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FINITE p-GROUPS WHOSE NON-ABELIAN SUBGROUPS HAVE THE SAME CENTER

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Abstract. For an odd prime p, finite p-groups whose non-abelian subgroups have the same center are classified in this paper.

1. Introduction

The center $Z(G)$ of a group G is a very important concept in group theory. In some sense, the size of $Z(G)$ can be regarded as a measure of how far G is from an abelian group. Clearly, $Z(G) = G$ if and only if G is an abelian group. If G is non-abelian, then, naturally, we hope to investigate finite groups with "large" center or abelian subgroups. As is well known, the center of a group may be trivial. However, the center of a finite p-group is always nontrivial. So we pay our attention to finite p-groups. Some scholars classified finite p-groups with "large" abelian subgroups. For example, Rédei [6] classified finite non-abelian groups G of order p^n all of whose maximal subgroups are abelian. Obviously, such groups have "large" center. In fact, $|Z(G)| = p^{n-2}$. Along Rédei's line, Zhang et al. [12, 13] classified finite non-abelian p-groups of all of whose subgroups of index at most p^3 are abelian. On the other hand, some scholars have studied the structure of finite p-groups with conditions on its center or centers of its subgroups. For example, Janko [4] studied finite non-abelian p-groups having exactly one maximal subgroup with a noncyclic center. Finogenov [2] studied finite p-groups with cyclic commutator group and cyclic center.

The start point in this paper is to study the influence of the relationship between the center of a finite p-group and the centers of its non-abelian subgroups on the structure of a finite p-groups. As is well known, $H \cap Z(G) \leq Z(H)$ for a group G and its each subgroup H. The extreme case is $H \cap Z(G) = Z(H)$. In other words, $Z(H) \leq Z(G)$. We try to classify such finite p-groups G with $Z(H) \leq Z(G)$ for each non-abelian subgroup H. However, by using the

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Magma project, we observed there are too many p-groups satisfying the condition. Moreover, it is not difficult to prove that all finite p-groups G with $|G : Z(G)| \leq p^3$ satisfy the condition. Hence it is quite difficult to classify such finite p-groups. A natural question is:

Is it possible to classify finite p-groups G with $Z(H) = Z(G)$ for each non-abelian subgroup H of G?

The answer is positive. For $p = 2$, such p-groups were classified in [8]. The present paper is devoted to the case of $p \neq 2$. Hence finite p-groups G with $Z(H) = Z(G)$ are completely classified. It is worth to be mentioned that the argument of the case of $p \neq 2$ is quite different from that of $p = 2$.

For convenience, we introduce the following notation and concepts.

$\mathcal{P}$-group: A finite p-group in which centers of all non-abelian subgroups coincide.

$\mathcal{Q}$-group: A $\mathcal{P}$-group all of whose non-abelian subgroups are generated by two elements.

$\mathcal{S}$-group: A $\mathcal{P}$-group which has at least one non-abelian subgroup H with $d(H) > 2$.

Obviously, $\mathcal{P} = \mathcal{Q} \cup \mathcal{S}$ and $\mathcal{Q} \cap \mathcal{S} = \emptyset$.

We notice that the non-abelian subgroup of a minimal non-abelian p-group is itself. We assume a $\mathcal{P}$-group is not minimal non-abelian in this paper.

Suppose that G is a finite p-group. If all subgroups of index p^t of G are abelian and at least one subgroup of index p^{t-1} of G is not abelian, then G is called an $\mathcal{A}_t$-group. Obviously, an $\mathcal{A}_1$-group is a minimal non-abelian p-group, and for arbitrary a fixed integer i, all $\mathcal{A}_i$-subgroups of a $\mathcal{P}$-group have the same order.

Groups in this paper are finite p-groups and p is an odd prime. We use $c(G)$ and $d(G)$ to denote the nilpotency class and the minimal number of generators of a group G respectively. Other notation and terminology are consistent with that in [3].

2. Preliminary

In this section, we give some lemmas which are useful in the proof of our results.

Lemma 2.1 ([9, Lemma 2]). *Let G be a metabelian p-group and $a, b \in G$. For any positive integer i and j, let*

$$[ia, jb] = [a, b, \underbrace{a, \ldots, a}_{i-1}, \underbrace{b, \ldots, b}_{j-1}].$$

Then

(1) *For any positive integers m and n,* $[a^m, b^n] = \prod_{i=1}^{m} \prod_{j=1}^{n} [ia, jb]^{\binom{m}{i}\binom{n}{j}}$.

(2) *Let n be a positive integer. Then* $(ab^{-1})^n = a^n \prod_{i+j \leq n} [ia, jb]^{\binom{n}{i+j}} b^{-n}$.

