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L^2 HARMONIC FORMS ON GRADIENT SHRINKING RICCI SOLITONS

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Abstract. In this paper, we study vanishing properties for L^2 harmonic 1-forms on a gradient shrinking Ricci soliton. We prove that if (M, g, f) is a complete oriented noncompact gradient shrinking Ricci soliton with potential function f, then there are no non-trivial L^2 harmonic 1-forms which are orthogonal to df. Second, we show that if the scalar curvature of the metric g is greater than or equal to $(n-2)/2$, then there are no non-trivial L^2 harmonic 1-forms on (M, g). We also show that any multiplication of the total differential df by a function cannot be an L^2 harmonic 1-form unless it is trivial. Finally, we derive various integral properties involving the potential function f and L^2 harmonic 1-forms, and handle their applications.

1. Introduction

A differential form ω on a Riemannian manifold (M, g) is said to be *harmonic* if it satisfies

$$\Delta\omega = (d\delta + \delta d)\omega = 0$$

and ω is said to be in L^2 if

$$\int_M \omega \wedge *\omega = \int_M |\omega|^2 \, dv_g < \infty,$$

where $*$ denotes the Hodge star operator and dv_g is the volume form of (M, g).

If ω is a harmonic 1-form, then its dual $\omega^\sharp$ is a harmonic vector field on M in the following sense: if we choose a local frame $e_1, \ldots, e_n$ such that $D_{e_i}e_j = 0$ at a point and if we denote $\omega^\sharp = \omega_i e_i$, then $D_{e_i}\omega_j = D_{e_j}\omega_i$ and $D_{e_i}\omega_i = 0$ at the point. Or, equivalently

$$\omega_{i;j} = \omega_{j;i} \quad \text{and} \quad \sum_i \omega_{i;i} = 0. \tag{1.1}$$

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It is well-known that if ω is an L^2 harmonic 1-form on a Riemannian manifold (M, g), then

$$d\omega = 0 \quad \text{and} \quad \delta\omega = 0. \tag{1.2}$$

The theory of L^2 harmonic differential forms can be used to study the geometry and topology of complete noncompact Riemannian manifolds.

In this paper, we study the structure of the space of L^2 harmonic 1-forms on a complete gradient shrinking Ricci soliton. A complete Riemannian metric g on a smooth manifold M^n is called a *Ricci soliton* if there exist a constant ρ and a smooth 1-form ω such that

$$2r_g + \mathcal{L}_{\omega^\sharp} g = 2\rho g, \tag{1.3}$$

where r_g is the Ricci tensor of the metric g, $\omega^\sharp$ is the vector field dual to ω, and $\mathcal{L}_{\omega^\sharp}$ denotes the Lie derivative along $\omega^\sharp$. Since $\mathcal{L}_{\omega^\sharp} g(X, Y) = D_X\omega(Y) + D_Y\omega(X)$ for any vector fields X and Y, (1.3) is equivalent to

$$2r_g(X, Y) + D_X\omega(Y) + D_Y\omega(X) = 2\rho g(X, Y). \tag{1.4}$$

Moreover if there is a smooth function f on M such that $\omega = df$, then g is called a *gradient Ricci soliton.* The Ricci soliton is said to be *shrinking, steady* and *expanding* according as $\rho > 0$, $\rho = 0$, $\rho < 0$. In case of gradient Ricci soliton, (1.3) becomes

$$r_g + Ddf = \rho g. \tag{1.5}$$

There are some books and expository articles on Ricci solitons and gradient Ricci solitons (cf. [3], [5], [6] and references are therein).

In [12], O. Munteanu and N. Sesum proved that if (M, g) is a gradient shrinking Kähler-Ricci soliton (see [12] for the definition of Kähler-Ricci soliton), or a gradient steady Ricci soliton, then there are no nontrivial harmonic functions with finite energy. Note that the total differential du of a nonconstant harmonic function u defined on a noncompact complete Riemannian manifold is a nontrivial harmonic 1-form. Furthermore, if u has finite energy, then the total differential becomes a nontrivial L^2 harmonic 1-form on M. Thus due to O. Munteanu and N. Sesum's result, there are no nontrivial L^2 harmonic 1-forms on a gradient shrinking Kähler-Ricci soliton or a gradient steady Ricci soliton. In case of shrinking Kähler-Ricci solitons (M, g, f), they proved that if u is a harmonic function with finite energy, then $\langle \nabla f, \nabla u\rangle = 0$.

Motivated by this property, we consider, in this paper, vanishing properties of L^2 harmonic 1-forms on a complete gradient shrinking Ricci soliton which is orthogonal to the total differential of the potential function as above. We prove a similar result as Munteanu and Sesum's result mentioned above holds in a complete gradient shrinking Ricci soliton.

Theorem A. *Let (M, g, f) be a complete oriented gradient shrinking Ricci soliton. Then there are no nontrivial L^2 harmonic 1-forms ω on M such that $\langle df, \omega\rangle = 0$.*

When the scalar curvature s_g of a complete gradient Ricci soliton (M, g, f) satisfies $(n-2)\rho \le s_g$, we can also show that there are no nontrivial L^2 harmonic 1-forms on (M, g).

Theorem B. *Let (M, g, f) be a complete noncompact oriented gradient shrinking Ricci soliton satisfying* (1.5) *with $(n-2)\rho \le s_g$. Then there are no nontrivial L^2 harmonic 1-forms on (M, g).*

In this paper, we also study various properties on the space of L^2 harmonic 1-forms on a complete gradient shrinking Ricci soliton, and derive various useful integral identities on L^2 harmonic 1-forms. Among them, we would like to mention the following property.

Theorem C. *Let (M, g, f) be a complete oriented gradient shrinking Ricci soliton satisfying* (1.5)*, and let ω be an L^2 harmonic 1-form on (M, g). Then*
(1) $\int_M e^{-f}\langle df, \omega\rangle^2 = \rho \int_M e^{-f}|\omega|^2 + \int_M e^{-f}|D\omega|^2$.
(2) $\int_M e^{-f}\langle Ddf, D\omega\rangle = 0$ *and* $\int_M e^{-f}\langle df, \omega\rangle = 0$.

Using (1) in Theorem C, we can recover the proof of Theorem A. And from (2), we can see a weaker version of Theorem A does hold. In fact, we can show that, on a complete oriented gradient Ricci soliton, there are no nontrivial L^2 harmonic 1-forms ω on M such that either $\langle df, \omega\rangle$ is nonnegative or constant.

