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COLORED PERMUTATIONS WITH NO MONOCHROMATIC CYCLES

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Abstract. An $(n_1, n_2, \dots, n_k)$-colored permutation is a permutation of $n_1 + n_2 + \cdots + n_k$ in which $1, 2, \dots, n_1$ have color 1, and $n_1 + 1$, $n_1 + 2$, $\dots, n_1 + n_2$ have color 2, and so on. We give a bijective proof of Steinhardt's result: the number of colored permutations with no monochromatic cycles is equal to the number of permutations with no fixed points after reordering the first n_1 elements, the next n_2 element, and so on, in ascending order. We then find the generating function for colored permutations with no monochromatic cycles. As an application we give a new proof of the well known generating function for colored permutations with no fixed colors, also known as multi-derangements.

1. Introduction

Let S_n denote the set of permutations of $[n] := \{1, 2, \dots, n\}$. Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be a permutation in S_n. An integer $i \in [n]$ is called a *fixed point* of π if $\pi_i = i$. A *derangement* is a permutation with no fixed points. An integer $i \in [n-1]$ is called a *descent* of π if $\pi_i > \pi_{i+1}$, and an *ascent* of π if $\pi_i < \pi_{i+1}$. If the set of descents of π is equal to $\{1, 3, 5, \dots\} \cap [n-1]$, then π is called an *alternating permutation*. There are many interesting properties of alternating permutations, see [10].

More generally, if $B = \{b_1, b_2, \dots, b_n\}$ is an n-set with $b_1 < b_2 < \cdots < b_n$, a rearrangement $\sigma = s_1 s_2 \cdots s_n$ of elements of B is called a permutation of B. Let S_B denote the set of all permutations of B. The statistics *ascent* in S_B can be defined as in S_n, i.e., i is an ascent of σ if $s_i < s_{i+1}$.

In [9, Conjecture 6.3] Stanley conjectured that for $n \geq 2$, the number of alternating permutations of $[2n]$ with maximum number of fixed points, which is n, is equal to the number of derangements of $[n]$. This conjecture was proved by Chapman and Williams [2]. Han and Xin [6, Theorem 1] generalized Stanley's conjecture by enumerating the number of permutations $\pi \in S_n$ such that the set of descents is J and the number of fixed points is $n - |J|$, which is the largest possible, for any set $J \in [n-1]$. They showed that this number is equal to the

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number of derangements with a certain condition on descents. They also found a formula for the generating function for the number of such derangements. To be more precise, we need some definitions.

Let $\mathrm{NFiA}(n_1, n_2, \ldots, n_k)$ (respectively $\mathrm{NFiD}(n_1, n_2, \ldots, n_k)$) be the set of permutations $\pi = \pi_1\pi_2 \cdots \pi_n$ of $n = n_1 + n_2 + \cdots + n_k$ such that if π' is the permutation obtained from π by rearranging the first n_1 elements $\pi_1\pi_2 \cdots \pi_{n_1}$, the next n_2 elements $\pi_{n_1+1}\pi_{n_1+2} \cdots \pi_{n_1+n_2}$, and so on, in ascending order (respectively in descending order), then π' has no fixed points. Here, NFiA stands for **N**o **F**ixed points **i**n **A**scending order and NFiD stands for **N**o **F**ixed points **i**n **D**escending order. Note that $|\mathrm{NFiD}(n_1, n_2, \ldots, n_k)|/n_1! \cdots n_k!$ is the number of derangements of $[n]$ such that the first n_1 elements are in ascending order, the next n_2 elements are in ascending order, and so on.

Using symmetric functions, Han and Xin [6, Theorem 9] showed that

$$\begin{aligned}(1)\qquad &\sum_{n_1,n_2,\ldots,n_k \geq 0} |\mathrm{NFiD}(n_1, n_2, \ldots, n_k)| \frac{x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}}{n_1!n_2! \cdots n_k!} \\ &= \frac{1}{(1 + x_1) \cdots (1 + x_k)(1 - x_1 - \cdots - x_k)}.\end{aligned}$$

Eriksen, Freij, and Wästlund [3, Section 2] found a combinatorial proof of (1). Steinhardt [12, Corollary 4.2] proved the following analogous result of (1):

$$\begin{aligned}(2)\qquad &\sum_{n_1,n_2,\ldots,n_k \geq 0} |\mathrm{NFiA}(n_1, n_2, \ldots, n_k)| \frac{x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}}{n_1!n_2! \cdots n_k!} \\ &= \frac{(1 - x_1) \cdots (1 - x_k)}{1 - x_1 - \cdots - x_k}.\end{aligned}$$

In this paper we show that the left hand side of (2) has a natural interpretation in terms of colored permutations defined below. The key idea is the compositional formula for multivariate exponential generating functions.