Lemma 2.2 ([3] or [7], Aufgabe 2, p. 259)**.** *Suppose that a finite non-abelian p-group G has an abelian normal subgroup A, and $G/A = \langle bA\rangle$ is cyclic. Then the map $a \mapsto [a,b], a \in A$ is an epimorphism from A to G' and $G' \cong A/A\cap Z(G)$. In particular, if a non-abelian p-group G has an abelian maximal subgroup, then $|G| = p|G'||Z(G)|$.*

Lemma 2.3 ([10, Lemma 2.2])**.** *Suppose that G is a finite non-abelian p-group. Then the following conditions are equivalent*:
(1) *G is minimal non-abelian*;
(2) *$d(G) = 2$ and $|G'| = p$*;
(3) *$d(G) = 2$ and $\Phi(G) = Z(G)$.*

The following lemma is simple but often used.

Lemma 2.4. *If $G = \langle x, y\rangle$ is a minimal non-abelian p-group, then $Z(G) = \langle x^p, y^p, [x, y]\rangle$.*

Lemma 2.5 ([8, Lemma 2.4])**.** *Let G be a $\mathcal{P}$-group. If $x, y \in G\backslash Z(G)$ and $[x, y] = 1$, then $C_G(x) = C_G(y)$.*

Some results about $\mathcal{P}$-groups are given in following lemmas.

Lemma 2.6. *Let G be a metacyclic p-group, and p an odd prime. Then G is not a $\mathcal{P}$-group.*

Proof. By [11, Theorem 2.1] or see [5] we have

$$G = \langle a, b | a^{p^{r+s+u}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, [a, b] = a^{p^r}\rangle,$$

where r, s, t, u are non-negative integers, $r \geq 1$ and $u \leq r$. Since

$$[a^{p^{s+u-1}}, b] = [a, b]^{p^{s+u-1}} = a^{p^{r+s+u-1}},$$

$H = \langle a^{p^{s+u-1}}, b\rangle$ is minimal non-abelian by Lemma 2.3. If G is a $\mathcal{P}$-group, then $b^p \in Z(H) = Z(G)$ by Lemma 2.4. Therefore, $1 = [a, b^p]$. On the other hand,

$$[a, b^p] = [a, b]^{\binom{p}{1}}[a, b]^{\binom{p}{2}} \cdots [a, b, \ldots, b]^{\binom{p}{p}} = a^{p^{r+1}+p^{2r}\binom{p}{2}+\cdots+p^{pr}\binom{p}{p}}$$

by Lemma 2.1(1). It follows that $a^{p^{r+1}} = 1$. Since $o(a) = p^{r+s+u}$, $s + u = 1$. Thus $|G'| = |\langle a^{p^r}\rangle| = p$. Hence G is minimal non-abelian by Lemma 2.3, which contradicts to the hypothesis. □

Lemma 2.7. *If G is a p-group of maximal class with an abelian maximal subgroup, then G is a $\mathcal{P}$-group.*

Proof. Let A be an abelian maximal subgroup of G and H any non-abelian subgroup of G. Then $G = AH$ and $A \cap H$ is an abelian maximal subgroup of H. Hence, $Z(H) \leq A\cap H$. Since A is abelian, $Z(H) \leq Z(G)$. By [10, Theorem 2.5], $|Z(G)| = p$. It follows that $Z(H) = Z(G)$. Hence, G is a $\mathcal{P}$-group. □

Lemma 2.8. *Let G be an $\mathcal{A}_2$-group. Then G is a $\mathcal{P}$-group if and only if $Z(G) \leq \Phi(G)$ and $|G : Z(G)| = p^3$.*

Proof. ($\Longrightarrow$) Take one non-abelian proper subgroup H of G. Since G is an $\mathcal{A}_2$-group, H is minimal non-abelian and $|G : H| = p$. By Lemma 2.3, we have $Z(H) = \Phi(H)$ and $|H : Z(H)| = p^2$. Since $Z(H) = Z(G)$ and $\Phi(H) \leq \Phi(G)$, $Z(G) \leq \Phi(G)$ and $|G : Z(G)| = p^3$.

($\Longleftarrow$) Let H be any non-abelian proper subgroup of G. Since G ia an $\mathcal{A}_2$-group, H is maximal in G and H is an $\mathcal{A}_1$-group. Since $Z(G) \leq \Phi(G)$, $Z(G) \leq H$ and $Z(G) \leq Z(H)$. It follows by $|G : Z(G)| = p^3$ that $|H : Z(G)| = p^2$. Since H is an $\mathcal{A}_1$-group, $|H : Z(H)| = p^2$ by Lemma 2.3. Hence $Z(G) = Z(H)$ and G is a $\mathcal{P}$-group. □

Lemma 2.9. *Let G be an $\mathcal{A}_2$-group. Then G is a $\mathcal{P}$-group if and only if G is isomorphic to one of the following pairwise non-isomorphic groups*:

(I) $d(G) = 2$. *In this case, G is a $\mathcal{Q}$-group.*

(I-1) *G is a group of maximal class of order p^4, that is, G is one of the groups* (ii)-(iv) *listed in* [12, Theorem 3.2(2)];

(I-2) *G is one of the groups listed in* [12, Theorems 3.5 and 3.9];

(II) $d(G) = 3$. *In this case, G is an $\mathcal{S}$-group and G is one of the groups* (5–7) *listed in* [12, Theorem 3.6].