2. Preliminaries and basic formulas

In this section, we shall state some basic well-known facts on Ricci solitons, and derive some integral properties involving differential 1-forms. First of all, taking the trace in (1.5), we have

$$\Delta f = n\rho - s_g, \quad d\Delta f = -ds_g, \quad \Delta s_g = -\Delta^2 f. \tag{2.1}$$

Note that the following identities on Riemannian manifolds hold without any condition:

$$\delta Ddf = -d\Delta f - r_g(\nabla f, \cdot) \tag{2.2}$$

and

$$\delta r_g = -\frac{1}{2}ds_g. \tag{2.3}$$

So, taking the divergence of both sides in (1.5) and using these identities, we obtain

$$-\frac{1}{2}ds_g - d\Delta f - r_g(\nabla f, \cdot) = 0,$$

which implies, from (2.1),

$$r_g(\nabla f, \cdot) = \frac{1}{2}ds_g \tag{2.4}$$

and

$$\delta Ddf = \frac{1}{2}ds_g. \tag{2.5}$$

Next, it is well-known ([2], [9]) that, for any gradient Ricci soliton (M, g, f),

$$s_g + |\nabla f|^2 - 2\rho f = C(\text{constant}). \tag{2.6}$$

In fact, using the Ricci soliton equation (1.5) and (2.4), we can easily show that

$$d\left(s_g + |\nabla f|^2 - 2\rho f\right) = 0.$$

By (2.3) and (2.4)

$$\frac{1}{2}\delta ds_g = -\frac{1}{2}\langle ds_g, df\rangle - \langle r_g, Ddf\rangle.$$

In fact, choosing an orthonormal basis $\{e_i\}$ such that $D_{e_i}e_j(p) = 0$ for some point $p \in M$, we have, at the point p,

$$\begin{aligned}\frac{1}{2}\delta ds_g &= -D_{e_i}(i_{\nabla f} r_g)(e_i) = -D_{e_i}(i_{\nabla f} r_g(e_i)) = -D_{e_i}(r_g(\nabla f, e_i))\\ &= -D_{e_i} r_g(\nabla f, e_i) - r_g(D_{e_i}\nabla f, e_i)\\ &= \delta r_g(\nabla f) - \langle r_g, Ddf\rangle\\ &= -\frac{1}{2}\langle ds_g, df\rangle - \langle r_g, Ddf\rangle.\end{aligned}$$

Thus, we obtain

$$\Delta s_g = \langle ds_g, df\rangle + 2\langle r_g, Ddf\rangle. \tag{2.7}$$

It follows from (1.5) that

$$\langle r_g, Ddf\rangle = \rho\Delta f - |Ddf|^2$$

and

$$\langle r_g, Ddf\rangle = \rho s_g - |r_g|^2.$$

Thus, we get

$$\Delta(2\rho f - s_g) + \langle ds_g, df\rangle = 2|Ddf|^2. \tag{2.8}$$

Note that the adjoint operator δ^* of the divergence operator δ on the space of symmetric 2-tensors is the composition of covariant derivative with symmetrization (cf. [1]). Thus on the space of 1-forms $\Omega^1(M)$, we have

$$\begin{aligned}\delta^*\alpha(X, Y) &= \frac{1}{2}\{D_X\alpha(Y) + D_Y\alpha(X)\}\\ &= \frac{1}{2}\mathcal{L}_{\alpha^\sharp} g(X, Y).\end{aligned}$$

Convention. When we are going to integrate some quantity on a gradient Ricci soliton (M, g, f), we omit the volume form dv_g. Thus $\int_M e^{-f}|r_g|^2$ just means $\int_M e^{-f}|r_g|^2\, dv_g$.

Lemma 2.1. *Let (M, g, f) be a compact gradient Ricci soliton satisfying* (1.5). *Then for any 1-form η,*

$$2\rho\int_M e^{-f}\langle df, \eta\rangle = \int_M e^{-f}\langle Ddf, \mathcal{L}_{\eta^\sharp} g\rangle.$$

Proof. Since $\delta^*\eta = \frac{1}{2}\mathcal{L}_{\eta^\sharp} g$, it follows from (1.5) that

$$\langle r_g, \delta^*\eta\rangle = \frac{\rho}{2}\langle g, \mathcal{L}_{\eta^\sharp} g\rangle - \frac{1}{2}\langle Ddf, \mathcal{L}_{\eta^\sharp} g\rangle.$$

From definition, we have

$$\langle g, \mathcal{L}_{\eta^\sharp} g\rangle = \mathrm{tr}_g \mathcal{L}_{\eta^\sharp} g = -2\delta\eta.$$

Thus

$$\langle r_g, \delta^*\eta\rangle = -\rho\delta\eta - \frac{1}{2}\langle Ddf, \mathcal{L}_{\eta^\sharp} g\rangle.$$

Since $\delta(e^{-f} r_g) = 0$ for gradient Ricci solitons, we have

$$\begin{aligned} 0 &= \int_M \langle \delta(e^{-f} r_g), \eta\rangle = \int_M e^{-f}\langle r_g, \delta^*\eta\rangle \\ &= -\rho\int_M e^{-f}\delta\eta - \frac{1}{2}\int_M e^{-f}\langle Ddf, \mathcal{L}_{\eta^\sharp} g\rangle \\ &= \rho\int_M e^{-f}\langle df, \eta\rangle - \frac{1}{2}\int_M e^{-f}\langle Ddf, \mathcal{L}_{\eta^\sharp} g\rangle. \end{aligned}$$

□

When $\eta = du$ for a function $u : M \to \mathbb{R}$, then

$$\rho\int_M e^{-f}\langle df, du\rangle = \int_M e^{-f}\langle Ddf, Ddu\rangle.$$

In particular, we have

$$\rho\int_M e^{-f}|df|^2 = \int_M e^{-f}|Ddf|^2. \tag{2.9}$$

Using (2.9), we can prove a well-known rigidity result which says that any compact gradient steady or expanding Ricci soliton is Einstein.

From now, assume that (M, g, f) be a complete noncompact gradient Ricci soliton.

Lemma 2.2. *Let (M, g, f) be a complete noncompact gradient Ricci soliton satisfying (1.5). Then for any 1-form η on M and any C^1 function ψ with compact support,*

$$2\rho\int_M \psi e^{-f}\langle df, \eta\rangle = \int_M \psi e^{-f}\langle Ddf, \mathcal{L}_{\eta^\sharp} g\rangle + 2\int_M e^{-f}\langle Ddf, d\psi\odot\eta\rangle,$$

where

$$d\psi\odot\eta(X, Y) = \frac{1}{2}\{d\psi(X)\eta(Y) + d\psi(Y)\eta(X)\}.$$

Proof. Applying Lemma 2.1 to the 1-form $\alpha := \psi\eta$ which has compact support, we have

$$2\rho\int_M \psi e^{-f}\langle df, \eta\rangle = \int_M e^{-f}\langle Ddf, \mathcal{L}_{\alpha^\sharp} g\rangle. \tag{2.10}$$

