_\beta,T_0],T_\alpha]$. From $[T_\beta,T_0]\subset L^0_\beta$ and Lemma 3.1(5), one gets $\{T_\beta,T_0,T_\alpha\}=0$, a contradiction.

Finally, to show $\beta+\gamma+\alpha\in\Lambda^1_{\alpha_0}\cup\{0\}$. Suppose $\beta+\gamma+\alpha=0$ and so $\beta+\gamma+\alpha\in\Lambda^1_{\alpha_0}\cup\{0\}$. Suppose $\beta+\gamma+\alpha\neq 0$, by $[T_\gamma,T_\alpha]\neq 0$, $\gamma+\alpha\in\Lambda^0\cup\{0\}$. If $\gamma+\alpha\neq 0$, then $\gamma+\alpha\in\Lambda^0$ and so $\{\alpha,\gamma,\beta\}$ is a connection from α to $\beta+\gamma+\alpha$. Hence $\beta+\gamma+\alpha\in\Lambda^1_{\alpha_0}$. If $\gamma+\alpha=0$, then necessarily $\beta\in\Lambda^1_{\alpha_0}\cup\{0\}$. Indeed, if $\beta\neq 0$ and α is not connected with β, by Lemma 3.1(5), $\{T_\beta,T_\gamma,T_\alpha\}=\{T_\beta,T_{-\alpha},T_\alpha\}=[[T_\beta,T_{-\alpha}],T_\alpha]=0$, contradicting $\{T_\beta,T_\gamma,T_\alpha\}\neq 0$. Therefore, we also have $\beta+\gamma+\alpha=\beta\in\Lambda^1_{\alpha_0}\cup\{0\}$.

Csse 2. If $[[T_\beta,T_\alpha],T_\gamma]\neq 0$, we will show β, γ, $\beta+\gamma+\alpha\in\Lambda^1_{\alpha_0}\cup\{0\}$. First to show $\beta\in\Lambda^1_{\alpha_0}\cup\{0\}$. The fact that $[T_\beta,T_\alpha]\neq 0$ implies by Lemma 3.1(1) that $\beta\sim\alpha$ in the case $\beta\neq 0$. From here, $\beta\in\Lambda^1_{\alpha_0}\cup\{0\}$.

Next to show $\gamma\in\Lambda^1_{\alpha_0}\cup\{0\}$. Indeed, if $\gamma\neq 0$ and α is not connected with γ, then β is not connected with γ in the case $\beta\neq 0$. By Lemma 3.1(1), $[T_\beta,T_\gamma]=0$ whenever $\beta\neq 0$, contradicting $\{T_\beta,T_\gamma,T_\alpha\}\neq 0$. Similarly, it is easy to show if $\gamma\neq 0$ and in the case $\beta=0$, we can obtain $\gamma\in\Lambda^1_{\alpha_0}$.

Finally, to show $\beta+\gamma+\alpha\in\Lambda^1_{\alpha_0}\cup\{0\}$. Suppose $\beta+\gamma+\alpha=0$ and so $\beta+\gamma+\alpha\in\Lambda^1_{\alpha_0}\cup\{0\}$. Suppose $\beta+\gamma+\alpha\neq 0$, by $[T_\beta,T_\alpha]\neq 0$, one has $\beta+\alpha\in\Lambda^0\cup\{0\}$. If $\beta+\alpha\neq 0$, then $\beta+\alpha\in\Lambda^0$ and so $\{\alpha,\beta,\gamma\}$ is a connection from α to $\beta+\gamma+\alpha$. Hence $\beta+\gamma+\alpha\in\Lambda^1_{\alpha_0}$. If $\beta+\alpha=0$, then necessarily $\gamma\in\Lambda^1_{\alpha_0}\cup\{0\}$. Indeed, if $\gamma\neq 0$ and α is not connected with γ, by Lemma 3.1(5), $\{T_\beta,T_\gamma,T_\alpha\}=\{T_{-\alpha},T_\gamma,T_\alpha\}=[[T_{-\alpha},T_\gamma],T_\alpha]=0$, contradicting $\{T_\beta,T_\gamma,T_\alpha\}\neq 0$. Therefore, we also have $\beta+\gamma+\alpha=\gamma\in\Lambda^1_{\alpha_0}\cup\{0\}$. □

Lemma 3.4. *Fix* $\alpha_0\in\Lambda^1$ *and suppose* Λ^0 *is symmetric. For* $\alpha,\beta,\gamma\in\Lambda^1_{\alpha_0}\cup\{0\}$ *with* $\alpha+\beta+\gamma=0$ *and* $\delta,\epsilon\in\Lambda^1\cup\{0\}$, *the following assertions hold.*

(1) *If* $\{\{T_\alpha,T_\beta,T_\gamma\},T_\delta,T_\epsilon\}\neq 0$, *then* δ, ϵ, $\delta+\epsilon\in\Lambda^1_{\alpha_0}\cup\{0\}$.

(2) *If* $\{T_\delta,\{T_\alpha,T_\beta,T_\gamma\},T_\epsilon\}\neq 0$, *then* δ, ϵ, $\delta+\epsilon\in\Lambda^1_{\alpha_0}\cup\{0\}$.

(3) *If* $\{T_\delta,T_\epsilon,\{T_\alpha,T_\beta,T_\gamma\}\}\neq 0$, *then* δ, ϵ, $\delta+\epsilon\in\Lambda^1_{\alpha_0}\cup\{0\}$.