Proof. $\mathcal{A}_2$-groups are classified in [12] and they are listed in [12, Theorems 3.1, 3.2, 3.5, 3.6, 3.9]. Next, we check the groups one by one.

The groups in [12, Theorem 3.1] are metacyclic. They are not the required groups by Lemma 2.6.

If G is one of the groups listed in [12, Theorem 3.2], then $|G| = p^4$. By Lemma 2.8, we get the groups (I-1).

If G is one of groups listed in [12], Theorems 3.5 and 3.9], then $Z(G) \leq \Phi(G)$ and $|G : Z(G)| = p^3$ by a simple checking. Thus we get the groups (I-2) by Lemma 2.8.

Assume that G is one of the groups (1–7) listed in [12, Theorem 3.6]. We have $Z(G) \nleq \Phi(G)$ for the groups (1–3). Hence they are not $\mathcal{Q}$-groups by Lemma 2.8. Since p is odd, the group (4) is not a $\mathcal{Q}$-group. On the other hand, by computation we have $Z(G) = \Phi(G)$ and $|G : Z(G)| = p^3$ for the groups (5–7), we get the groups (II) by Lemma 2.8. □

3. Main results

In this section, we give the classification of $\mathcal{P}$-groups. We know that $\mathcal{P} = \mathcal{Q} \cup \mathcal{S}$ and $\mathcal{Q} \cap \mathcal{S} = \emptyset$. It is enough to classify $\mathcal{Q}$-groups and $\mathcal{S}$-groups, respectively.

Theorem 3.1. *Let G be a finite non-abelian p-group. Then G is a $\mathcal{Q}$-group if and only if G is isomorphic to one of the following pairwise non-isomorphic groups*:

(I) *G is one of non-metacyclic $\mathcal{A}_2$-groups of order $\geq p^5$ with* $d(G) = 2$, *that is, G is one of the groups listed in* [12, Theorems 3.5 and 3.9].

(II) *G is one of groups of maximal class with an abelian maximal subgroup*;

(III) *G is one of the groups listed in* [10, Theorem 3.13].

Proof. Let G be a $\mathcal{Q}$-group. Due to the classification of finite p-groups whose non-abelian proper subgroups are generated by two elements in [10], what we need to do is to check those groups to be $\mathcal{Q}$-groups in the groups (1–7) listed in [10, Main Theorem].

Assume that G is one of the groups (1), i.e., G is an $\mathcal{A}_2$-groups. Since G is a $\mathcal{Q}$-group, $d(G) = 2$. By Lemma 2.9, we get the groups (I) and the groups of maximal class of order p^4 which are contained in (II).

Assume that G is one of the groups (2), i.e., G is metacyclic. It follows by Lemma 2.6 that G is not a $\mathcal{Q}$-group.

Assume that G is one of the groups (3), i.e., G is of maximal class with an abelian maximal subgroup. Then we get the groups (II) by Lemma 2.7.

Assume that G is one of the groups (4), i.e., G is a 3-group of maximal class. Let G_1 be the fundamental subgroup of G. Then G_1 is abelian or minimal non-abelian by [1, §9, Excise 10]. If G_1 is minimal non-abelian, then $|G_1 : Z(G_1)| = 9$. Since G is a $\mathcal{Q}$-group, $Z(G) = Z(G_1)$. Now we have $|G : Z(G)| = 27$. It follows that $|G| = 3^4$. Hence G has an abelian maximal subgroup. However, all maximal subgroups except G_1 are of maximal class by [1, Theorem 9.6(e)], a contradiction. Thus G_1 is abelian. We get G is one of the groups (II).

Assume that G is one of the groups (5), i.e., G is a $D_p'(2)$-group. It follows by [10, Lemma 3.1(4)] that G is a $\mathcal{Q}$-group. We get the groups (III).