An $(n_1, n_2, \ldots, n_k)$-*colored permutation* is a permutation in $S_{n_1+n_2+\cdots+n_k}$ such that $1, 2, \ldots, n_1$ have color 1, and $n_1 + 1, n_1 + 2, \ldots, n_1 + n_2$ have color 2, and so on. A cycle of an $(n_1, n_2, \ldots, n_k)$-colored permutation is called *monochromatic* if the elements of the cycle have the same color. We denote by $\mathrm{NMCy}(n_1, n_2, \ldots, n_k)$ the set of $(n_1, n_2, \ldots, n_k)$-colored permutations with no monochromatic cycles (NMCy stands for **N**o **M**onochromatic **Cy**cles).

In Section 2 we show that

$$\begin{aligned}(3)\qquad &\sum_{n_1,n_2,\ldots,n_k \geq 0} |\mathrm{NMCy}(n_1, n_2, \ldots, n_k)| \frac{x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}}{n_1!n_2! \cdots n_k!} \\ &= \frac{(1 - x_1) \ldots (1 - x_k)}{1 - x_1 - \cdots - x_k}.\end{aligned}$$

In fact we will show a more general formula using permutation statistics, see Theorem 2.1.

For an application of (3) we consider the set $\mathrm{NFCo}(n_1, n_2, \ldots, n_k)$ of $(n_1, n_2, \ldots, n_k)$-colored permutations π such that i and π_i have different colors for every i. Here, NFCo stands for **N**o **F**ixed **Co**lors. Such permutations are also called multi-derangements. By finding a simple relation between the generating functions for $|\mathrm{NMCy}(n_1, n_2, \ldots, n_k)|$ and $|\mathrm{NFCo}(n_1, n_2, \ldots, n_k)|$, we obtain a new proof of the following well known formula

$$\sum_{n_1,n_2,\ldots,n_k \geq 0} |\mathrm{NFCo}(n_1, n_2, \ldots, n_k)| \frac{x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}}{n_1! n_2! \cdots n_k!} = \frac{1}{1 - e_2 - 2e_3 - \cdots - (k-1)e_k}, \tag{4}$$

where e_i is the i-th elementary symmetric function on $x_1, x_2, \ldots, x_k$, which is defined by

$$e_i := \sum_{1 \leq j_1 < \cdots < j_i \leq k} x_{j_1} \cdots x_{j_i}.$$

We will show a more general formula using permutation statistics, see Theorem 3.1.

Note that by (2) and (3) we have

$$|\mathrm{NFiA}(n_1, n_2, \ldots, n_k)| = |\mathrm{NMCy}(n_1, n_2, \ldots, n_k)|. \tag{5}$$

Steinhardt [12, Theorem 6.2] also proved (5) but his proof is not bijective, see Remark 1. In Section 4 we give a bijective proof of (5).

2. The generating function for $\mathrm{NMCy}(n_1, n_2, \ldots, n_k)$

For a permutation $\pi = \pi_1\pi_2 \ldots \pi_n$ of $[n]$, an *excedance* of π is an integer $i \in \{1, 2, \ldots, n\}$ such that $\pi_i > i$. We will denote by $\mathrm{exc}(\pi)$ and $\mathrm{cyc}(\pi)$ the number of excedances of π and the number of cycles of π respectively. Define a generating function for $\mathrm{NMCy}(n_1, n_2, \ldots, n_k)$ by

$$f_{\mathrm{NMCy}}(x_1, x_2, \ldots, x_k; y, z) := \sum_{n_1,n_2,\ldots,n_k \geq 0} \left(\sum_{\pi \in \mathrm{NMCy}(n_1,n_2,\ldots,n_k)} y^{\mathrm{exc}(\pi)} z^{\mathrm{cyc}(\pi)} \right) \frac{x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}}{n_1! n_2! \cdots n_k!}.$$

In this section we show the following theorem.

Theorem 2.1. *We have*

$$f_{\mathrm{NMCy}}(x_1, x_2, \ldots, x_k; y, z) = \left((1-y)^{1-k} \frac{\left(1 - ye^{(1-y)x_1}\right) \cdots \left(1 - ye^{(1-y)x_k}\right)}{1 - ye^{(1-y)(x_1 + \cdots + x_k)}} \right)^z.$$

Note that if $y \to 1$ and $z \to 1$ in Theorem 2.1, we obtain (3).

Recall that for a permutation $\pi = \pi_1\pi_2 \cdots \pi_n$, an *ascent* of π is an integer $i \in \{1, 2, \ldots, n-1\}$ such that $\pi_i < \pi_{i+1}$. Let $\mathrm{asc}(\pi)$ denote the number of ascent of π. It is well known that the two statistics $\mathrm{exc}(\pi)$ and $\mathrm{asc}(\pi)$ are equidistributed in S_n, see [11, Proposition 1.4.3]. Let $A_n(y)$ be the *Eulerian polynomial* defined by

$$A_n(y) := \sum_{\pi \in S_n} y^{\mathrm{exc}(\pi)} = \sum_{\pi \in S_n} y^{\mathrm{asc}(\pi)}.$$

We denote by C_n the set of n-cycles formed with $1, 2, \ldots, n$.