Note that

$$\mathcal{L}_{(\psi\eta)^\sharp} g = \psi\mathcal{L}_{\eta^\sharp} g + 2d\psi\odot\eta. \tag{2.11}$$

Therefore,

$$2\rho\int_M \psi e^{-f}\langle df,\eta\rangle = \int_M \psi e^{-f}\langle Ddf,\mathcal{L}_{\eta^\sharp}g\rangle + 2\int_M e^{-f}\langle Ddf, d\psi\odot\eta\rangle. \qquad \square$$

If $\eta = du$ for a function $u : M \to \mathbb{R}$, then

$$\rho\int_M \psi e^{-f}\langle df,du\rangle = \int_M \psi e^{-f}\langle Ddf,Ddu\rangle + \int_M e^{-f}Ddf(\nabla u,\nabla\psi).$$

In particular,

$$\rho\int_M \psi e^{-f}|df|^2 = \int_M \psi e^{-f}|Ddf|^2 + \int_M e^{-f}Ddf(\nabla f,\nabla\psi). \tag{2.12}$$

Notation. From now, for convenience we will use some confused notations for vector fields and 1-forms if there is no ambiguity. For example, we use ω for both 1-form ω and vector field $\omega^\sharp$ which is dual to ω, and df for both vector field ∇f and the total differential df as a 1-form. This means that

$$r_g(\omega,\omega) = r_g(\omega^\sharp,\omega^\sharp), \quad r_g(df,df) = r_g(\nabla f,\nabla f)$$

and

$$Ddf(\omega,\omega) = Ddf(\omega^\sharp,\omega^\sharp), \quad Ddf(df,\omega) = Ddf(\nabla f,\omega^\sharp)$$

etc. And, by a definition, a cut-off function φ means that

$$0\le\varphi\le 1, \quad |\nabla\varphi|\le\frac{2}{r}, \quad \varphi = 1 \text{ on } B\left(\frac{r}{2}\right)$$

and

$$\operatorname{supp}(\varphi)\subset B(r)$$

for a geodesic ball $B(r)$ at a point in M.

3. Vanishing property of L^2 harmonic 1-forms

In this section, we are going to show vanishing properties of L^2 harmonic 1-forms on a complete oriented gradient shrinking Ricci soliton (M,g,f) by using Bochner formula for f-Hodge Laplancian. Let

$$\delta_f = \delta + \iota_{\nabla f},$$

where $\iota_{\nabla f}$ is the interior product with the vector field ∇f. The f-Hodge Laplacian is defined by

$$\Delta_f = -\left(d\delta_f + \delta_f d\right).$$

Then it is well-known that, for a 1-form ω on a smooth metric measure space $(M,g,e^{-f}dv_g)$,

$$\frac{1}{2}\Delta_f|\omega|^2 = |D\omega|^2 + \langle\Delta_f\omega,\omega\rangle + \mathrm{Ric}_f(\omega,\omega), \tag{3.1}$$

where $\mathrm{Ric}_f = r_g + Ddf$ (cf. [11] or [16]).

Theorem 3.1. *Let (M,g,f) be a complete noncompact oriented gradient shrinking Ricci soliton satisfying* (1.5). *Then there are no nontrivial L^2 harmonic 1-forms ω on M such that $\langle df,\omega\rangle = 0$.*

Proof. Let ω be an L^2 harmonic 1-forms ω on (M, g, f) such that $\langle df, \omega\rangle = 0$. First of all, in case of gradient Ricci soliton, we have

$$\mathrm{Ric}_f(\omega,\omega) = \rho|\omega|^2.$$

Since $d\omega = 0 = \delta\omega, \Delta\omega = 0$ and

$$\Delta_f = \Delta - (d\iota_{\nabla f} + \iota_{\nabla f} d),$$

we have

$$\Delta_f \omega = -d\langle\omega, df\rangle = 0$$

by assumption. Thus, from (3.1) and Kato's inequality, we obtain

$$\frac{1}{2}\Delta_f|\omega|^2 = |D\omega|^2 + \rho|\omega|^2 \geq |\nabla|\omega||^2 + \rho|\omega|^2. \tag{3.2}$$

Let φ be a cut-off function on M. Multiplying (3.2) by $\varphi^2 e^{-f}$ and integrating it over M, we have

$$\begin{aligned}\int_M \varphi^2 e^{-f}|\nabla|\omega||^2 + \rho\int_M \varphi^2 e^{-f}|\omega|^2 &\leq \frac{1}{2}\int_M \varphi^2 e^{-f}\Delta_f|\omega|^2\\ &= -\frac{1}{2}\int_M e^{-f}\langle\nabla\varphi^2, \nabla|\omega|^2\rangle\\ &\leq 2\int_M e^{-f}\varphi|\omega||\nabla\varphi||\nabla|\omega||\\ &\leq \int_M e^{-f}\varphi^2|\nabla|\omega||^2 + \int_M e^{-f}|\nabla\varphi|^2|\omega|^2.\end{aligned}$$

Thus, we obtain

$$\rho\int_M \varphi^2 e^{-f}|\omega|^2 \leq \frac{4}{r^2}\int_M |\omega|^2.$$

Letting $r \to \infty$, we have $\omega = 0$. □

Remark 3.2. Applying Theorem 4.6 in [14] or Theorem 4.2 in [15] to (3.2), we have $|\omega|$ is constant. It is well-known that a complete oriented noncompact gradient shrinking Ricci soliton has an infinite volume (cf. [13]). Thus ω should be trivial.

Theorem 3.1 can be reformulated as follows:

Theorem 3.3. *Let (M, g, f) be a complete oriented noncompact gradient shrinking Ricci soliton. Then there are no nontrivial L^2 harmonic* 1*-forms ω on M such that, on each level hypersurface $f^{-1}(c)$ with a regular value c of f, the vector field $\omega^\sharp$ dual to ω is tangent to $f^{-1}(c)$.*

Next, we are going to show vanishing property of L^2 harmonic 1-forms on a complete oriented noncompact gradient shrinking Ricci solition (M, g, f) satisfying (1.5) with $(n-2)\rho \leq s_g$. Let φ be a cut-off function and let ω be an L^2

harmonic 1-form on a complete oriented noncompact gradient shrinking Ricci solition (M, g, f) satisfying (1.5). Then

$$\int_M \varphi Ddf(\omega,\omega) = \int_M f_{;ij}\omega_i\omega_j\varphi = -\int_M f_{;i}(\omega_i\omega_j\varphi)_{;j} \tag{3.3}$$
$$= -\int_M f_{;i}\omega_{i;j}\omega_j\varphi - \int_M f_{;i}\omega_i\omega_{j;j}\varphi - \int_M f_{;i}\omega_i\omega_j(\varphi)_{;j}$$
$$= -\int_M f_{;i}\omega_{i;j}\omega_j\varphi - \int_M f_{;i}\omega_i\omega_j(\varphi)_{;j}.$$