Assume that G is one of the groups (6) and (7). Clearly, there exists a subgroup H of G such that $Z(H) \neq Z(G)$. Hence, G is not a $\mathcal{Q}$-group. □

Theorem 3.2. *Let G be a finite non-abelian p-group. Then G is an $\mathcal{S}$-group if and only if G is isomorphic to one of the following pairwise non-isomorphic groups*:

(1) $G = \langle a, b, c \mid a^{p^n} = b^{p^2} = c^{p^2} = 1, [a, b] = c^p, [c, a] = b^{-\nu p}, [c, b] = 1\rangle$, *where ν is a fixed quadratic non-residue modulo p and $n \geq 1$;*

(2) $G = \langle a, b, c \mid a^{p^n} = b^{p^2} = c^{p^2} = 1, [a, b] = c^p, [c, a] = b^{up}c^p, [c, b] = 1\rangle$, *where $4u = 1 - \rho^{2r+1}$ with $1 \leq r \leq \frac{1}{2}(p-1)$ and ρ the smallest positive integer which is a primitive root modulo p and $n \geq 1$;*

(3) $G = \langle a_1, a_2, b \mid a_1^{p^2} = a_i^{p^{q+1}} = a_j^{p^q} = b^{p^n} = 1,\ a_1^p = a_{r+1}^{p^q},\ [a_1, b] = b^{p^{n-1}},$ $[a_k, b] = a_{k+1},\ [a_p, b] = \prod_{t=2}^{p} a_t^{-\binom{p}{t-1}},\ [a_u, a_v] = 1\rangle$, *where* $2 \leq c = (p-1)q + r$, $1 \leq r \leq p-1$, $2 \leq i \leq r+1$, $r+2 \leq j \leq p$, $2 \leq k \leq p-1$, $1 \leq u, v \leq p$, $n \geq 2$ *and* $|G| = p^{n+c+1}$.

In brief, each $\mathcal{S}$-group is an extension of an abelian p-group by a cyclic group.

Proof. First, we prove that the groups listed in the theorem are $\mathcal{S}$-groups. Suppose that G is (1) or (2). Then G is an $\mathcal{A}_2$-group by [12, Theorem 3.6]. It follows by Lemma 2.9(II) that G is an $\mathcal{S}$-group.

Suppose that G is one of the groups (3). Then

$$Z(G) = \langle a_1^p, b^p\rangle \text{ and } G' = \langle a_1^p, a_2^p, a_3, \ldots, a_p, b^{p^{n-1}}\rangle.$$

Moreover, $M = \langle a_1, a_2, \ldots, a_p, b^p\rangle$ is the unique abelian maximal subgroup of G.

Let H be a non-abelian subgroup of G. We prove $Z(H) = Z(G)$.

First, $Z(H) \leq H \cap M$ since $H \cap M$ is an abelian maximal subgroup of H. Notice that $G = MH$ and M is abelian. Thus $Z(H) \leq Z(G)$. On the other hand, since every non-abelian subgroup H contains a minimal non-abelian subgroup K, it is enough to prove $Z(G) \leq Z(K)$.

Clearly, there exists an element $k \in K \setminus M$. Since $G = M\langle b\rangle$, we can assume that $k = bm$ and $K = \langle k, m'\rangle$, where $m, m' \in M$. Since

$$k^p \in C_M(k) = C_M(bm) = C_M(b) = \langle a_1^p, b^p\rangle,$$

$k^p = (bm)^p = a_1^{ip}b^{jp}$ for some i and j. By Lemma 2.1, we have

$$k^p = (bm)^p = b^p m^p [b, m^{-1}]^{\binom{p}{2}} [2b, m^{-1}]^{\binom{p}{3}} \cdots [(p-1)b, m^{-1}]^{\binom{p}{p}}.$$

Hence, $j \equiv 1(\text{mod } p)$ and $k^p = a_1^{ip}b^{(1+vp)p}$. Moreover, $b^{p^2} \in K$.

It is clear that

$$[b, m'] = [k, m'] \in C_{G'}(k) = C_{G'}(b) = \langle a_1^p, b^{p^{n-1}}\rangle.$$

Hence, we can assume that $[b, m'] = a_1^{i'p}b^{j'p^{n-1}}$, where $p \nmid i'$ or $p \nmid j'$. Since

$$[a_1, b] = b^{p^{n-1}} \text{ and } [a_r^{p^q}, b] = a_{r+1}^{p^q} = a_1^p,$$

$[b, m'a_1^{j'}a_r^{i'p^q}] = 1$ and $m'a_1^{j'}a_r^{i'p^q} \in C_M(b) = \langle a_1^p, b^p\rangle$. Thus

$$m' = a_1^{-j_1'+sp}a_r^{-i'p^q}b^{pl}$$

for some integers s, t and l.

If $p \nmid j'$, then $m'^p = a_1^{-pj'}b^{p^2l}$. Since $b^{p^2} \in K$, $a_1^p \in K$. If $p \mid j'$, then $p \nmid i'$ and $[k, m'] = a_1^{i'p}$. Hence $a_1^p \in K$.