Proof. (1) From the fact that $\alpha+\beta+\gamma=0$, $\{T_0,T_0,T_0\}=0$ and $\{T_\alpha,T_{-\alpha},T_0\}$ $=0$ whenever $\alpha\in\Lambda^1$, one may suppose that at least two distinct elements in $\{\alpha,\beta,\gamma\}$ are nonzero and one may consider the case $\{T_\alpha,T_\beta,T_\gamma\}\neq 0$, $\alpha+\beta\neq 0$

and $\gamma \neq 0$. By (2.2), one gets

$$0 \neq \{\{T_\alpha, T_\beta, T_\gamma\}, T_\delta, T_\epsilon\} \subset \{T_\alpha, T_\beta, \{T_\gamma, T_\delta, T_\epsilon\}\} + \{\{T_\alpha, T_\beta, T_\delta\}, T_\gamma, T_\epsilon\} + \{\{T_\alpha, T_\beta, T_\epsilon\}, T_\gamma, T_\delta\} + \{\{T_\alpha, T_\beta, T_\epsilon\}, T_\delta, T_\gamma\},$$

any of the above four summands is nonzero. In order to prove δ, ϵ, $\delta + \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$, we will consider four cases.

Case 1. Suppose $\{T_\alpha, T_\beta, \{T_\gamma, T_\delta, T_\epsilon\}\} \neq 0$. As $\gamma \neq 0$ and $\{T_\gamma, T_\delta, T_\epsilon\} \neq 0$, Lemma 3.3(1) shows that $\delta, \epsilon, \gamma+\delta+\epsilon$ are connected with γ in the case of being nonzero roots and so $\delta, \epsilon, \gamma + \delta + \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$. If $\gamma + \delta + \epsilon = 0$, then $\delta + \epsilon = -\gamma \in \Lambda^1_{\alpha_0}$. If $\gamma + \delta + \epsilon \neq 0$, taking into account $0 \neq \{T_\alpha, T_\beta, \{T_\gamma, T_\delta, T_\epsilon\}\} \subset \{T_\alpha, T_\beta, T_{\gamma+\delta+\epsilon}\}$, Lemma 3.3(3) gives us that $\alpha + \beta + \gamma + \delta + \epsilon = \delta + \epsilon \in \Lambda^1_{\alpha_0}$.

Case 2. Suppose $\{\{T_\alpha, T_\beta, T_\delta\}, T_\gamma, T_\epsilon\} \neq 0$. It is clear that $\{T_\alpha, T_\beta, T_\delta\} \neq 0$. As $\alpha + \beta \neq 0$, one gets either $\alpha \in \Lambda^1_{\alpha_0}$ or $\beta \in \Lambda^1_{\alpha_0}$. By Lemma 3.3(1) and (2), one gets $\delta \in \Lambda^1_{\alpha_0} \cup \{0\}$. It is obvious that $0 \neq \{\{T_\alpha, T_\beta, T_\delta\}, T_\gamma, T_\epsilon\} \subset \{T_{\alpha+\beta+\delta}, T_\gamma, T_\epsilon\}$. As $\gamma \neq 0$, $\gamma \in \Lambda^1_{\alpha_0}$, by Lemma 3.3(2), one gets $\epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$ and $\alpha + \beta + \gamma + \delta + \epsilon = \delta + \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Case 3. Suppose $\{\{T_\alpha, T_\beta, T_\epsilon\}, T_\gamma, T_\delta\} \neq 0$. It is easy to see that $\{T_\alpha, T_\beta, T_\epsilon\} \neq 0$. As $\alpha+\beta \neq 0$, we get either $\alpha \in \Lambda^1_{\alpha_0}$ or $\beta \in \Lambda^1_{\alpha_0}$. By Lemma 3.3(1) and (2), one gets $\epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$. Note that $0 \neq \{\{T_\alpha, T_\beta, T_\epsilon\}, T_\gamma, T_\delta\} \subset \{T_{\alpha+\beta+\epsilon}, T_\gamma, T_\delta\}$. As $\gamma \neq 0$, $\gamma \in \Lambda^1_{\alpha_0}$, by Lemma 3.3(2), one gets $\delta \in \Lambda^1_{\alpha_0} \cup \{0\}$ and $\alpha + \beta + \gamma + \delta + \epsilon = \delta + \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Case 4. Suppose $\{\{T_\alpha, T_\beta, T_\epsilon\}, T_\delta, T_\gamma\} \neq 0$. It is clear that $\{T_\alpha, T_\beta, T_\epsilon\} \neq 0$. As $\alpha + \beta \neq 0$, one gets either $\alpha \in \Lambda^1_{\alpha_0}$ or $\beta \in \Lambda^1_{\alpha_0}$. By Lemma 3.3(1) and (2), one gets $\epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$. It is clear that $0 \neq \{\{T_\alpha, T_\beta, T_\epsilon\}, T_\delta, T_\gamma\} \subset \{T_{\alpha+\beta+\epsilon}, T_\delta, T_\gamma\}$. As $\gamma \neq 0$, $\gamma \in \Lambda^1_{\alpha_0}$, by Lemma 3.3(3), one gets $\delta \in \Lambda^1_{\alpha_0} \cup \{0\}$ and $\alpha + \beta + \epsilon + \delta + \gamma = \delta + \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$.