Lemma 2.2. *We have*

$$\sum_{n \geq 0} A_n(y) \frac{x^n}{n!} = \frac{(1-y)e^{(1-y)x}}{1 - ye^{(1-y)x}}, \tag{6}$$

$$\sum_{n \geq 1} \left(\sum_{\pi \in C_n} y^{\mathrm{exc}(\pi)} \right) \frac{x^n}{n!} = \log \frac{1-y}{1 - ye^{(1-y)x}}. \tag{7}$$

Proof. Equation (6) is well known, see [11, Proposition 1.4.5]. For (7), observe that if we write an n-cycle $\pi \in C_n$ as $\pi = (n, a_1, a_2, \ldots, a_{n-1})$, then $\mathrm{exc}(\pi) = 1 + \mathrm{asc}(a_1 a_2 \cdots a_{n-1})$. Thus we have

$$\sum_{\pi \in C_n} y^{\mathrm{exc}(\pi)} = \sum_{\sigma \in S_{n-1}} y^{1+\mathrm{asc}(\sigma)} = yA_{n-1}(y).$$

Integrating both sides of (6) with respect to x, we obtain

$$\sum_{n \geq 1} A_{n-1}(y) \frac{x^n}{n!} = \frac{1}{y} \log \frac{1-y}{1 - ye^{(1-y)x}},$$

which finishes the proof of (7). □

We now prove Theorem 2.1.

Proof of Theorem 2.1. We claim that

$$\begin{aligned} &\sum_{n \geq 0} \frac{X^n}{n!} \sum_{n_1 + \cdots + n_k = n} \binom{n}{n_1, \ldots, n_k} x_1^{n_1} \ldots x_k^{n_k} \sum_{\pi \in \mathrm{NMCy}(n_1, n_2, \ldots, n_k)} y^{\mathrm{exc}(\pi)} z^{\mathrm{cyc}(\pi)} \\ &= \exp\left(\sum_{n \geq 1} \frac{X^n}{n!} \left((x_1 + \cdots + x_k)^n - x_1^n - \cdots - x_k^n \right) \sum_{\pi \in C_n} y^{\mathrm{exc}(\pi)} z \right). \end{aligned} \tag{8}$$

A *k-colored permutation* is a permutation in which every integer has color i for some $i = 1, 2, \ldots, k$. Then the left hand side of (8) is equal to

$$\sum_{n \geq 0} \frac{X^n}{n!} \sum_{\substack{\pi\text{: a } k\text{-colored permutation of } [n] \\ \text{with no monochromatic cycles}}} \mathrm{wt}(\pi), \tag{9}$$

where

$$\mathrm{wt}(\pi) = \prod_{i=1}^{k} x_i^{(\#\text{ elements of color } i \text{ in } \pi)} y^{\mathrm{exc}(\pi)} z^{\mathrm{cyc}(\pi)}.$$

Since a k-colored permutation π is divided into cycles, by the exponential formula [8, Corollary 5.1.6], (9) is equal to

$$\exp\left(\sum_{n\geq 1}\frac{X^n}{n!}\sum_{\substack{\pi:\text{ a }k\text{-colored cycle of }[n]\\ \text{with at least two colors}}}\mathrm{wt}(\pi)\right),$$

which is equal to the right hand side of (8).

Setting $X = 1$ in (8) and using (7), we get the desired formula. □

3. The generating function for $\mathrm{NFCo}(n_1, n_2, \ldots, n_k)$

Define a generating function for $\mathrm{NFCo}(n_1, n_2, \ldots, n_k)$ by

$$f_{\mathrm{NFCo}}(x_1, x_2, \ldots, x_k; y, z) := \sum_{n_1,n_2,\ldots,n_k\geq 0}\left(\sum_{\pi\in\mathrm{NFCo}(n_1,n_2,\ldots,n_k)} y^{\mathrm{exc}(\pi)}z^{\mathrm{cyc}(\pi)}\right)\frac{x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}}{n_1!n_2!\cdots n_k!}.$$

In this section we will prove the following theorem.

Theorem 3.1. *We have*

$$\begin{aligned} &f_{\mathrm{NFCo}}(x_1, x_2, \ldots, x_k; y, z)\\ &= \left(1 - ye_2 - (y + y^2)e_3 - \cdots - (y + y^2 + \cdots + y^{k-1})e_k\right)^{-z}. \end{aligned} \tag{10}$$

Askey and Ismail [1] showed (10) when $z = 1$ using MacMahon's master theorem. Foata and Zeilberger [4] showed (10) when $y = 1$ using the β-extension of MacMahon's master theorem. Kim and Zeng [7] found a combinatorial proof of (10) when $z = 1$. Zeng [13] showed (10) without restriction using the β-extension of MacMahon's master theorem. Zeng [14] proved (10) by decomposing multi-derangements into "wave segments".

We will show (10) by finding a relation between $f_{\mathrm{NMCy}}(x_1, x_2, \ldots, x_k)$ and $f_{\mathrm{NFCo}}(x_1, x_2, \ldots, x_k)$. We need a multivariate analog of the compositional formula [8, Theorem 5.1.4].