It follows by $k^p = a_1^{ip}b^{p(1+vp)}$ that $b^p \in K$. Therefore, $Z(G) = \langle a_1^p, b^p\rangle \leq K$, and hence $Z(G) \leq Z(K)$. Thus G is an $\mathcal{S}$-group.

Now we prove $\mathcal{S}$-groups are exactly the groups listed in the theorem.

Let G be an $\mathcal{S}$-group. Then G has one non-abelian subgroup H with $d(H) > 2$. Assume H is the subgroup of G with the smallest order such that $d(H) > 2$. Let $|G : H| = p^s$. We prove the result by induction on s.

If $s = 0$, then $H = G$. It follows that all non-abelian proper subgroups H of G are generated by two elements. Hence, $d(G) = d(H) = 3$. By [10, Main Results], G is an $\mathcal{A}_2$-group with an abelian maximal subgroup. It follows by Lemma 2.9 that G is one of the groups (5–7) listed in [12, Theorem 3.6], that is, one of the groups (1–2) and (3) with $c = 2$. In other words, the theorem is true for $s = 0$. Now, let M be a maximal subgroup of G such that $H \leq M$. Then $|M : H| = p^{s-1}$. By induction hypothesis, M is an $\mathcal{S}$-group. Thus M is isomorphic to one of the groups listed in the theorem. Let $x \in G\setminus M$. Then G is a cyclic extension of M by $\langle x\rangle$. We will prove G is exactly the group (3) with $c > 2$.

Case 1. M is isomorphic to the group (1) in the theorem. That is, $M \cong \langle a, b, c \mid a^{p^n} = b^{p^2} = c^{p^2} = 1, [a, b] = c^p, [c, a] = b^{-\nu p}, [c, b] = 1\rangle$, where ν is a fixed quadratic non-residue modulo p, $n \geq 1$.

We will prove there is no $\mathcal{S}$-group G which contains M as its maximal subgroup in this case. Otherwise, we will deduce a contradiction.

First we have $Z(M) = \langle a^p, b^p, c^p\rangle$ and M has exactly one abelian maximal subgroup $A = \langle a^p, b, c\rangle$. Hence, $A \trianglelefteq G$ and $G' \leq A$ since $|G/A| = p^2$. By hypotheses,

$$Z(G) = Z(M) = \langle a^p, b^p, c^p\rangle.$$

Let $x \in G\backslash M$. Then $G = \langle x, a, b, c\rangle$. We will deduce a contradiction by the following steps.

(1) $[b, x] = [c, x] = 1$.

Since

$$[M, A, G] \leq [M', G] \leq [Z(M), G] = [Z(G), G] = 1 \text{ and } [G, M, A] \leq [G', A] = 1,$$

$[A, G, M] = 1$ by the Three Subgroups Lemma. Hence,

$$[A, G] \leq C_A(M) = Z(M) = \langle a^p, b^p, c^p\rangle.$$

Let $[b, x] = a^{ps}b^{pt}c^{pu}$. Since $1 = [b^p, x] = [b, x]^p = a^{p^2 s}$, $p^{n-2}|s$. Hence we can assume that $[b, x] = a^{p^{n-1}i_2}b^{pj_2}c^{pk_2}$. Since $[b, xa^{k_2}] = a^{p^{n-1}i_2}b^{pj_2}$, we can assume by replacing x with xa^{k_2} that

$$[b, x] = a^{p^{n-1}i_2}b^{pj_2}.$$

If $[b, x] \neq 1$, then $p \nmid i_2$ or $p \nmid j_2$. It follows from Lemma 2.3 that $\langle b, x\rangle$ is minimal non-abelian. By Lemma 2.4,

$$x^p \in Z(\langle b, x\rangle) = Z(G) = \langle a^p, b^p, c^p\rangle.$$

Let $x^p = a^{pi}b^{pj}c^{pk}$. Since

$$[b, xc^{-k}] = [b, x] = a^{p^{n-1}i_2}b^{pj_2} \neq 1,$$

$\langle b, xc^{-k}\rangle$ is minimal non-abelian by Lemma 2.3. Since $(xc^{-k})^p = x^p c^{-pk} = a^{pi}b^{pj}$, we have, by Lemma 2.4,

$$Z(\langle b, xc^{-k}\rangle) = \langle b^p, (xc^{-k})^p, [b, xc^{-k}]\rangle = \langle b^p, a^{pi}b^{pj}, a^{p^{n-1}i_2}b^{pj_2}\rangle.$$

Since $c^p \in Z(G)$ and $c^p \notin Z(\langle b, xc^{-k}\rangle)$,

$$Z(\langle b, xc^{-k}\rangle) \neq Z(G).$$

This is a contradiction. Hence, $[b, x] = 1$. By Lemma 2.5, we also have $[c, x] = 1$.

(2) $[a, x] = a^{p^{n-1}i_1}$.