(2) By Proposition 2.5(3), we obtain that

$$0 \neq \{T_\delta, \{T_\alpha, T_\beta, T_\gamma\}, T_\epsilon\} \subset \{\{T_\delta, T_\gamma, T_\epsilon\}, T_\beta, T_\alpha\} + \{\{T_\delta, T_\gamma, T_\epsilon\}, T_\alpha, T_\beta\} + \{\{T_\delta, T_\beta, T_\alpha\}, T_\gamma, T_\epsilon\} + \{\{T_\delta, T_\alpha, T_\beta\}, T_\gamma, T_\epsilon\} + \{T_\delta, T_\gamma, \{T_\alpha, T_\beta, T_\epsilon\}\},$$

any of the above five summands is nonzero.

Suppose $\{\{T_\delta, T_\gamma, T_\epsilon\}, T_\beta, T_\alpha\} \neq 0$, it is obvious $\{T_\delta, T_\gamma, T_\epsilon\} \neq 0$. As $\gamma \neq 0$, $\gamma \in \Lambda^1_{\alpha_0}$, by Lemma 3.3(2), one gets $\delta, \delta + \gamma + \epsilon, \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$. Note that $0 \neq \{\{T_\delta, T_\gamma, T_\epsilon\}, T_\beta, T_\alpha\} \subset \{T_{\delta+\gamma+\epsilon}, T_\beta, T_\alpha\}$. As $\alpha + \beta \neq 0$, we get either $\alpha \in \Lambda^1_{\alpha_0}$ or $\beta \in \Lambda^1_{\alpha_0}$. By Lemma 3.3(2) and (3), one gets $\delta + \gamma + \epsilon + \beta + \alpha = \delta + \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$.

If $\{\{T_\delta, T_\gamma, T_\epsilon\}, T_\alpha, T_\beta\} \neq 0$, $\{\{T_\delta, T_\beta, T_\alpha\}, T_\gamma, T_\epsilon\} \neq 0$, $\{\{T_\delta, T_\alpha, T_\beta\}, T_\gamma, T_\epsilon\} \neq 0$ or $\{T_\delta, T_\gamma, \{T_\alpha, T_\beta, T_\epsilon\}\} \neq 0$, a similar argument gives us δ, ϵ, $\delta + \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$.

(3) By Proposition 2.5(3), we obtain that

$$0 \neq \{T_\delta, T_\epsilon, \{T_\alpha, T_\beta, T_\gamma\}\} \subset \{\{T_\delta, T_\epsilon, T_\gamma\}, T_\beta, T_\alpha\} + \{\{T_\delta, T_\epsilon, T_\gamma\}, T_\alpha, T_\beta\} + \{\{T_\delta, T_\beta, T_\alpha\}, T_\epsilon, T_\gamma\} + \{\{T_\delta, T_\alpha, T_\beta\}, T_\epsilon, T_\gamma\} + \{T_\delta, \{T_\alpha, T_\beta, T_\epsilon\}, T_\gamma\},$$

any of the above five summands is nonzero.

Suppose $\{\{T_\delta, T_\epsilon, T_\gamma\}, T_\beta, T_\alpha\} \neq 0$. One easily gets $\{T_\delta, T_\epsilon, T_\gamma\} \neq 0$. As $\gamma \neq 0$, $\gamma \in \Lambda^1_{\alpha_0}$, by Lemma 3.3(3), one has $\delta, \delta+\epsilon+\gamma, \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$. Note that $0 \neq \{\{T_\delta, T_\epsilon, T_\gamma\}, T_\beta, T_\alpha\} \subset \{T_{\delta+\epsilon+\gamma}, T_\beta, T_\alpha\}$. As $\alpha+\beta \neq 0$, one gets either $\alpha \in \Lambda^1_{\alpha_0}$ or $\beta \in \Lambda^1_{\alpha_0}$. By Lemma 3.3(2) and (3), one has $\delta+\epsilon+\gamma+\beta+\alpha = \delta+\epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$.

If $\{\{T_\delta, T_\epsilon, T_\gamma\}, T_\alpha, T_\beta\} \neq 0$, $\{\{T_\delta, T_\beta, T_\alpha\}, T_\epsilon, T_\gamma\} \neq 0$, $\{\{T_\delta, T_\alpha, T_\beta\}, T_\epsilon, T_\gamma\} \neq 0$ or $\{T_\delta, \{T_\alpha, T_\beta, T_\epsilon\}, T_\gamma\} \neq 0$, a similar argument gives us $\delta, \epsilon, \delta+\epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$. □

Lemma 3.5. *Fix $\alpha_0 \in \Lambda^1$ and suppose Λ^0 is symmetric. If $\alpha_1, \alpha_2, \alpha_3 \in \Lambda^1_{\alpha_0} \cup \{0\}$ with $\alpha_1+\alpha_2+\alpha_3 = 0$ and $\overline{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$, then the following assertions hold.*

(1) $[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\overline{\epsilon}}] = 0$.

(2) *In case $\overline{\epsilon} \in \Lambda^0$, then* $[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, L^0_{\overline{\epsilon}}] = 0$.

(3) $[[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_0], T_{\overline{\epsilon}}] = 0$.