Note that

$$\int_M f_{;i}\omega_{i;j}\omega_j\varphi = \int_M f_{;i}\omega_{j;i}\omega_j\varphi = -\int_M \omega_j(f_{;i}\omega_j\varphi)_{;i}$$
$$= -\int_M \omega_j f_{;ii}\omega_j\varphi - \int_M \omega_j f_{;i}\omega_{j;i}\varphi - \int_M \omega_j f_{;i}\omega_j(\varphi)_{;i}.$$

Thus

$$\int_M f_{;i}\omega_{i;j}\omega_j\varphi = -\frac{1}{2}\int_M (\Delta f)|\omega|^2\varphi - \frac{1}{2}\int_M |\omega|^2\langle df, d\varphi\rangle.$$

Plugging this into (3.3), we obtain

$$\int_M \varphi Ddf(\omega,\omega) \tag{3.4}$$
$$= \frac{1}{2}\int_M (\Delta f)|\omega|^2\varphi + \frac{1}{2}\int_M |\omega|^2\langle df, d\varphi\rangle - \int_M \langle df,\omega\rangle\langle\omega, d\varphi\rangle.$$

To prove Theorem B, we need the following property on a complete noncompact gradient shrinking Ricci soliton (M, g, f) which is well-known ([4], [12]):

$$\frac{1}{4}(r(x)-c)^2 \le f(x) \le \frac{1}{4}(r(x)+C)^2 \tag{3.5}$$

for some positive constants c and C. Here $r(x) = \mathrm{dist}(p, x)$ is the distance function from a fixed point $p \in M$. Thus, we have

$$|df| = \mathrm{O}(r) \tag{3.6}$$

as $r \to \infty$.

Lemma 3.4. *Let (M, g, f) be a complete noncompact oriented gradient shrinking Ricci soliton satisfying* (1.5). *Then for any L^2 harmonic* 1*-form on M,*

$$\int_M Ddf(\omega,\omega) = \frac{1}{2}\int_M (\Delta f)|\omega|^2. \tag{3.7}$$

Proof. Let φ be a cut-off function on M. Then, by (3.6),

$$\left|\int_M |\omega|^2\langle df, d\varphi\rangle\right| \le \int_M |\omega|^2|df||d\varphi| \le C\int_{B(r)\setminus B\left(\frac{r}{2}\right)} |\omega|^2.$$

Since ω is in L^2, this tends to 0 as $r \to \infty$. The same argument also shows

$$\lim_{r\to\infty} \int_M \langle df, \omega\rangle\langle \omega, d\varphi\rangle = 0.$$

So, the proof follows from (3.4). □

Theorem 3.5. *Let* (M, g, f) *be a complete noncompact oriented gradient shrinking Ricci soliton satisfying* (1.5) *with* $(n-2)\rho \le s_g$. *Then there are no nontrivial* L^2 *harmonic* 1*-forms on* (M, g).

Proof. It follows from Lemma 3.4 together with the Ricci soliton equation (1.5) and (2.1) that

$$\int_M r_g(\omega,\omega) = \frac{1}{2}\int_M [s_g - (n-2)\rho]\,|\omega|^2 \ge 0. \tag{3.8}$$

Recall the usual Bochner-Weitzenböck formula

$$\frac{1}{2}\Delta|\omega|^2 = |D\omega|^2 + r_g(\omega,\omega) \tag{3.9}$$

for harmonic 1-forms ω. Since

$$\frac{1}{2}\Delta|\omega|^2 = |\omega|\Delta|\omega| + |\nabla|\omega||^2$$

and $|D\omega|^2 \ge |\nabla|\omega||^2$ by Kato's inequality, we have

$$|\omega|\Delta|\omega| \ge r_g(\omega,\omega). \tag{3.10}$$

Let φ be a cut-off function on M. Multiplying (3.10) by φ^2 and integrating it over M, we have

$$\begin{aligned}\int_M \varphi^2 r_g(\omega,\omega) &\le \int_M \varphi^2|\omega|\Delta|\omega| \\ &= -\int_M \varphi^2|\nabla|\omega||^2 - 2\int_M \varphi|\omega|\langle\nabla\varphi, \nabla|\omega|\rangle \\ &\le -\int_M \varphi^2|\nabla|\omega||^2 + 2\int_M \varphi|\omega||\nabla\varphi||\nabla|\omega||.\end{aligned}$$

By the inequality $\epsilon a^2 + \frac{1}{\epsilon}b^2 \ge 2ab$ for $a, b > 0$, we have

$$2\int_M \varphi|\omega||\nabla\varphi||\nabla|\omega|| \le \frac{1}{4}\int_M \varphi^2|\nabla|\omega||^2 + 4\int_M |\nabla\varphi|^2|\omega|^2.$$

Thus,

$$\begin{aligned}\int_M \varphi^2 r_g(\omega,\omega) &\le -\frac{3}{4}\int_M \varphi^2|\nabla|\omega||^2 + 4\int_M |\nabla\varphi|^2|\omega|^2 \\ &\le -\frac{3}{4}\int_M \varphi^2|\nabla|\omega||^2 + \frac{16}{r^2}\int_M |\omega|^2.\end{aligned}$$

Letting $r \to \infty$, $|\omega|$ should be constant by (3.8). Since (M, g) has an infinite volume, $\omega = 0$. □

Remark 3.6. Lack of examples, the condition $s_g \geq (n-2)\rho$ on a gradient shrinking Ricci soliton looks a little strong. For instance, $M = \mathbb{R} \times S^{n-1}$ or $M = \mathbb{R}^2 \times S^{n-2}$ with product metric and $f(x) = \frac{\rho}{2}|x|^2$ for $x \in \mathbb{R}$ or $\mathbb{R}^2$ satisfies this condition. Of course, it is easy to see that those manifolds do not admit nontrivial L^2 harmonic 1-forms. We don't know whether other gradient shrinking Ricci solitons satisfying the scalar curvature condition $(n-2)\rho \leq s_g$ exist.

The next result shows that the total differential of the potential function on a complete noncompact gradient Ricci soliton cannot be an L^2 harmonic 1-form unless it is constant.

Theorem 3.7. *Let (M, g, f) be a gradient shrinking Ricci soliton which is not Einstein. Assume that the scalar curvature s_g satisfies*

$$s_g(x) \leq Cr(x) \tag{3.11}$$

for some positive constant C, where $r(x) = \mathrm{dist}(p, x)$ for a fixed point p. Then for any smooth function α, $\xi := \alpha df$ cannot be L^2 harmonic 1-form except $\alpha = 0$.