Let $[a, x] = a^{pm}b^s c^t$. By Lemma 2.1, $[a^p, x] = \prod_{i=1}^{p}[ia, x]^{\binom{p}{i}}$. Moreover, since

$$[a, x, a] = [a^{pm}b^s c^t, a] = b^{-tvp}c^{-sp} \text{ and } G_4 = 1,$$

we have

$$[a^p, x] = \prod_{i=1}^{p} [ia, x]^{\binom{p}{i}} = [a, x]^p = a^{p^2 m} b^{ps} c^{pt}.$$

On the other hand, $[a^p, x] = 1$ since $a^p \in Z(G)$. Thus

$$a^{p^2 m} b^{ps} c^{pt} = 1.$$

It follows that $p^{n-2}|m$, $p|s$ and $p|t$. Hence, we can assume that

$$[a, x] = a^{p^{n-1} i_1} b^{p j_1} c^{p k_1}.$$

Since

$$[a, xb^{-k_1} c^{-j_1 v^{-1}}] = a^{p^{n-1} i_1},\ [b, xb^{-k_1} c^{-j_1 v^{-1}}] = [b, x] \text{ and}$$
$$[c, xb^{-k_1} c^{-j_1 v^{-1}}] = [c, x],$$

we can assume by replacing x with $xb^{-k_1} c^{-j_1 v^{-1}}$ that

$$[a, x] = a^{p^{n-1} i_1}.$$

(3) a final contradiction

Since $x \notin Z(G)$, $[a, x] \neq 1$. Thus $\langle a, x \rangle$ is minimal non-abelian. Moreover, by Lemma 2.6 we have

$$Z(\langle a, x \rangle) = \langle a^p, x^p, [a, x] \rangle = \langle a^p, x^p \rangle.$$

On the other hand, $Z(G) = \langle a^p, b^p, c^p \rangle$. Thus

$$Z(\langle a, x \rangle) \neq Z(G).$$

This is a contradiction.

Case 2. M is isomorphic to the group (2) in the theorem. That is, $M = \langle a, b, c \mid a^{p^n} = b^{p^2} = c^{p^2} = 1, [a, b] = c^p, [c, a] = b^{up} c^p, [c, b] = 1 \rangle$, where $4u = 1 - \rho^{2r+1}$ with $1 \leq r \leq \frac{1}{2}(p-1)$ and ρ the smallest positive integer which is a primitive root modular p, $n \geq 1$.

Using the same argument as that of Case 1, we also prove that there is no $\mathcal{S}$-group which contains M as its maximal subgroup in this case. The details are omitted.

Case 3. M is isomorphic to the group (3) in the theorem. That is,

$M = \langle a_1, a_2, b \mid a_1^{p^2} = a_i^{p^{q+1}} = a_j^{p^q} = b^{p^n} = 1,\ a_1^p = a_{r+1}^{p^q},\ [a_1, b] = b^{p^{n-1}},$ $[a_k, b] = a_{k+1},\ [a_p, b] = \prod_{t=2}^{p} a_t^{-\binom{p}{t-1}},\ [a_u, a_v] = 1 \rangle$, where $2 \leq c = (p-1)q + r$, $1 \leq r \leq p-1$, $2 \leq i \leq r+1$, $r+2 \leq j \leq p$, $2 \leq k \leq p-1$, $1 \leq u, v \leq p$, $n \geq 2$.

It is clear that $Z(M) = \langle a_1^p, b^p \rangle$ and M has exactly one abelian maximal subgroup $A = \langle b^p, a_1, a_2, \ldots, a_p \rangle$. Hence, $A \trianglelefteq G$ and $G' \leq A$ since $|G/A| = p^2$. Take $x \in G \backslash M$. Then $G = \langle x, a, b, c \rangle$. We prove there exists an $\mathcal{S}$-group G such that M is a maximal subgroup of G by the following steps.

(1) $[x, a_i] = 1$ for $1 \leq i \leq p$, i.e., $x \in C_G(A)$.

Since

$$[M_{c-1}, M, G] \le [Z(M),\ G] = 1 \text{ and } [M, G, M_{c-1}] \le [A, A] = 1,$$

$[G, M_{c-1}, M] = 1$ by the Three Subgroup Lemma. Hence $[x, M_{c-1}] \le Z(M) = Z(G)$. Since $M_{c-1} = \langle a_r^{p^q}, a_{r+1}^{p^q}\rangle$, $[x, a_r^{p^q}] = a_1^{pi_c} b^{pj_c}$. Notice that $a_r^{p^{q+1}} = 1$. Thus

$$1 = [x, a_r^{p^{q+1}}] = [x, a_r^{p^q}]^p = b^{p^2 j_c}.$$

It follows that

$$p^{n-2} \mid j_c \text{ and } [x, a_r^{p^q}] = a_1^{pi_c} b^{p^{n-1} j'_c}.$$

Since $[a_r^{p^q}, b] = a_1^p$, we can assume that $[x, a_r^{p^q}] = b^{j'_c p^{n-1}}$ by replacing x with xb^{i_c}. We will prove $[x, a_r^{p^q}] = 1$.