Let $\Pi(n)$ be the set of partitions of $\{1, 2, \ldots, n\}$. For $\mu \in \Pi(n)$, the number of blocks of μ is denoted by $|\mu|$. We use the convention that the empty product is 1. For instance, if $S = \emptyset$, then $\prod_{i\in S} g(i) = 1$ for any function g. Lemma 3.2 is a multivariate compositional formula. This can be shown by the same arguments as in the proof of [8, Theorem 5.1.4].

Lemma 3.2 (A multivariate compositional formula)**.** *Suppose that*

$$G(x_1, x_2, \ldots, x_k) = \sum_{n_1,n_2,\ldots,n_k\geq 0} g(n_1, n_2, \ldots, n_k)\frac{x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}}{n_1!n_2!\cdots n_k!}$$

is a multivariate formal power series, and for $i = 1, 2, \ldots, k$*,*

$$F_i(x) = \sum_{n\geq 1} f_i(n)\frac{x^n}{n!}$$

is a formal power series. Let

$$H(x_1, x_2, \ldots, x_k) = \sum_{n_1, n_2, \ldots, n_k \geq 0} h(n_1, n_2, \ldots, n_k) \frac{x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}}{n_1! n_2! \cdots n_k!}$$

be the multivariate formal power series, where

$$h(n_1, n_2, \ldots, n_k) = \sum_{\substack{\mu_i \in \Pi(n_i) \\ i=1,2,\ldots,k}} g(|\mu_1|, |\mu_2|, \ldots, |\mu_k|) \prod_{\substack{B \in \mu_i \\ i=1,2,\ldots,k}} f_i(|B|).$$

Then we have

$$H(x_1, x_2, \ldots, x_k) = G(F_1(x_1), F_2(x_2), \ldots, F_k(x_k)).$$

Proposition 3.3. *We have*

$$\begin{aligned} &f_{\mathrm{NMCy}}(x_1, x_2, \ldots, x_k; y, z) \\ &= f_{\mathrm{NFCo}}\left(\frac{e^{(1-y)x_1} - 1}{1 - ye^{(1-y)x_1}}, \ldots, \frac{e^{(1-y)x_k} - 1}{1 - ye^{(1-y)x_k}}; y, z\right), \end{aligned} \tag{11}$$

$$\begin{aligned} &f_{\mathrm{NFCo}}(x_1, x_2, \ldots, x_k; y, z) \\ &= f_{\mathrm{NMCy}}\left(\frac{1}{1-y} \log \frac{1 + x_1}{1 + yx_1}, \ldots, \frac{1}{1-y} \log \frac{1 + x_k}{1 + yx_k}; y, z\right). \end{aligned} \tag{12}$$

Proof. The second identity is obtained from the first one by substituting $x_i' = \frac{e^{(1-y)x_i} - 1}{1 - ye^{(1-y)x_i}}$, which is equivalent to $x_i = \frac{1}{1-y} \log \frac{1+x_i'}{1+yx_i'}$. Thus it suffices to show (11).

Let $\pi \in \mathrm{NMCy}(n_1, n_2, \ldots, n_k)$, and consider a cycle γ of π. Since π has no monochromatic cycles, the cycle γ contains more than one colors. We split γ into intervals, $\sigma_1, \sigma_2, \ldots, \sigma_r$, in such a way that γ is the concatenation of $\sigma_1, \sigma_2, \ldots, \sigma_r$, and each σ_i is monochromatic, and for each i the color of σ_i differs from that of σ_{i+1} with convention $\sigma_{r+1} = \sigma_1$. We call each σ_i a *maximal monochromatic interval* in γ, and regard it, being a sequence of distinct integers, as a permutation of its elements. Then γ can be regarded as an r-cycle $(\sigma_1, \sigma_2, \ldots, \sigma_r)$ of permutations $\sigma_1, \sigma_2, \ldots, \sigma_r$.

We now identify γ with the pair (T, τ), where $T = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ is the set of maximal monochromatic intervals defined above and τ is the r-cycle $(\sigma_1, \sigma_2, \ldots, \sigma_r)$. It is easy to see that

$$\mathrm{exc}(\gamma) = \mathrm{exc}(\tau) + \sum_{i=1}^{r} \mathrm{asc}(\sigma_i), \tag{13}$$

where $\mathrm{exc}(\tau)$ is defined based on the linear order on $\sigma_1, \ldots, \sigma_r$ by $\sigma_i > \sigma_j$ if the first element of σ_i is bigger than that of σ_j.

Let $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ be the set of disjoint cycles of $\pi \in \mathrm{NMCy}(n_1, n_2, \ldots, n_k)$, where each γ_i is identified with (T_i, τ_i). Then $\{\tau_1, \tau_2, \ldots, \tau_m\}$, regarded as a disjoint cycle decomposition, is a permutation of $T_1 \cup T_2 \cup \cdots \cup T_m$.

Thus we can identify π as a pair (U, ρ) satisfying the following:

- $U := T_1 \cup T_2 \cup \cdots \cup T_m$ is the set of all monochromatic permutations, i.e., maximal monochromatic intervals from disjoint cycles of π,
- every element $j \in [n_1 + \cdots + n_k]$ appears in exactly one σ in U and
- $\rho := \{\tau_1, \tau_2, \ldots, \tau_m\}$ is a permutation of U such that σ and $\rho(\sigma)$ have different colors for every $\sigma \in U$, i.e., ρ is a permutation of no fixed color.