Proof. (1) From the fact $\alpha_1+\alpha_2+\alpha_3 = 0$, $\{T_0, T_0, T_0\} = 0$ and $\{T_\alpha, T_{-\alpha}, T_0\} = 0$ for $\alpha \in \Lambda^1$, one gets if $\alpha_3 = 0$, then it is clear that $[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\overline{\epsilon}}] = 0$. Let us consider the case $\alpha_3 \neq 0$. By Leibniz identity, we have

$$
\begin{aligned}
[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\overline{\epsilon}}] &= [[[T_{\alpha_1}, T_{\alpha_2}], T_{\alpha_3}], T_{\overline{\epsilon}}] \\
&\subset [[T_{\alpha_1}, T_{\alpha_2}], [T_{\alpha_3}, T_{\overline{\epsilon}}]] + [[[T_{\alpha_1}, T_{\alpha_2}], T_{\overline{\epsilon}}], T_{\alpha_3}].
\end{aligned} \tag{3.5}
$$

Let us consider the first summand in (3.5). As $\alpha_3 \neq 0$, one has $\alpha_3 \in \Lambda^1_{\alpha_0}$. For $\overline{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$ and Lemma 3.1(5), one easily gets $[T_{\alpha_3}, T_{\overline{\epsilon}}] = 0$. Therefore $[[T_{\alpha_1}, T_{\alpha_2}], [T_{\alpha_3}, T_{\overline{\epsilon}}]] = 0$.

Let us now consider the second summand in (3.5), it is sufficient to verify that

$$[[[T_{\alpha_1}, T_{\alpha_2}], T_{\overline{\epsilon}}], T_{\alpha_3}] = 0.$$

To do so, we first assert that $[[T_{\alpha_1}, T_{\alpha_2}], T_{\overline{\epsilon}}] = 0$. Indeed, by Leibniz identity, we have

$$[[T_{\alpha_1}, T_{\alpha_2}], T_{\overline{\epsilon}}] \subset [T_{\alpha_1}, [T_{\alpha_2}, T_{\overline{\epsilon}}]] + [[T_{\alpha_1}, T_{\overline{\epsilon}}], T_{\alpha_2}], \tag{3.6}$$

where $\alpha_1, \alpha_2 \in \Lambda^1_{\alpha_0} \cup \{0\}$, $\overline{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$. In the following, we distinguish three cases.

Case 1. $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. As $\alpha_1 \in \Lambda^1_{\alpha_0}$ and $\overline{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$, by Lemma 3.1(1), one gets $[T_{\alpha_1}, T_{\overline{\epsilon}}] = 0$. As $\alpha_2 \in \Lambda^1_{\alpha_0}$ and $\overline{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$, by Lemma 3.1(1), one gets $[T_{\alpha_2}, T_{\overline{\epsilon}}] = 0$. Therefore by (3.6), one can show that $[[T_{\alpha_1}, T_{\alpha_2}], T_{\overline{\epsilon}}] = 0$.

Case 2. $\alpha_1 \neq 0$ and $\alpha_2 = 0$. As $\alpha_1 \in \Lambda^1_{\alpha_0}$ and $\overline{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$, by Lemma 3.1(1), one gets $[T_{\alpha_1}, T_{\overline{\epsilon}}] = 0$. That is $[[T_{\alpha_1}, T_{\overline{\epsilon}}], T_{\alpha_2}] = 0$. As $\alpha_2 = 0$, $[T_{\alpha_2}, T_{\overline{\epsilon}}] = [T_0, T_{\overline{\epsilon}}] \subset L^0_{\overline{\epsilon}}$. By Lemma 3.1(5), one gets $[T_{\alpha_1}, [T_{\alpha_2}, T_{\overline{\epsilon}}]] = 0$. Therefore by (3.6), one can show that $[[T_{\alpha_1}, T_{\alpha_2}], T_{\overline{\epsilon}}] = 0$.

Case 3. $\alpha_1 = 0$ and $\alpha_2 \neq 0$. As $\alpha_2 \in \Lambda^1_{\alpha_0}$ and $\overline{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$, by Lemma 3.1(1), one gets $[T_{\alpha_2}, T_{\overline{\epsilon}}] = 0$. That is $[T_{\alpha_1}, [T_{\alpha_2}, T_{\overline{\epsilon}}]] = 0$. As $\alpha_1 = 0$, $[T_{\alpha_1}, T_{\overline{\epsilon}}] = [T_0, T_{\overline{\epsilon}}] \subset L^0_{\overline{\epsilon}}$. By Lemma 3.1(5), we get $[[T_{\alpha_1}, T_{\overline{\epsilon}}], T_{\alpha_2}] = 0$. Therefore by (3.6), one can show that $[[T_{\alpha_1}, T_{\alpha_2}], T_{\overline{\epsilon}}] = 0$.

So $[[[T_{\alpha_1}, T_{\alpha_2}], T_{\overline{\epsilon}}], T_{\alpha_3}]=0$ is a consequence of $[[T_{\alpha_1}, T_{\alpha_2}], T_{\overline{\epsilon}}] = 0$. By (3.5), one gets $[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\overline{\epsilon}}] = 0$. The proof is complete.