Proof. First, assume that $\xi := \alpha df$ is an L^2 harmonic 1-form with $\alpha > 0$. Then we have

$$\int_M \alpha^2 |df|^2 < \infty \tag{3.12}$$

and

$$d\xi = d\alpha \wedge df = 0, \quad \delta\xi = -\langle d\alpha, df\rangle - \alpha\Delta f = 0. \tag{3.13}$$

Thus, we have the following PDE:

$$\Delta f + \langle \nabla \log \alpha, \nabla f\rangle = 0. \tag{3.14}$$

Recall that $f \sim \mathrm{O}(r^2)$ and so $|\nabla f| \sim \mathrm{O}(r)$ from (2.6). Since $\nabla \log \alpha$ is parallel to ∇f by (3.13), (3.14) together with the fact $\Delta f = n\rho - s_g$ and our assumption (3.11) shows that

$$|\nabla \log \alpha|$$

is bounded. By (3.5), f should attain its local minimum at somewhere point. It follows from maximum principle (cf. [8], Theorem 3.5) that f should be constant on a geodesic ball, which means that f is constant on M. This contradicts that (M, g, f) is not Einstein.

Now assume that α is arbitrary. Let

$$\Omega^+ = \{x \in M \, : \, \alpha(x) > 0\}.$$

Replacing αdf by $-\alpha df$ if necessary, we may assume that Ω^+ is unbounded open subset of M. Choose a geodesic ball $B \subset \Omega^+$ containing a local minimum point of f. Applying arguments mentioned above to B, f should be constant on B and so constant on M since $\Delta f = n\rho - s_g$. Consequently, αdf cannot be an L^2 harmonic 1-form unless it is trivial. □

4. Integral properties and generalizations

In this section, we shall derive various integral identities including L^2 harmonic 1-forms on a complete gradient shrinking Ricci soliton, and consider generalizations of results mentioned in previous section. To do this, we need, first, the following Ricci identity which is well-known.

Lemma 4.1. *Let ω be an L^2 harmonic 1-form on a Riemannian manifold (M, g). Then*

$$-D^*D\omega = r_g(\omega, \cdot), \tag{4.1}$$

where D^* is the adjoint of the covariant derivative D and $D^*D\omega$ is given by

$$D^*D\omega = -\sum_{i=1}^{n} (D_{e_i}D_{e_i}\omega - D_{e_i}e_i\omega)$$

and $\{e_i\}$ is a local orthonormal frame.

The following property on a complete noncompact gradient shrinking Ricci soliton (M, g, f) is well-known ([12]):

$$\int_M e^{-f}|r_g|^2 < \infty. \tag{4.2}$$

Theorem 4.2. *Let (M, g, f) be a complete oriented gradient shrinking Ricci soliton. Then for any L^2 harmonic 1-form ω,*

$$\int_M e^{-f}\langle df, \omega\rangle^2 = \rho\int_M e^{-f}|\omega|^2 + \int_M e^{-f}|D\omega|^2. \tag{4.3}$$

Proof. Let φ be a cut-off function. Since $\delta(e^{-f}\omega) = e^{-f}\langle df, \omega\rangle$, we get

$$\begin{aligned}\int_M \varphi^2 e^{-f}\langle df, \omega\rangle^2 &= \int_M \varphi^2\langle df, \omega\rangle\delta(e^{-f}\omega)\\ &= \int_M e^{-f}\langle d(\varphi^2\langle df, \omega\rangle), \omega\rangle\\ &= 2\int_M e^{-f}\varphi\langle df, \omega\rangle\langle\omega, d\varphi\rangle\\ &\quad+\int_M e^{-f}\varphi^2\left[Ddf(\omega, \omega) + D\omega(df, \omega)\right].\end{aligned} \tag{4.4}$$

Next, from the Ricci identity (4.1) together with the Ricci soliton equation (1.5), we have

$$-D^*D\omega(\omega) = r_g(\omega, \omega) = \rho|\omega|^2 - Ddf(\omega, \omega). \tag{4.5}$$

Multiplying both sides by $e^{-f}\varphi^2$ and integrating it over M, we get

$$\begin{aligned}&\rho\int_M e^{-f}\varphi^2|\omega|^2 - \int_M e^{-f}\varphi^2 Ddf(\omega, \omega)\\ &= -\int_M \langle D^*D\omega, e^{-f}\varphi^2\omega\rangle\end{aligned}$$

$$= -\int_M \langle D\omega, D(e^{-f}\varphi^2\omega)\rangle$$
$$= \int_M e^{-f}\varphi^2 D\omega(df,\omega) - 2\int_M e^{-f}\varphi D\omega(d\varphi,\omega) - \int_M e^{-f}\varphi^2|D\omega|^2.$$

Thus

$$\int_M e^{-f}\varphi^2 Ddf(\omega,\omega) + \int_M e^{-f}\varphi^2 D\omega(df,\omega)$$
$$= \rho\int_M e^{-f}\varphi^2|\omega|^2 + \int_M e^{-f}\varphi^2|D\omega|^2 + 2\int_M e^{-f}\varphi D\omega(d\varphi,\omega).$$

This together with (4.4) shows that

$$\int_M \varphi^2 e^{-f}\langle df,\omega\rangle^2 - 2\int_M e^{-f}\varphi\langle df,\omega\rangle\langle\omega,d\varphi\rangle$$
$$= \rho\int_M e^{-f}\varphi^2|\omega|^2 + \int_M e^{-f}\varphi^2|D\omega|^2 + 2\int_M e^{-f}\varphi D\omega(d\varphi,\omega). \tag{4.6}$$

Finally, using the usual Bochner-Weitzenböck formula (3.9) together with the fact (3.5), we can see that $\int_M e^{-f}|D\omega|^2$ is bounded and so by letting $r\to\infty$, the third term in (4.6) tends to 0. By (4.2), the second term also tends to 0 as $r\to\infty$. Consequently, we obtain

$$\int_M e^{-f}\langle df,\omega\rangle^2 = \rho\int_M e^{-f}|\omega|^2 + \int_M e^{-f}|D\omega|^2.$$

□

As a direct consequence of Theorem 4.2, we can recover Theorem 3.1.