Clearly $\mathrm{cyc}(\pi) = \mathrm{cyc}(\rho)$. Also, from (13), we get

$$\mathrm{exc}(\pi) = \mathrm{exc}(\rho) + \sum_{\sigma \in U} \mathrm{asc}(\sigma).$$

Thus we have

$$\sum_{\pi \in \mathrm{NMCy}(n_1, n_2, \ldots, n_k)} y^{\mathrm{exc}(\pi)} z^{\mathrm{cyc}(\pi)}$$
$$= \sum_{\substack{\mu_i \in \Pi(n_i) \\ i=1,2,\ldots,k}} \left(\sum_{\rho \in \mathrm{NFCo}(|\mu_1|, |\mu_2|, \ldots, |\mu_k|)} y^{\mathrm{exc}(\rho)} z^{\mathrm{cyc}(\rho)} \right) \prod_{\substack{B \in \mu_i \\ i=1,2,\ldots,k}} \sum_{\sigma \in S_B} y^{\mathrm{asc}(\sigma)}.$$

Since

$$\sum_{\sigma \in S_B} y^{\mathrm{asc}(\sigma)} = \sum_{\sigma \in S_{|B|}} y^{\mathrm{asc}(\sigma)},$$

by Lemma 3.2 and (6), we obtain (11). □

We are ready to give a new proof of Theorem 3.1.

Proof of Theorem 3.1. By Proposition 3.3 and Theorem 2.1 we have

$$f_{\mathrm{NFCo}}(x_1, x_2, \ldots, x_k; y, z)$$
$$= f_{\mathrm{NMCy}}\left(\frac{1}{1-y} \log \frac{1+x_1}{1+yx_1}, \ldots, \frac{1}{1-y} \log \frac{1+x_k}{1+yx_k}; y, z \right)$$
$$= \left((1-y)^{1-k} \frac{\prod_{i=1}^k \left(1 - y \exp\left[(1-y)\frac{1}{1-y} \log \frac{1+x_i}{1+yx_i}\right]\right)}{1 - y \exp\left[(1-y) \sum_{i=1}^k \frac{1}{1-y} \log \frac{1+x_i}{1+yx_i}\right]} \right)^z$$
$$= \left(\frac{1-y}{\prod_{i=1}^k (1+yx_i) - y \prod_{i=1}^k (1+x_i)} \right)^z.$$

Using the fact

$$\prod_{i=1}^k (1 + x_i y) = \sum_{i=0}^k e_i y^i,$$

one can easily see that

$$\prod_{i=1}^k (1 + x_i y) - y \prod_{i=1}^k (1 + x_i)$$
$$= (1-y)\left(1 - ye_2 - (y+y^2)e_3 - \cdots - (y + y^2 + \cdots + y^{k-1})e_k\right).$$

Thus we get

$$\begin{aligned} & f_{\mathrm{NFCo}}(x_1, x_2, \ldots, x_k; y, z) \\ = & \left(1 - ye_2 - (y+y^2)e_3 - \cdots - (y + y^2 + \cdots + y^{k-1})e_k\right)^{-z}, \end{aligned}$$

which completes the proof. □

4. Bijections

In this section we give a bijective proof of (5). We will follow Steinhardt's approach [12] using Gessel and Reutenauer's map.

Let $A(n_1, n_2, \ldots, n_k)$ be the set of derangements $\pi = \pi_1\pi_2\cdots\pi_n$ of $n = n_1 + n_2 + \cdots + n_k$ such that each of the k intervals

$$\pi_1\pi_2 \ldots \pi_{n_1},\ \pi_{n_1+1}\pi_{n_1+2} \ldots \pi_{n_1+n_2}, \text{ and so on,}$$

is in ascending order. Note that we can consider $\mathrm{NFiA}(n_1, n_2, \ldots, n_k)$ as the set $A(n_1, n_2, \ldots, n_k) \times S_{n_1} \times \cdots \times S_{n_k}$.

For example, let $(n_1, n_2, \ldots, n_k) = (8, 5, 1)$ and

$$\pi = |\ 8\ 7\ 9\ 12\ 6\ 5\ 11\ 10\ |\ 2\ 3\ 4\ 1\ 14\ |\ 13\ | \in \mathrm{NFiA}(n_1, n_2, \ldots, n_k),$$

where we put a bar '|' between $\pi_{n_1+\cdots+n_i}$ and $\pi_{n_1+\cdots+n_i+1}$ for each $i = 1, 2, \ldots, k-1$, and at the beginning and at the end for visibility. Then π' is the permutation obtained from π by rearranging the integers between two consecutive bars in ascending order:

$$(14)\quad \pi' = |\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ |\ 1\ 2\ 3\ 4\ 14\ |\ 13\ | \in A(n_1, n_2, \ldots, n_k).$$