(2) From the fact $\alpha_1 + \alpha_2 + \alpha_3 = 0$, $\{T_0, T_0, T_0\} = 0$ and $\{T_\alpha, T_{-\alpha}, T_0\} = 0$ for $\alpha \in \Lambda^1$, one gets if $\alpha_3 = 0$, then it is clear that $[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, L^0_{\overline{\epsilon}}] = 0$. Let us consider the case $\alpha_3 \neq 0$. Note that

$$[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, L^0_{\overline{\epsilon}}] \subset [[T_{\alpha_1}, T_{\alpha_2}], [T_{\alpha_3}, L^0_{\overline{\epsilon}}]] + [[[T_{\alpha_1}, T_{\alpha_2}], L^0_{\overline{\epsilon}}], T_{\alpha_3}]. \tag{3.7}$$

Let us consider the first summand in (3.7). As $\alpha_3 \neq 0$, one gets

$$[[T_{\alpha_1}, T_{\alpha_2}], [T_{\alpha_3}, L^0_{\overline{\epsilon}}]] = 0$$

by Lemma 3.1(5). Let us now consider the second summand in (3.7). As either $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$, Leibniz identity, the fact $[T_0, L^0_{\overline{\epsilon}}] \subset T_{\overline{\epsilon}}$ and Lemma 3.1(5), we obtain that $[[[T_{\alpha_1}, T_{\alpha_2}], L^0_{\overline{\epsilon}}], T_{\alpha_3}] = 0$. So, the second summand in (3.7) is also zero and then $[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, L^0_{\overline{\epsilon}}] = 0$.

(3) It is a consequence of Lemma 3.5(1), (2) and

$$[[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_0], T_{\overline{\epsilon}}] \subset [\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, [T_0, T_{\overline{\epsilon}}]] + [[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\overline{\epsilon}}], T_0].$$

□

Definition 3.6. A Leibniz triple system T is said to be *simple* if its product is nonzero and its only ideals are $\{0\}$, J and T.

It should be noted that the above definition agrees with the definition of a simple Lie triple system, since $J = \{0\}$ in this case.

Theorem 3.7. *Suppose Λ^0 is symmetric, the following assertions hold.*

(1) *For any $\alpha_0 \in \Lambda^1$, the subsystem*

$$T_{\Lambda^1_{\alpha_0}} = T_{0,\Lambda^1_{\alpha_0}} \oplus V_{\Lambda^1_{\alpha_0}}$$

of T associated to the root subsystem $\Lambda^1_{\alpha_0}$ is an ideal of T.

(2) *If T is simple, then there exists a connection from α to β for any α, $\beta \in \Lambda^1$.*

Proof. (1) Recall that

$$T_{0,\Lambda^1_{\alpha_0}} := \text{span}_{\mathbb{K}}\{\{T_\alpha, T_\beta, T_\gamma\} : \alpha + \beta + \gamma = 0;\ \alpha, \beta, \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}\} \subset T_0$$

and $V_{\Lambda^1_{\alpha_0}} := \oplus_{\gamma \in \Lambda^1_{\alpha_0}} T_\gamma$. In order to complete the proof, it is sufficient to show that

$$\{T_{\Lambda^1_{\alpha_0}}, T, T\} + \{T, T_{\Lambda^1_{\alpha_0}}, T\} + \{T, T, T_{\Lambda^1_{\alpha_0}}\} \subset T_{\Lambda^1_{\alpha_0}}.$$

We first check that $\{T_{\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$. It is easy to see that

$$\{T_{\Lambda^1_{\alpha_0}}, T, T\} = \{T_{0,\Lambda^1_{\alpha_0}} \oplus V_{\Lambda^1_{\alpha_0}}, T, T\} = \{T_{0,\Lambda^1_{\alpha_0}}, T, T\} + \{V_{\Lambda^1_{\alpha_0}}, T, T\}.$$

Next, we will show that $\{T_{0,\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$. Note that

$$\begin{aligned}\{T_{0,\Lambda^1_{\alpha_0}}, T, T\} &= \{T_{0,\Lambda^1_{\alpha_0}}, T_0 \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha), T_0 \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha)\} \\ &= \{T_{0,\Lambda^1_{\alpha_0}}, T_0, T_0\} + \{T_{0,\Lambda^1_{\alpha_0}}, T_0, \oplus_{\alpha \in \Lambda^1} T_\alpha\}\end{aligned}$$

$$+\{T_{0,\Lambda^1_{\alpha_0}}, \oplus_{\alpha\in\Lambda^1} T_\alpha, T_0\} + \{T_{0,\Lambda^1_{\alpha_0}}, \oplus_{\alpha\in\Lambda^1} T_\alpha, \oplus_{\beta\in\Lambda^1} T_\beta\}.$$

Here, it is clear that $\{T_{0,\Lambda^1_{\alpha_0}}, T_0, T_0\} \subset \{T_0, T_0, T_0\} = 0$. Taking into account $\{T_{0,\Lambda^1_{\alpha_0}}, T_0, T_\alpha\}$ for $\alpha \in \Lambda^1$, Lemma 3.4(1) and the fact that either $\alpha \in \Lambda^1_{\alpha_0}$ or $\alpha \notin \Lambda^1_{\alpha_0}$, give us that $\{T_{0,\Lambda^1_{\alpha_0}}, T_0, T_\alpha\} \subset V_{\Lambda^1_{\alpha_0}}$ or $\{T_{0,\Lambda^1_{\alpha_0}}, T_0, T_\alpha\} = 0$. Similarly, one gets that $\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_0\} \subset V_{\Lambda^1_{\alpha_0}}$ or $\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_0\} = 0$. Next, we will consider $\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\}$, where $\alpha, \beta \in \Lambda^1$. We treat five cases.