Lemma 4.3. *Let (M,g,f) be a gradient shrinking Ricci soliton. Then for any L^2 harmonic 1-form ω on M, we have*

$$\int_M e^{-f}\langle Ddf, D\omega\rangle = 0 \quad \textit{and} \quad \int_M e^{-f}\langle df,\omega\rangle = 0. \tag{4.7}$$

Proof. Let φ be a cut-off function on M. Since $\delta(e^{-f}r_g)=0$, we have

$$0 = \int_M \langle\varphi\omega, \delta(e^{-f}r_g)\rangle = \int_M \langle\delta^*(\varphi\omega), e^{-f}r_g\rangle. \tag{4.8}$$

Since ω is harmonic, we have

$$\delta^*(\varphi\omega) = d\varphi\odot\omega + \varphi D\omega$$

and so

$$\langle\delta^*(\varphi\omega), r_g\rangle = \langle d\varphi\odot\omega + \varphi D\omega, \rho g - Ddf\rangle$$
$$= \rho\langle d\varphi\odot\omega, g\rangle - \langle d\varphi\odot\omega, Ddf\rangle - \varphi\langle D\omega, Ddf\rangle.$$

By Cauchy-Schwarz inequality,

$$\int_M \varphi e^{-f}\langle D\omega, Ddf\rangle = \int_M \rho e^{-f}\langle d\varphi\odot\omega, g\rangle - \int_M e^{-f}\langle d\varphi\odot\omega, Ddf\rangle$$

$$\leq \rho\left(\int_M e^{-f}\right)^{\frac{1}{2}}\left(\int_M |d\varphi|^2|\omega|^2\right)^{\frac{1}{2}} + \left(\int_M e^{-f}|Ddf|^2\right)^{\frac{1}{2}}\left(\int_M |d\varphi|^2|\omega|^2\right)^{\frac{1}{2}}. \tag{4.9}$$

By letting $r \to \infty$, we have

$$\int_M e^{-f}\langle D\omega, Ddf\rangle = 0. \tag{4.10}$$

Next, recall, by (2.5), that

$$\delta Ddf = r_g(df, \cdot).$$

So, we have

$$\begin{aligned}\int_M e^{-f}\varphi^2\langle \delta Ddf, \omega\rangle &= \int_M e^{-f}\varphi^2 r_g(df, \omega)\\ &= \rho\int_M e^{-f}\varphi^2\langle df, \omega\rangle - \int_M e^{-f}\varphi^2 Ddf(df, \omega).\end{aligned}$$

On the other hand, integration by parts shows

$$\begin{aligned}\int_M e^{-f}\varphi^2\langle \delta Ddf, \omega\rangle &= \int_M \langle Ddf, \delta^*(e^{-f}\varphi^2\omega)\rangle\\ &= -\int_M e^{-f}\varphi^2 Ddf(df, \omega) + 2\int_M e^{-f}\varphi Ddf(d\varphi, \omega)\\ &\quad + \int_M e^{-f}\varphi^2\langle Ddf, D\omega\rangle.\end{aligned}$$

Comparing these two equalities, we have

$$\rho\int_M e^{-f}\varphi^2\langle df, \omega\rangle = 2\int_M e^{-f}\varphi Ddf(d\varphi, \omega) + \int_M e^{-f}\varphi^2\langle Ddf, D\omega\rangle.$$

Since

$$\left|\int_M e^{-f}\varphi Ddf(d\varphi, \omega)\right| \leq \left(\int_M e^{-f}\varphi^2|Ddf|^2\right)^{\frac{1}{2}}\left(\int_M e^{-f}|\omega|^2\,|\nabla\varphi|^2\right)^{\frac{1}{2}},$$

we have, from (4.10),

$$\int_M e^{-f}\langle df, \omega\rangle = 0.$$

□

Using Lemma 4.3, we can generalize Theorem 3.1 as follows.

Theorem 4.4. *Let (M, g, f) be a gradient shrinking Ricci soliton. If ω is an L^2 harmonic 1-form on M satisfying either*

(i) *$\langle df, \omega\rangle$ is nonnegative or nonpositive on the whole M, or*
(ii) *$\langle df, \omega\rangle$ is constant,*

then ω is trivial.

Remark 4.5. We can prove the second part of Theorem 4.4 by using Lemma 4.3 and the co-area formula. In fact, we may assume that f is not constant. Suppose that ω is an L^2 harmonic 1-form such that the angle θ between ∇f and $\omega^\sharp$ is constant. By Lemma 4.3 and the co-area formula,

$$\begin{aligned} 0 &= \int_M e^{-f}\langle df, \omega\rangle = \int_M e^{-f}|df||\omega|\cos\theta \\ &= \int_0^\infty e^{-t}\left(\int_{f^{-1}(t)} |\omega|\cos\theta\, d\sigma\right) dt \\ &= \cos\theta \int_0^\infty e^{-t}\int_{f^{-1}(t)} |\omega|\, d\sigma\, dt. \end{aligned}$$

If $\cos\theta = 0$, then $\langle df, \omega\rangle = 0$ and so $\omega = 0$ by Theorem 3.1. If $\cos\theta \neq 0$, then $\omega = 0$.

Next, we will derive a formula on the Laplacian of the function $\langle df, \omega\rangle$ for an L^2 harmonic 1-form on a gradient shrinking Ricci soliton (M, g, f). Recall that the Bochner-Weitzenböck formulas

$$\frac{1}{2}\Delta|\nabla f|^2 = |Ddf|^2 + \langle d\Delta f, df\rangle + r_g(df, df)$$

and

$$\frac{1}{2}\Delta|\omega|^2 = |D\omega|^2 + r_g(\omega, \omega)$$

for a harmonic 1-form ω. Since $d\Delta f = -ds_g = -2r_g(df, \cdot)$,

$$\begin{aligned} \frac{1}{2}\Delta|\omega + df|^2 &= |D\omega + Ddf|^2 + r_g(\omega + df, \omega + df) + \langle \Delta(\omega + df), \omega + df\rangle \\ &= \frac{1}{2}\Delta|\omega|^2 + \frac{1}{2}\Delta|\nabla f|^2 + 2\langle D\omega, Ddf\rangle. \end{aligned}$$

On the other hand,

$$\frac{1}{2}\Delta|\omega + df|^2 = \frac{1}{2}\Delta|\omega|^2 + \Delta\langle\omega, df\rangle + \frac{1}{2}\Delta|df|^2. \tag{4.11}$$

Comparing these two identities, we get

$$\Delta\langle\omega, df\rangle = 2\langle D\omega, Ddf\rangle. \tag{4.12}$$

From the Ricci identity in Lemma 4.1, we also have $D^*D\omega(df) = -r_g(df, \omega) = -\frac{1}{2}ds_g(\omega)$, i.e.,

$$\frac{1}{2}\langle ds_g, \omega\rangle = -\langle df, D^*D\omega\rangle.$$

Theorem 4.6. *Let (M, g, f) be a complete oriented gradient Ricci soliton. Let ω be an L^2 harmonic 1-form. If $\langle Ddf, D\omega\rangle \geq 0$ or $\langle Ddf, D\omega\rangle \leq 0$ on the whole M, then $\langle df, \omega\rangle$ is a harmonic function.*

Proof. It is obvious by Lemma 4.3 and (4.12). □

We are going to mention a few more properties related to L^2 harmonic 1-forms. First of all, it follows from (4.4) by letting $r \to \infty$ that

$$\int_M e^{-f} \langle df, \omega \rangle^2 = \int_M e^{-f} \left[Ddf(\omega, \omega) + D\omega(df, \omega) \right]. \tag{4.13}$$