We divide π into the k subwords of lengths $n_1, n_2, \ldots, n_k$ and then consider them as permutations in $S_{n_1}, S_{n_2}, \ldots, S_{n_k}$ to get $\sigma_1, \sigma_2, \ldots, \sigma_k$:

$$8\ 7\ 9\ 12\ 6\ 5\ 11\ 10 \ \cong\ 4\ 3\ 5\ 8\ 2\ 1\ 7\ 6 = \sigma_1,$$

$$2\ 3\ 4\ 1\ 14 \ \cong\ 2\ 3\ 4\ 1\ 5 = \sigma_2,$$

$$z = 13 \ \cong\ 1 = \sigma_3.$$

Here, for two words $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_n$ of integers, we write $u \cong v$ if $u_i < u_j$ implies $v_i < v_j$ and vice versa for all i, j. Then we identify π with $(\pi', \sigma_1, \sigma_2, \ldots, \sigma_k)$.

We now review Gessel and Reutenauer's map [5].

A *necklace* is a cycle of integers with possible repetitions. An *ornament* is a multiset of necklaces. Let $\Omega(n_1, n_2, \ldots, n_k)$ denote the set of ornaments ω such that i appears n_i times in the necklaces of ω for each i. Let $\eta = (b_1, b_2, \ldots, b_m)$ be a necklace. Define b_i for all integers i so that $b_i = b_j$ if $i \equiv j \mod m$. A *period* of η is an integer d such that $b_{i+d} = b_i$ for all i. We say that η is *r-repeating* if $r = m/d$, where d is the smallest period of η. A *primitive* necklace is a 1-repeating necklace. An ornament is called *primitive* if all of its necklaces are primitive. Let $\Omega_0(n_1, n_2, \ldots, n_k)$ be the set of primitive ornaments in $\Omega(n_1, n_2, \ldots, n_k)$ with no necklaces containing only one element.

For a permutation π, we define $\phi_{n_1,n_2,\dots,n_k}(\pi) \in \Omega(n_1, n_2, \dots, n_k)$ to be the ornament obtained from the cycles of π by replacing j with i if

$$n_1 + \cdots + n_{i-1} + 1 \leq j \leq n_1 + \cdots + n_{i-1} + n_i$$

for all $j \in [n]$. In other words, $\phi_{n_1,n_2,\dots,n_k}(\pi)$ is the ornament obtained from the cycles of π by replacing each element with its color. For example, the permutation π' in (14) has the cycles

$$(1, 5, 9), (2, 6, 10), (3, 7, 11), (4, 8, 12), (13, 14).$$

Thus the image of π' under this map is

$$\phi_{8,5,1}(\pi') = \{(1, 1, 2), (1, 1, 2), (1, 1, 2), (1, 1, 2), (2, 3)\}. \tag{15}$$

Proposition 4.1 ([5, Lemma 3.4])**.** *The map $\phi_{n_1,n_2,\dots,n_k}$ is a bijection between $A(n_1, n_2, \dots, n_k)$ and $\Omega_0(n_1, n_2, \dots, n_k)$.*

By Proposition 4.1, (5) is equivalent to

$$n_1!n_2! \cdots n_k!|\Omega_0(n_1, n_2, \dots, n_k)| = |\operatorname{NMCy}(n_1, n_2, \dots, n_k)|. \tag{16}$$

Remark 1. In the sketch of proof of [12, Theorem 6.2] Steinhardt states (16) without explanation. However, (16) is nontrivial since $\operatorname{NMCy}(n_1, n_2, \dots, n_k)$ has no obvious symmetries giving the factor $n_1!n_2! \cdots n_k!$.

We will give a bijective proof of (16). We define the map

$$\psi : \Omega_0(n_1, n_2, \dots, n_k) \times S_{n_1} \times \cdots \times S_{n_k} \to \operatorname{NMCy}(n_1, n_2, \dots, n_k)$$

as follows.

(1) Let $(\omega, \sigma_1, \dots, \sigma_k) \in \Omega_0(n_1, n_2, \dots, n_k) \times S_{n_1} \times \cdots \times S_{n_k}$. Any necklace in ω can be represented by the word that is the smallest in lexicographic order among the words read from it. Let $\gamma_1, \dots, \gamma_m$ be the sequence of words obtained by reading the necklaces in ω such that each γ_i is the smallest word which makes the corresponding necklace and $\gamma_1 \leq \cdots \leq \gamma_m$ in lexicographic order.

(2) For a permutation σ and an integer j, let $\sigma + j$ denote the word obtained from σ by increasing each integer by j. For $1 \leq i \leq k$, let $\sigma'_i = \sigma_i + (n_1 + \cdots + n_{i-1})$, where $n_0 = 0$.

(3) Note that, for each i, the integer i appears n_i times in $\gamma_1, \dots, \gamma_m$. Let $\rho_1, \dots, \rho_m$ be the sequence of words obtained from the sequence $\gamma_1, \dots, \gamma_m$ by replacing the n_i i's with the elements of σ'_i for $1 \leq i \leq k$. More precisely, the j-th occurrence of i is replaced with the element in the j-th position in σ'_i.