Case 1. If $\alpha \in \Lambda^1_{\alpha_0}$, $\beta \in \Lambda^1_{\alpha_0}$ and $\alpha + \beta = 0$. One has

$$\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\} \subset T_{0,\Lambda^1_{\alpha_0}}.$$

Case 2. If $\alpha \in \Lambda^1_{\alpha_0}$, $\beta \in \Lambda^1_{\alpha_0}$ and $\alpha + \beta \neq 0$. By $\Lambda^1_{\alpha_0}$ is a root subsystem, one gets

$$\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\} \subset V_{\Lambda^1_{\alpha_0}}.$$

Case 3. If $\alpha \in \Lambda^1_{\alpha_0}$ and $\beta \notin \Lambda^1_{\alpha_0}$. By Lemma 3.4(1), one has

$$\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\} = 0.$$

Case 4. If $\beta \in \Lambda^1_{\alpha_0}$ and $\alpha \notin \Lambda^1_{\alpha_0}$. By Lemma 3.4(1), one has

$$\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\} = 0.$$

Case 5. If $\beta \notin \Lambda^1_{\alpha_0}$ and $\alpha \notin \Lambda^1_{\alpha_0}$. By Lemma 3.4(1), one has

$$\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\} = 0.$$

Therefore, $\{T_{0,\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$.

Next, we will show that $\{V_{\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$. It is obvious that

$$\begin{aligned}\{V_{\Lambda^1_{\alpha_0}}, T, T\} &= \{\oplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha), T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha)\}\\ &= \{\oplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, T_0, T_0\} + \{\oplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, T_0, \oplus_{\alpha\in\Lambda^1} T_\alpha\}\\ &\quad + \{\oplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, \oplus_{\alpha\in\Lambda^1} T_\alpha, T_0\} + \{\oplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, \oplus_{\alpha\in\Lambda^1} T_\alpha, \oplus_{\beta\in\Lambda^1} T_\beta\}.\end{aligned}$$

Here, it is clear that $\{T_\gamma, T_0, T_0\} \subset V_{\Lambda^1_{\alpha_0}}$ for $\gamma \in \Lambda^1_{\alpha_0}$. Next, we will consider $\{T_\gamma, T_0, T_\alpha\}$ for $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1$. We treat three cases.

Case 1. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \notin \Lambda^1_{\alpha_0}$. By Lemma 3.3(1), one has

$$\{T_\gamma, T_0, T_\alpha\} = 0.$$

Case 2. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1_{\alpha_0}$ and $\gamma + \alpha \neq 0$. By $\Lambda^1_{\alpha_0}$ is a root subsystem, one has

$$\{T_\gamma, T_0, T_\alpha\} \subset V_{\Lambda^1_{\alpha_0}}.$$

Case 3. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1_{\alpha_0}$ and $\gamma + \alpha = 0$. It is clear that

$$\{T_\gamma, T_0, T_\alpha\} \subset T_{0,\Lambda^1_{\alpha_0}}.$$

Hence, $\{T_\gamma, T_0, T_\alpha\} \subset T_{\Lambda^1_{\alpha_0}}$ for $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1$. Similarly, it is easy to get $\{T_\gamma, T_\alpha, T_0\} \subset T_{\Lambda^1_{\alpha_0}}$ for $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1$. At last, we will consider

$\{\oplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, \oplus_{\alpha\in\Lambda^1} T_\alpha, \oplus_{\beta\in\Lambda^1} T_\beta\}$ for $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1$ and $\beta \in \Lambda^1$. We treat five cases.

Case 1. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1_{\alpha_0}$, $\beta \in \Lambda^1_{\alpha_0}$ and $\gamma+\alpha+\beta = 0$, one gets

$$\{T_\gamma, T_\alpha, T_\beta\} \subset T_{0,\Lambda^1_{\alpha_0}}.$$

Case 2. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1_{\alpha_0}$, $\beta \in \Lambda^1_{\alpha_0}$ and $\gamma+\alpha+\beta \neq 0$, one gets

$$\{\oplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, \oplus_{\alpha\in\Lambda^1} T_\alpha, \oplus_{\beta\in\Lambda^1} T_\beta\} \subset V_{\Lambda^1_{\alpha_0}}.$$

Case 3. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1_{\alpha_0}$ and $\beta \notin \Lambda^1_{\alpha_0}$. By Lemma 3.3(1) and (2), one gets

$$\{T_\gamma, T_\alpha, T_\beta\} = 0.$$

Case 4. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \notin \Lambda^1_{\alpha_0}$ and $\beta \in \Lambda^1_{\alpha_0}$. By Lemma 3.3(1) and (3), one gets

$$\{T_\gamma, T_\alpha, T_\beta\} = 0.$$

Case 5. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \notin \Lambda^1_{\alpha_0}$ and $\beta \notin \Lambda^1_{\alpha_0}$. By Lemma 3.3(1), one gets

$$\{T_\gamma, T_\alpha, T_\beta\} = 0.$$

So, $\{V_{\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$. Therefore $\{T_{\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$ is a consequence of $\{T_{0,\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$ and $\{V_{\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$.

A similar argument gives us $\{T, T_{\Lambda^1_{\alpha_0}}, T\} \subset T_{\Lambda^1_{\alpha_0}}$ and $\{T, T, T_{\Lambda^1_{\alpha_0}}\} \subset T_{\Lambda^1_{\alpha_0}}$. Consequently, this proves that $T_{\Lambda^1_{\alpha_0}}$ is an ideal of T.