Lemma 4.7. *Let ω be an L^2 harmonic 1-form on a gradient shrinking Ricci soliton (M, g, f). Then*

$$\int_M e^{-f} [Ddf(df, \omega) + D\omega(df, df)] = 0 \tag{4.14}$$

and

$$\frac{1}{2} \int_M e^{-f} s_g \langle df, \omega \rangle = \int_M e^{-f} D\omega(df, df). \tag{4.15}$$

Proof. Since $\Delta \langle df, \omega \rangle = 2 \langle Ddf, D\omega \rangle$, we have

$$\begin{aligned} 2 \int_M e^{-f} \varphi^2 \langle Ddf, D\omega \rangle &= \int_M e^{-f} \varphi^2 \Delta \langle df, \omega \rangle = - \int_M \langle d(e^{-f} \varphi^2), d\langle df, \omega \rangle \rangle \\ &= \int_M e^{-f} \varphi^2 [Ddf(df, \omega) + D\omega(df, df)] \\ &\quad - 2 \int_M e^{-f} \varphi [Ddf(d\varphi, \omega) + D\omega(d\varphi, df)]. \end{aligned}$$

By letting $r \to \infty$, both the third and fourth terms tend to 0, respectively, and so we have

$$2 \int_M e^{-f} \langle Ddf, D\omega \rangle = \int_M e^{-f} [Ddf(df, \omega) + D\omega(df, df)]. \tag{4.16}$$

By Lemma 4.3, we have

$$\int_M e^{-f} [Ddf(df, \omega) + D\omega(df, df)] = 0. \tag{4.17}$$

Next, since $\int_M e^{-f} \langle df, \omega \rangle = 0$, we have

$$\int_M e^{-f} Ddf(df, \omega) = - \int_M e^{-f} r_g(df, \omega)$$

from the Ricci soliton equation. So, (4.14) is equivalent to

$$\int_M e^{-f} r_g(df, \omega) = \int_M e^{-f} D\omega(df, df). \tag{4.18}$$

Also since

$$\int_M e^{-f} r_g(df, \omega) = \frac{1}{2} \int_M e^{-f} \langle ds_g, \omega \rangle = \frac{1}{2} \int_M s_g \delta(e^{-f} \omega) = \frac{1}{2} \int_M e^{-f} s_g \langle df, \omega \rangle,$$

we have

$$\frac{1}{2} \int_M e^{-f} s_g \langle df, \omega \rangle = \int_M e^{-f} D\omega(df, df). \tag{4.19}$$

□

By (4.14) and (4.17), we obtain

$$\frac{1}{2}\int_M e^{-f}s_g\langle df,\omega\rangle + \int_M e^{-f}Ddf(df,\omega) = 0. \tag{4.20}$$

We can show (4.20) from the following identity which can be obtain from the Ricci soliton equation (1.5),

$$\frac{1}{2}\langle ds_g,\omega\rangle = \rho\langle df,\omega\rangle - Ddf(df,\omega), \tag{4.21}$$

by multiplying (4.21) by $e^{-f}\varphi^2$ and integrating it over M.

Finally, we would like to mention an integral identity which is similar as Lemma 3.4. To derive this, we need, first, the following.

Lemma 4.8 ([7], [10])**.** *Let ω be an L^2 harmonic 1-form on a Riemannian manifold. Then for any smooth bounded domain $D \subset\subset M$ and smooth vector field X, we have*

$$\int_D \left\{\langle (DX)\omega,\omega\rangle - \frac{1}{2}(\mathrm{div}X)|\omega|^2\right\} = \int_{\partial D}\left\{\langle i_X\omega, i_\nu\omega\rangle - \frac{1}{2}\langle X,\nu\rangle|\omega|^2\right\}. \tag{4.22}$$

Here $(DX)\omega(Y)$ is defined by $(DX)\omega(Y) = \omega(D_Y X)$.

Lemma 4.9. *Let ω be an L^2 harmonic 1-form on a complete noncompact gradient shrinking Ricci soliton (M,g,f). Then*

$$\int_M\left\{e^{-f}Ddf(\omega,\omega) - \frac{1}{2}e^{-f}(\Delta f)|\omega|^2\right\} = \int_M\left\{e^{-f}\langle df,\omega\rangle^2 - \frac{1}{2}e^{-f}|df|^2|\omega|^2\right\}. \tag{4.23}$$

Proof. Let $D = B(r)$ be a geodesic ball and let

$$X := e^{-f}\nabla f.$$

Then we have

$$DX = -e^{-f}df\otimes df + e^{-f}Ddf \tag{4.24}$$

and so

$$\langle (DX)\omega,\omega\rangle = -e^{-f}\langle df,\omega\rangle^2 + e^{-f}Ddf(\omega,\omega).$$

Note also that

$$\frac{1}{2}(\mathrm{div}X)|\omega|^2 = -\frac{1}{2}e^{-f}|df|^2|\omega|^2 + \frac{1}{2}e^{-f}(\Delta f)|\omega|^2$$

and

$$\langle i_X\omega, i_\nu\omega\rangle - \frac{1}{2}\langle X,\nu\rangle|\omega|^2 = e^{-f}\langle df,\omega\rangle\langle\omega,\nu\rangle - \frac{1}{2}e^{-f}\langle df,\nu\rangle|\omega|^2.$$

Substituting these into (4.22) in Lemma 4.8, we obtain

$$\int_{B(r)}\left\{-e^{-f}\langle df,\omega\rangle^2 + e^{-f}Ddf(\omega,\omega)\right\} \tag{4.25}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{B(r)} e^{-f}(|df|^2-\Delta f)|\omega|^2 \\
&= \int_{\partial B(r)} \left\{ e^{-f}\langle df,\omega\rangle\langle\omega,\nu\rangle - \frac{1}{2}e^{-f}\langle df,\nu\rangle|\omega|^2 \right\}.
\end{aligned}
$$

Letting $r\to\infty$, the right hand side tends to 0, and so we have (4.23). □

5. Decomposition of L^2 harmonic 1-forms

In this section, we consider an L^2 closed 1-form, but not necessarily harmonic on a complete gradient shrinking Ricci soliton (M, g, f). We shall derive some conditions so that such a form vanishes, and apply this to the decomposition of an L^2 harmonic 1-form on (M, g, f).

Let η be a closed 1-form on a complete noncompact oriented gradient shrinking Ricci soliton (M, g, f) satisfying

$$
\langle df,\eta\rangle = 0 \quad \text{and} \quad \int_M |\eta|^2 < \infty. \tag{5.1}
$$

We have the following Ricci identity for a closed 1-form which is similar as Lemma 4.1 for harmonic 1-forms.