(4) Let $S \subset [m]$ be a maximal set subject to $\gamma_i = \gamma_j$ for all $i, j \in S$. Then $S = \{s+1, s+2, \dots, s+r\}$ for some integers s and r. Let $\tau = \tau_1 \cdots \tau_r \in S_r$ be the permutation such that $\tau_i < \tau_j$ if and only if $\rho_{s+i} < \rho_{s+j}$ in lexicographic order. In this case we say that τ and $\rho_{s+1}, \dots, \rho_{s+r}$ are *order-isomorphic*. Let C_S be the set of cycles obtained from the cycles of τ by replacing τ_i with ρ_{s_i} for all i. We define $\psi(\omega, \sigma_1, \dots, \sigma_k)$ to be

the permutation whose cycles are the elements of the union of C_S for all S.

Example 1. Let $(n_1, n_2, \dots, n_k) = (8, 5, 1)$. Let

$$\omega = \{(1,1,2),(1,1,2),(1,1,2),(1,1,2),(2,3)\}$$

be the ornament in (15) and $\sigma_1 = 43582176$, $\sigma_2 = 23415$ and $\sigma_3 = 1$ as before. Note that

$$\gamma_1, \dots, \gamma_5 = 112, \quad 112, \quad 112, \quad 112, \quad 23, \tag{17}$$

and

$$\sigma'_1 = \sigma_1 = 4 \;\; 3 \;\; 5 \;\; 8 \;\; 2 \;\; 1 \;\; 7 \;\; 6,$$
$$\sigma'_2 = \sigma_2 + n_1 = 10 \;\; 11 \;\; 12 \;\; 9 \;\; 13,$$
$$\sigma'_3 = \sigma_3 + (n_1 + n_2) = 14.$$

By replacing the eight 1's with σ'_1, the five 2's with σ'_2, and the one 3 with σ'_3 in (17), we have

$$\rho_1, \dots, \rho_5 = \mathbf{4} \;\; \mathbf{3} \;\; 10, \quad \mathbf{5} \;\; \mathbf{8} \;\; 11, \quad \mathbf{2} \;\; \mathbf{1} \;\; 12, \quad \mathbf{7} \;\; \mathbf{6} \;\; 9, \quad 13 \;\; 14,$$

where the elements of σ'_1 are written in bold face. Since $\gamma_1 = \cdots = \gamma_4$, we consider $\rho_1, \dots, \rho_4$ which is order-isomorphic to $2314 = (123)(4) \in S_4$. Thus we construct the cycles

$$(\rho_1, \rho_2, \rho_3) = (4,3,10,5,8,11,2,1,12), \qquad (\rho_4) = (7,6,9).$$

Thus,

$$\psi(\omega, \sigma_1, \sigma_2, \sigma_3) = (4,3,10,5,8,11,2,1,12)(7,6,9)(13,14).$$

Theorem 4.2. *The map*

$$\psi : \Omega_0(n_1, n_2, \dots, n_k) \times S_{n_1} \times \cdots \times S_{n_k} \to \mathrm{NMCy}(n_1, n_2, \dots, n_k)$$

is a bijection.

Proof. We will show this theorem by constructing the inverse map of ψ.

Let $\pi \in \mathrm{NMCy}(n_1, n_2, \dots, n_k)$. We define a map $\pi \mapsto (\omega, \sigma_1, \dots, \sigma_k)$ as follows.

(1) Let H be the set of words γ on $\{1, 2, \dots, k\}$ such that

- $\phi_{n_1,n_2,\dots,n_k}(\pi)$ contains the necklace $(\overbrace{\gamma, \dots, \gamma}^{j})$ for some integer $j \geq 1$,
- (γ) is primitive and γ is the smallest word among all of its cyclic shifts in lexicographic order,

where we regard a word γ as a sequence of integers in the natural way.

(2) For $\gamma \in H$, we define T_γ to be the set of all words ρ satisfying that ρ is a consecutive subsequence in some cycle of π and $\phi_{n_1,n_2,\ldots,n_k}(\rho) = \gamma$. Here, $\phi_{n_1,n_2,\ldots,n_k}(\rho)$ denotes the word obtained from ρ by replacing each number in ρ, say j, with i if

$$n_1 + \cdots + n_{i-1} + 1 \leq j \leq n_1 + \cdots + n_{i-1} + n_i.$$

(3) For $\gamma \in H$, let

$$\rho_1^\gamma < \rho_2^\gamma < \cdots < \rho_{m_\gamma}^\gamma$$

be the elements of T_γ ordered by lexicographic order. Consider the cycles of π containing the words in T_γ as consecutive subsequences. In these cycles, if we replace the consecutive subsequence which forms ρ_i^γ by i for each i, we obtain cycles consisting of $1, 2, \ldots, m_\gamma$. The resulting cycles form a permutation, which we denote by

$$\tau^\gamma = \tau_1^\gamma \tau_2^\gamma \cdots \tau_{m_\gamma}^\gamma.$$

Then we define W_γ to be the sequence of the elements in T_γ according to the permutation τ^γ, that is,

$$W_\gamma = \rho^\gamma_{\tau_1^\gamma}, \rho^\gamma_{\tau_2^\gamma}, \ldots, \rho^\gamma_{\tau^\gamma_{m_\gamma}}.$$

(4) Let

$$W = \rho_1, \rho_2, \ldots, \rho_m$$

be the concatenation of the sequence W_γ for all $\gamma \in H$ where we start with the lexicographically smallest γ and proceed with the next smallest one, and so on.