(2) The simplicity of T implies $T_{\Lambda^1_{\alpha_0}} \in \{J, T\}$ for any $a \in \Lambda^1$. If $\alpha \in \Lambda^1$ is such that $T_{\Lambda^1_{\alpha_0}} = T$. Then $\Lambda^1_{\alpha_0} = \Lambda^1$. Hence, T has all its nonzero roots connected. Otherwise, if $T_{\Lambda^1_{\alpha_0}} = J$ for any $\alpha \in \Lambda^1$, then $\Lambda^1_{\alpha_0} = \Lambda^1_{\beta_0}$ for any α_0, $\beta_0 \in \Lambda^1$ and so $\Lambda^1_{\alpha_0} = \Lambda^1$. We also conclude that T has all its nonzero roots connected. □

Theorem 3.8. *Suppose Λ^0 is symmetric. Then for a vector space complement U of* $\mathrm{span}_{\mathbb{K}}\{\{T_\alpha, T_\beta, T_\gamma\} : \alpha+\beta+\gamma = 0$*, where* $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}\}$ *in* T_0*, we have*

$$T = U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]},$$

where any $I_{[\alpha]}$ is one of the ideals $T_{\Lambda^1_{\alpha_0}}$ of T described in Theorem 3.7. Moreover $\{I_{[\alpha]}, T, I_{[\beta]}\} = \{I_{[\alpha]}, I_{[\beta]}, T\} = \{T, I_{[\alpha]}, I_{[\beta]}\} = 0$ *if* $[\alpha] \neq [\beta]$.

Proof. Let us denote $\xi_0 :=\mathrm{span}_{\mathbb{K}}\{\{T_\alpha, T_\beta, T_\gamma\} : \alpha+\beta+\gamma = 0$, where $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}\}$ in T_0. By Proposition 2.13, we can consider the quotient set $\Lambda^1/\sim := \{[\alpha] : \alpha \in \Lambda^1\}$. By denoting $I_{[\alpha]} := T_{\Lambda^1_\alpha}$, $T_{0,[\alpha]} := T_{0,\Lambda^1_\alpha}$ and $V_{[\alpha]} := V_{\Lambda^1_\alpha}$, one gets $I_{[\alpha]} := T_{0,[\alpha]} \oplus V_{[\alpha]}$. From

$$T = T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha) = (U + \xi_0) \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha),$$

it follows

$$\oplus_{\alpha\in\Lambda^1} T_\alpha = \oplus_{[\alpha]\in\Lambda^1/\sim} V_{[\alpha]}, \quad \xi_0 = \sum_{[\alpha]\in\Lambda^1/\sim} T_{0,[\alpha]},$$

which implies

$$T = U + \xi_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha) = U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]},$$

where each $I_{[\alpha]}$ is an ideal of T by Theorem 3.7.

Next, it is sufficient to show that $\{I_{[\alpha]}, T, I_{[\beta]}\} = 0$ if $[\alpha] \neq [\beta]$. Note that,

$$\begin{aligned}\{I_{[\alpha]}, T, I_{[\beta]}\} &= \{T_{0,[\alpha]} \oplus V_{[\alpha]}, T_0 \oplus (\oplus_{\gamma\in\Lambda^1} T_\gamma), T_{0,[\beta]} \oplus V_{[\beta]}\}\\ &= \{T_{0,[\alpha]}, T_0, T_{0,[\beta]}\} + \{T_{0,[\alpha]}, T_0, V_{[\beta]}\} + \{T_{0,[\alpha]}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{0,[\beta]}\}\\ &\quad + \{T_{0,[\alpha]}, \oplus_{\gamma\in\Lambda^1} T_\gamma, V_{[\beta]}\} + \{V_{[\alpha]}, T_0, T_{0,[\beta]}\} + \{V_{[\alpha]}, T_0, V_{[\beta]}\}\\ &\quad + \{V_{[\alpha]}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{0,[\beta]}\} + \{V_{[\alpha]}, \oplus_{\gamma\in\Lambda^1} T_\gamma, V_{[\beta]}\}.\end{aligned}$$

Here, it is clear that $\{T_{0,[\alpha]}, T_0, T_{0,[\beta]}\} \subset \{T_0, T_0, T_0\} = 0$. If $[\alpha] \neq [\beta]$, by Lemmas 3.3 and 3.4, it is easy to see

$$\begin{aligned}&\{T_{0,[\alpha]}, T_0, V_{[\beta]}\} = 0,\\ &\{T_{0,[\alpha]}, \oplus_{\gamma\in\Lambda^1} T_\gamma, V_{[\beta]}\} = 0,\\ &\{V_{[\alpha]}, T_0, T_{0,[\beta]}\} = 0,\\ &\{V_{[\alpha]}, T_0, V_{[\beta]}\} = 0,\\ &\{V_{[\alpha]}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{0,[\beta]}\} = 0,\\ &\{V_{[\alpha]}, \oplus_{\gamma\in\Lambda^1} T_\gamma, V_{[\beta]}\} = 0.\end{aligned}$$