Otherwise, $\langle x, a_r^{p^q}\rangle$ is minimal non-abelian. Hence

$$Z(\langle x, a_r^{p^q}\rangle) = \langle x^p, [x, a_r^{p^q}]\rangle = \langle x^p, b^{j'_c p^{n-1}}\rangle = Z(G) = \langle a_1^p, b^p\rangle.$$

It follows that $n = 2$ and $x^p = a_1^{ip} b^{jp}$, where $p \nmid i$.

Let $H = \langle a_r^{p^q}, a_1^{-i}x\rangle$. Then $[a_1^{-i}x, a_r^{p^q}] = b^{j'_c p}$ and H is minimal non-abelian. By Lemma 2.4, we get $Z(H) = \langle (a_1^{-i}x)^p, [a_1^{-i}x, a_r^{p^q}]\rangle$. Moreover, we have

$$Z(H) = \langle b^p\rangle \ne Z(G)$$

since $(a_1^{-i}x)^p = b^{jp}$ and $[a_1^{-i}x, a_r^{p^q}] = b^{j'_c p}$. This is a contradiction. Hence, $[x, a_r^{p^q}] = 1$. It follows by Lemma 2.5 that

$$[x, a_1] = [x, a_2] = \cdots = [x, a_p] = 1.$$

(2) $[x, b] = a_2$.

Let

$$[x, b] = a_1^{i_1} a_2^{i_2} \cdots a_p^{i_p} b^{pj}.$$

Since $[a_k, b] = a_{k+1}$ for $k = 2, \ldots, p-1$, we can assume by replacing x with $xa_2^{-i_3} \cdots a_{p-1}^{-i_p}$ that

$$[x, b] = a_1^{i_1} a_2^{i_2} b^{pj}.$$

Hence, $[x, 2b] = b^{i_1 p^{n-1}} a_3^{i_2}$, $[x, 3b] = a_4^{i_2}, \ldots,$ $[x, (p-1)b] = a_p^{i_2}$, $[x, pb] = \prod_{t=2}^{p} a_t^{-\binom{p}{t-1} i_2}$.

Since $b^p \in Z(G)$, we have by Lemma 2.1

$$\begin{aligned} 1 = [x, b^p] &= [x, b]^{\binom{p}{1}} [x, 2b]^{\binom{p}{2}} \cdots [x, pb]^{\binom{p}{p}} \\ &= a_1^{pi_1} a_2^{pi_2} b^{p^2 j} a_3^{i_2\binom{p}{2}} \cdots a_p^{i_2\binom{p}{2}} \prod_{t=2}^{p} a_t^{-\binom{p}{t-1} i_2} \\ &= a_1^{pi_1} b^{p^2 j}. \end{aligned}$$

It follows that $p|i_1$ and $p^{n-2}|j$. Thus

$$[x, b] = a_1^{pi_1} a_2^{i_2} b^{p^{n-1} j}.$$

Moreover, by Lemma 2.2 we can do a suitable replacement such that

$$[x, b] = a_2^{i_2}.$$

If $p \mid i_2$, then $[x, b] = a_2^{pi'_2}$. By Lemma 2.2, there exists $a \in A$ such that $[a, b] = a_2^p$. Hence, $[xa^{-i'_2}, b] = 1$ and $xa^{-i'_2} \in Z(G) = Z(M) \leq M$. It follows that $x \in M$, a contradiction. Thus, $p \nmid i_2$. Replacing x by $x^{i_2^{-1}}$, we can assume that $[x, b] = a_2$.

(3) $x^p = \prod_{t=2}^{p} a_t^{-\binom{p}{t}}$.

Since

$$\begin{aligned}[a_p, b] = \prod_{t=2}^{p} a_t^{-\binom{p}{t-1}} &= a_2^{-\binom{p}{1}} a_3^{-\binom{p}{2}} \cdots a_p^{-\binom{p}{p-1}} \\ &= a_2^{-p}[a_2, b]^{-\binom{p}{2}} \cdots [a_{p-1}, b]^{-\binom{p}{p-1}} \\ &= a_2^{-p}[a_2^{-\binom{p}{2}} \cdots a_{p-1}^{-\binom{p}{p-1}}, b],\end{aligned}$$

we have

$$a_2^p = [a_2^{-\binom{p}{2}} \cdots a_{p-1}^{-\binom{p}{p-1}} a_p^{-\binom{p}{p}}, b].$$