(5) We now define ω to be the ornament $\{(\gamma_1), \ldots, (\gamma_m)\}$ where $\gamma_i = \phi_{n_1,n_2,\ldots,n_k}(\rho_i)$. Here, we consider γ_i as a sequence of integers as before.

(6) For $1 \leq i \leq k$, we define σ_i to be the permutation in S_{n_i} which is order-isomorphic to the word obtained from W by taking the integers from $n_1 + \cdots + n_{i-1} + 1$ to $n_1 + \cdots + n_{i-1} + n_i$.

It is easy to see that $\pi \mapsto (\omega, \sigma_1, \ldots, \sigma_k)$ is the inverse map of ψ. □

Combining $\phi_{n_1,n_2,\ldots,n_k}$ and ψ, we obtain a bijective proof of (5).

Example 2. Let $(n_1, n_2, \ldots, n_k) = (8, 5, 1)$ and consider

$$\pi = (4, 3, 10, 5, 8, 11, 2, 1, 12)(7, 6, 9)(13, 14) \in \mathrm{NMCy}(n_1, n_2, \ldots, n_k).$$

The map $\pi \mapsto (\omega, \sigma_1, \ldots, \sigma_k)$ in the proof of Theorem 4.2 is constructed as follows. Since

$$\phi_{n_1,n_2,\ldots,n_k}(\pi) = (1, 1, 2, 1, 1, 2, 1, 1, 2)(1, 1, 2)(2, 3) \in \mathrm{NMCy}(n_1, n_2, \ldots, n_k),$$

we have $H = \{112, 23\}$,

$$T_{112} = \{\rho_1^{112} = 2\ \ 1\ \ 12, \quad \rho_2^{112} = 4\ \ 3\ \ 10, \quad \rho_3^{112} = 5\ \ 8\ \ 11, \quad \rho_4^{112} = 7\ \ 6\ \ 9\},$$

$$T_{23} = \{13\ \ 14\}.$$

The cycles of π containing the elements in T_{112} are

$$(4,3,10,5,8,11,2,1,12), \quad (7,6,9).$$

If we replace the consecutive subsequences "2,1,12", "4,3,10", "5,8,11", "7,6,9" with 1, 2, 3, 4 respectively in these cycles, we obtain $(2,3,1)=(1,2,3)$ and (4). Thus

$$\tau^{112}=(1,2,3)(4)=2314,$$

and

$$W_{112}=\rho_2^{112},\rho_3^{112},\rho_1^{112},\rho_4^{112}=4\ \ 3\ \ 10, \quad 5\ \ 8\ \ 11, \quad 2\ \ 1\ \ 12, \quad 7\ \ 6\ \ 9.$$

Similarly, we have $\tau^{23}=(1)=1$ and $W_{23}=13\ \ 14$. Thus,

$$W=W_{112},W_{23}=4\ \ 3\ \ 10, \quad 5\ \ 8\ \ 11, \quad 2\ \ 1\ \ 12, \quad 7\ \ 6\ \ 9, \quad 13\ \ 14.$$

Finally we obtain that

$$\omega=\{(1,1,2),(1,1,2),(1,1,2),(1,1,2),(2,3)\}$$

and $\sigma_1=43582176$, $\sigma_2=23415$ and $\sigma_3=1$.

5. Final remarks

As $\mathrm{NFiA}(n_1,n_2,\dots,n_k)$ has a counterpart $\mathrm{NMCy}(n_1,n_2,\dots,n_k)$, the set $\mathrm{NFiD}(n_1,n_2,\dots,n_k)$ has a combinatorial counterpart as follows.

Let $\mathrm{EMCy}(n_1,n_2,\dots,n_k)$ be the set of $(n_1,n_2,\dots,n_k)$-colored permutations in which the sum of the lengths of the monochromatic cycles of each color is even (EMCy stands for **E**venly **M**onochromatic **Cy**cles). Using the exponential formula, one can show that

$$\begin{aligned}\sum_{n_1,n_2,\dots,n_k\ge 0} &|\,\mathrm{EMCy}(n_1,n_2,\dots,n_k)|\frac{x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}}{n_1!n_2!\cdots n_k!}\\ &=\frac{1}{(1+x_1)\dots(1+x_k)(1-x_1-\dots-x_k)}.\end{aligned} \tag{18}$$

Thus from (1) and (18) we get

$$|\,\mathrm{NFiD}(n_1,n_2,\dots,n_k)|=|\,\mathrm{EMCy}(n_1,n_2,\dots,n_k)|. \tag{19}$$

We can also prove (19) bijectively, by using the same idea as in Theorem 4.2.

It will be interesting to find a refinement of (18) which is analogous to Theorem 2.1.

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