Next, we will show $\{T_{0,[\alpha]}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{0,[\beta]}\} = 0$. Indeed, for $\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\} \in T_{0,[\alpha]}$ with $\alpha_1, \alpha_2, \alpha_3 \in \Lambda^1_\alpha \cup \{0\}$, $\alpha_1 + \alpha_2 + \alpha_3 = 0$, and for $\{T_{\beta_1}, T_{\beta_2}, T_{\beta_3}\} \in T_{0,[\beta]}$ with $\beta_1, \beta_2, \beta_3 \in \Lambda^1_\beta \cup \{0\}$, $\beta_1 + \beta_2 + \beta_3 = 0$, by Proposition 2.5(3), one gets

$$\begin{aligned}&\{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, \oplus_{\gamma\in\Lambda^1} T_\gamma, \{T_{\beta_1}, T_{\beta_2}, T_{\beta_3}\}\}\\ \subset\ &\{\{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{\beta_3}\}, T_{\beta_2}, T_{\beta_1}\}\\ &+ \{\{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{\beta_3}\}, T_{\beta_1}, T_{\beta_2}\}\\ &+ \{\{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\beta_2}, T_{\beta_1}\}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{\beta_3}\}\\ &+ \{\{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\beta_1}, T_{\beta_2}\}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{\beta_3}\}\\ &+ \{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, \{T_{\beta_1}, T_{\beta_2}, \oplus_{\gamma\in\Lambda^1} T_\gamma\}, T_{\beta_3}\}.\end{aligned}$$

By Lemma 3.4, it is easy to see that

$$\begin{aligned}&\{\{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{\beta_3}\}, T_{\beta_2}, T_{\beta_1}\} = 0,\\ &\{\{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{\beta_3}\}, T_{\beta_1}, T_{\beta_2}\} = 0,\\ &\{\{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\beta_2}, T_{\beta_1}\}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{\beta_3}\} = 0,\\ &\{\{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\beta_1}, T_{\beta_2}\}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{\beta_3}\} = 0,\\ &\{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, \{T_{\beta_1}, T_{\beta_2}, \oplus_{\gamma\in\Lambda^1} T_\gamma\}, T_{\beta_3}\} = 0\end{aligned}$$

for $\alpha_1, \alpha_2, \alpha_3 \in \Lambda^1_\alpha \cup \{0\}$, $\alpha_1 + \alpha_2 + \alpha_3 = 0$, $\beta_1, \beta_2, \beta_3 \in \Lambda^1_\beta \cup \{0\}$, $\beta_1 + \beta_2 + \beta_3 = 0$, $[\alpha] \neq [\beta]$. So $\{I_{[\alpha]}, T, I_{[\beta]}\} = 0$ if $[\alpha] \neq [\beta]$ is proved.

A similar argument gives us $\{I_{[\alpha]}, I_{[\beta]}, T\} = \{T, I_{[\alpha]}, I_{[\beta]}\} = 0$ if $[\alpha] \neq [\beta]$. □

Definition 3.9. The *annihilator* of a Leibniz triple system T is the set $\mathrm{Ann}(T) = \{x \in T : \{x, T, T\} + \{T, x, T\} + \{T, T, x\} = 0\}$.

Corollary 3.10. *Suppose* Λ^0 *is symmetric. If* $\mathrm{Ann}(T) = 0$*, and* $\{T, T, T\} = T$*, then* T *is the direct sum of the ideals given in Theorem 3.8,*

$$T = \oplus_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}.$$

Proof. From $\{T, T, T\} = T$ and Theorem 3.8, we have

$$\{U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}, U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}, U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}\} = U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}.$$

Taking into account $U \subset T_0$, Lemma 3.3 and the fact that $\{I_{[\alpha]}, T, I_{[\beta]}\} = \{I_{[\alpha]}, I_{[\beta]}, T\} = \{T, I_{[\alpha]}, I_{[\beta]}\} = 0$ if $[\alpha] \neq [\beta]$ (see Theorem 3.8) give us that $U = 0$. That is,

$$T = \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}.$$

To finish, it is sufficient to show the direct character of the sum. For $x \in I_{[\alpha]} \cap \sum_{\substack{[\beta]\in\Lambda^1/\sim \\ \beta\not\sim\alpha}} I_{[\beta]}$, using again the equation $\{I_{[\alpha]}, T, I_{[\beta]}\} = 0$ for $[\alpha] \neq [\beta]$, we obtain

$$\{x, T, I_{[\alpha]}\} = \{x, T, \sum_{\substack{[\beta]\in\Lambda^1/\sim \\ \beta\not\sim\alpha}} I_{[\beta]}\} = 0.$$

So $\{x, T, T\} = \{x, T, I_{[\alpha]} + \sum_{\substack{[\beta]\in\Lambda^1/\sim \\ \beta\not\sim\alpha}} I_{[\beta]}\} = \{x, T, I_{[\alpha]}\} + \{x, T, \sum_{\substack{[\beta]\in\Lambda^1/\sim \\ \beta\not\sim\alpha}} I_{[\beta]}\} = 0 + 0 = 0$. We argue similarly. Using the equations $\{T, I_{[\alpha]}, I_{[\beta]}\} = 0$ and $\{I_{[\alpha]}, I_{[\beta]}, T\} = 0$ for $[\alpha] \neq [\beta]$, one gets $\{T, x, T\} = 0$ and $\{T, T, x\} = 0$. That is, $x \in \mathrm{Ann}(T) = 0$. Thus $x = 0$, as desired. □

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Yan Cao
School of Mathematics and Statistics
Northeast Normal University
Changchun 130024, P. R. China
and
Department of Basic Education
Harbin University of Science and Technology
Rongcheng Campus, Rongcheng 264300, P. R. China
E-mail address: 48069607@qq.com

Liangyun Chen
School of Mathematics and Statistics
Northeast Normal University
Changchun 130024, P. R. China
E-mail address: chenly640@nenu.edu.cn