Lemma 5.1. *Let η be a closed 1-form on a gradient shrinking Ricci soliton (M, g, f). Then*

$$
-D^*D\eta = r_g(\eta,\cdot) - d\delta\eta. \tag{5.2}
$$

Proof. Let $\{e_i\}$ be a local frame which is normal at a point. Writing $\omega = \sum \omega_i e_i$, from $d\eta = 0$, we have $\eta_{i,j} = \eta_{j,i}$ for each i, j. So, denoting $r_g(e_i, e_j) = r_{ij}$ and applying the Einstein convention,

$$
\begin{aligned}
D^*D\eta(e_k) &= -D_{e_i}D_{e_i}\eta(e_k) = -D_{e_i}D_{e_k}\eta(e_i) \\
&= -D_{e_k}D_{e_i}\eta(e_i) - R(e_i,e_k)\eta(e_i) \\
&= -e_k(\delta\eta) - \eta(R(e_i,e_k)e_i) \\
&= d\delta\eta(e_k) - r_g(\eta,\cdot).
\end{aligned}
$$

This implies that

$$
-D^*D\omega = r_g(\omega,\cdot) - d\delta\eta. \qquad \square
$$

Let η be a closed 1-form on a gradient shrinking Ricci soliton (M, g, f) satisfying (5.1). From Lemma 5.1 together with the Ricci soliton equation (1.5), we have

$$
-D^*D\eta(\eta) = \rho|\eta|^2 - Ddf(\eta,\eta) - \langle d\delta\eta,\eta\rangle.
$$

Let φ be a cut-off function on M. Multiplying by $e^{-f}\varphi^2$ and integrating it over M, we get

$$
\rho\int_M e^{-f}\varphi^2|\eta|^2 - \int_M e^{-f}\varphi^2 Ddf(\eta,\eta) - \int_M e^{-f}\varphi^2\langle d\delta\eta,\eta\rangle
$$

$$= -\int_M e^{-f}\varphi^2 D^*D\eta(\eta) = -\int_M \langle D\eta, D(e^{-f}\varphi^2\eta)\rangle$$
$$= \int_M e^{-f}\varphi^2 D\eta(df,\eta) - 2\int_M e^{-f}\varphi D\eta(d\varphi,\eta) - \int_M e^{-f}\varphi^2|D\eta|^2.$$

Thus,

$$\begin{aligned}
&\int_M e^{-f}\varphi^2\left[Ddf(\eta,\eta) + D\eta(df,\eta)\right]\\
&= \rho\int_M e^{-f}\varphi^2|\eta|^2 - \int_M \delta\eta\left(\delta(e^{-f}\varphi^2\eta)\right)\\
&\quad +2\int_M e^{-f}\varphi D\eta(d\varphi,\eta) + \int_M e^{-f}\varphi^2|D\eta|^2\\
&= \rho\int_M e^{-f}\varphi^2|\eta|^2 + 2\int_M e^{-f}\varphi\langle d\varphi,\eta\rangle\delta\eta + \int_M e^{-f}\varphi^2(\delta\eta)^2\\
&\quad +2\int_M e^{-f}\varphi D\eta(d\varphi,\eta) + \int_M e^{-f}\varphi^2|D\eta|^2.
\end{aligned} \tag{5.3}$$

Now assume that

$$D\eta(df,\eta) = D\eta(\eta,df). \tag{5.4}$$

Since η is not harmonic, this is not true in general. Since $\langle df,\eta\rangle = 0$, we have

$$Ddf(\eta,\cdot) + D\eta(\cdot,df) = 0.$$

So, from (5.4)

$$Ddf(\eta,\eta) + D\eta(df,\eta) = 0. \tag{5.5}$$

Therefore, it follows from (5.3) that

$$\begin{aligned}
&\rho\int_M e^{-f}\varphi^2|\eta|^2 + \int_M e^{-f}\varphi^2(\delta\eta)^2 + \int_M e^{-f}\varphi^2|D\eta|^2\\
&= -2\int_M e^{-f}\varphi\langle d\varphi,\eta\rangle\delta\eta - 2\int_M e^{-f}\varphi D\eta(d\varphi,\eta)\\
&\le \int_M e^{-f}\varphi^2(\delta\eta)^2 + \int_M e^{-f}|d\varphi|^2|\eta|^2\\
&\quad + \int_M e^{-f}\varphi^2|D\eta|^2 + \int_M e^{-f}|d\varphi|^2|\eta|^2.
\end{aligned}$$

Since η is in L^2, by letting $r\to\infty$, we obtain

$$\rho\int_M e^{-f}\varphi^2|\eta|^2 = 0,$$

which implies that $\eta = 0$. Thus we have the following.

Lemma 5.2. *Let η be an L^2 closed 1-form on a complete noncompact oriented gradient shrinking Ricci soliton (M,g,f). Suppose that $\langle df,\eta\rangle = 0$ and*

$$D\eta(df,\eta) = D\eta(\eta,df). \tag{5.6}$$

Then $\eta = 0$.

Lemma 5.2 can be considered as a generalization of Theorem 3.1 because harmonic 1-forms satisfy (5.6).

Proposition 5.3. *Let* ω *be an* L^2 *harmonic* 1*-form on a complete noncompact oriented gradient shrinking Ricci soliton* (M, g, f). *If there is a function* $\alpha : M \to \mathbb{R}$ *such that* $d\alpha$ *and* df *are parallel and satisfying*

$$\langle \omega - \alpha df, df \rangle = 0,$$

then $\omega = 0$.

Proof. Let $\eta := \omega - \alpha df$ so that η is a closed 1-form satisfying

$$\langle df, \eta \rangle = 0 \quad \text{and} \quad \int_M |\eta|^2 < \infty$$

and

$$\langle d\alpha, \eta \rangle = 0.$$

In particular,

$$Ddf(\eta, \eta) + D\eta(\eta, df) = 0.$$

Moreover, since $0 = \langle df, \eta \rangle = \langle df, \omega \rangle - \alpha |df|^2$, we have

$$\begin{aligned} |df|^2 \langle d\alpha, \eta \rangle &= \langle d\langle df, \omega \rangle, \eta \rangle - \alpha \langle d|df|^2, \eta \rangle \\ &= Ddf(\omega, \eta) + D\omega(df, \eta) - 2\alpha Ddf(df, \eta) \\ &= Ddf(\eta, \eta) + \alpha Ddf(df, \eta) + d\alpha \otimes df(df, \eta) + \alpha Ddf(df, \eta) \\ &\quad + D\eta(df, \eta) - 2\alpha Ddf(df, \eta) \\ &= Ddf(\eta, \eta) + D\eta(df, \eta) \\ &= 0. \end{aligned}$$

Thus,

$$D\eta(\eta, df) = D\eta(df, \eta).$$

By Lemma 5.2, we have $\eta = 0$ and so $\omega = \alpha df$. Finally, by Theorem 3.7, $\omega = 0$. ☐

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