On the other hand, $a_2^p = [x, b]^p = [x^p, b]$ by (2). It follows that

$$[x^p a_2^{\binom{p}{2}} \cdots a_{p-1}^{\binom{p}{p-1}} a_p^{\binom{p}{p}}, b] = 1$$

and

$$x^p a_2^{\binom{p}{2}} \cdots a_{p-1}^{\binom{p}{p-1}} a_p^{\binom{p}{p}} \in Z(G) = \langle a_1^p, b^p \rangle.$$

Hence, we can assume that

$$x^p = a_1^{ip} b^{jp} a_2^{-\binom{p}{2}} \cdots a_{p-1}^{-\binom{p}{p-1}} a_p^{-\binom{p}{p}}.$$

If $p \nmid j$, then $(x^{j^{-1}} b^{-1})^p = a_1^{pj^{-1}i}$ by Lemma 2.1. Let $H = \langle a_r^{p^q}, x^{j^{-1}} b^{-1} \rangle$. Then $H' = \langle a_1^p \rangle$. By Lemmas 2.3 and 2.6, we have H is minimal non-abelian and $Z(H) = \langle a^p \rangle \neq Z(G)$, a contradiction. Hence $p \mid j$. Thus

$$x^p = a_1^{ip} b^{p^2 j'} a_2^{-\binom{p}{2}} \cdots a_{p-1}^{-\binom{p}{p-1}} a_p^{-\binom{p}{p}}.$$

Replacing x by $xa_1^{-i} b^{-pj'}$, we have

$$x^p = a_2^{-\binom{p}{2}} \cdots a_{p-1}^{-\binom{p}{p-1}} a_p^{-\binom{p}{p}}.$$

Let

$$a'_1 = a_1, a'_2 = x, a'_3 = a_2 b^{-ip^{n-1}}, a'_4 = a_3, \ldots, a'_p = a_{p-1}.$$

Then, by an argument above, now we have

$$a_1'^{p^2} = b^{p^n} = 1, [a'_1, b] = b^{p^{n-1}}, [a'_2, b] = a'_3, \ldots, [a'_{p-1}, b] = a'_p, [a'_p, b] = \prod_{t=2}^{p} a_t^{-\binom{p}{t-1}}.$$

Moreover, by the hypothesis of M and the relations between a_i and a_i' we get the following:

If $r \leq p-2$, then

$$a_2'^{p^{q+1}} = a_3'^{p^{q+1}} = \cdots = a_{r+2}'^{p^{q+1}} = a_{r+3}'^{p^q} = \cdots = a_p'^{p^q} = 1, a_1^p = a_{r+2}'^{p^q}.$$

Thus

$G = \langle a_1', a_2', x \mid a_1'^{p^2} = a_2'^{p^{q+1}} = a_3'^{p^{q+1}} = \cdots = a_{r+2}'^{p^{q+1}} = a_{r+3}'^{p^q} = \cdots = a_p'^{p^q} = b^{p^n} = 1, a_1^p = a_{r+2}'^{p^q}, [a_1', b] = b^{p^{n-1}}, [a_2', b] = a_3', \ldots, [a_{p-1}', b] = a_p', [a_p', b] = \prod_{t=2}^{p} a_t^{\binom{p}{t-1}}, [a_u', a_v'] = 1\rangle$, where $1 \leq u, v \leq p$, which is a group (3) in the theorem with $q' = q$ and $r' = r+1$.

If $r = p-1$, then

$$a_2'^{p^{q+2}} = a_3'^{p^{q+1}} = \cdots = a_p'^{p^{q+1}} = 1, a_1^p = a_2'^{p^{q+1}}.$$

Hence

$G = \langle a_1', a_2', x \mid a_1'^{p^2} = a_2'^{p^{q+2}} = a_3'^{p^{q+1}} = \cdots = a_p'^{p^{q+1}} = b^{p^n} = 1$, $a_1^p = a_2'^{p^{q+1}}$, $[a_1', b] = b^{p^{n-1}}$, $[a_2', b] = a_3', \ldots$, $[a_{p-1}', b] = a_p'$, $[a_p', b] = \prod_{t=2}^{p} a_t^{\binom{p}{t-1}}$, $[a_u', a_v'] = 1\rangle$, where $1 \leq u, v \leq p$, which is a group (3) in the theorem with $q' = q+1$ and $r' = 1$.

Finally, we prove that the groups listed in the theorem are pairwise non-isomorphic. It is clear that $d(Z(G)) = 3$ for the groups (1–2) and $d(Z(G)) = 2$ for the groups (3). Thus, the groups (1) and (2) are not isomorphic to the groups (3). By [12, Theorem 3.6], the groups (1) are not isomorphic to the groups (2). □

Remark. From Theorem 3.2, we observe that an $\mathcal{S}$-group is generated by three elements and has a unique abelian maximal subgroup.

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