J. Korean Math. Soc. **54** (2017), No. 4, pp. 1099–1108
https://doi.org/10.4134/JKMS.j160075
pISSN: 0304-9914 / eISSN: 2234-3008

ON ϕ-SEMIPRIME SUBMODULES

Mahdieh Ebrahimpour and Fatemeh Mirzaee

Abstract. Let R be a commutative ring with non-zero identity and M be a unitary R-module. Let $S(M)$ be the set of all submodules of M and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. We say that a proper submodule P of M is a ϕ-semiprime submodule if $r \in R$ and $x \in M$ with $r^2x \in P \setminus \phi(P)$ implies that $rx \in P$. In this paper, we investigate some properties of this class of sub-modules. Also, some characterizations of ϕ-semiprime submodules are given.

1. Introduction

Throughout this paper R is a commutative ring with non-zero identity and M is a unitary R-module. We will denote the set of submodules of M by $S(M)$. Let I be an ideal of R and N be a submodule of M. Then $\sqrt{I}$ denotes the radical of I and $(N : M) = \{r \in R \,|\, rM \subseteq N\}$, which is clearly an ideal of R.

Various generalizations of prime (resp., primary) ideals are studied in [2–6, 9, 11–13, 21]. the class of prime submodules as a generalization of the class of prime ideals has been studied by many authors. For example see [1, 14, 17]. Then many generalizations of prime submodules were studied such as weakly prime (resp., primary) submodules in [10, 18], almost prime (resp., primary) submodules in [16], 2-absorbing submodules in [22], classical prime (resp., primary) submodules in [7, 8] and semiprime submodules in [19].

In this paper we extend the concept of semiprime submodules. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. A proper submodule P of M is called ϕ-semiprime if whenever $r \in R$ and $x \in M$ with $r^2x \in P \setminus \phi(P)$ implies $rx \in P$. Since $P \setminus \phi(P) = P \setminus (P \cap \phi(P))$, without loss of generality throughout the paper we will assume that $\phi(P) \subseteq P$. For two functions $\psi_1, \psi_2 : S(M) \to S(M) \cup \{\emptyset\}$ we write $\psi_1 \leq \psi_2$ if $\psi_1(N) \subseteq \psi_2(N)$ for each $N \in S(M)$.

In the rest of the paper we use the functions $\phi_\emptyset(N) = \emptyset$ for semiprime submodules, $\phi_0(N) = 0$ for weakly semiprime submodules, $\phi_1(N) = N$ for any submodule, $\phi_2(N) = (N : M)N$ for almost semiprime submodules, $\phi_n(N) = (N : M)^{n-1}N, (n \geq 3)$ for n-almost semiprime submodules and $\phi_\omega(N) =$

Received February 2, 2016; Revised July 11, 2016.
2010 *Mathematics Subject Classification.* Primary 13C05, Secondary 13C13.
Key words and phrases. semiprime submodules, ϕ-semiprime submodules, weakly semiprime submodules, m-almost semiprime submodules, flat modules.

$\cap_{i=1}^{\infty}(N:M)^iN$ for ω-semiprime submodules. Observe that $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1$.

Among many results concerning the properties of ϕ-semiprime submodules some characterizations of these submodules will be investigated in Theorems 2.5 and 2.12. In Theorem 2.22, it is proved that if F is a flat R-module and P is a weakly semiprime submodule of M such that $F\otimes P \neq F\otimes M$, then $F\otimes P$ is a weakly semiprime submodule of $F\otimes M$. Also we show that if F is a faithfully flat R-module and N is a submodule of M, then N is a weakly semiprime submodule of M if and only if $F\otimes N$ is a weakly semiprime submodule of $F\otimes M$.

2. ϕ-semiprime submodules

2.1. Some properties of ϕ-semiprime submodules

Every semiprime submodule is ϕ-semiprime. But the converse is not true in general. For example, consider the $\mathbb{Z}$-module $M=\mathbb{Z}_{24}$ and the submodule $N=8\mathbb{Z}$. Also let $\phi=\phi_2$. Since $\phi_2(N)=(N:M)N=N$, so N is a ϕ-semiprime submodule of M. But N is not semiprime. Because $2^2\overline{2}\in N$ but $2\overline{2}\notin N$.

Next we assert that under some conditions ϕ-semiprime submodules are semiprime.

Theorem 2.1. *Let R be a commutative ring and M be an R-module. Let $\phi: S(M)\to S(M)\cup\{\emptyset\}$ be a function and P be a ϕ-semiprime submodule of M. If $(x+(P:M))^2P \nsubseteq \phi(P)$ for all $x\in R\setminus(P:M)$, then P is a semiprime submodule of M.*

Proof. Let $r^2m\in P$, where $r\in R$ and $m\in M$. If $r^2m\notin\phi(P)$, then $rm\in P$ since P is ϕ-semiprime. So assume that $r^2m\in\phi(P)$. First, suppose that $r^2P\nsubseteq\phi(P)$. So there exists $n_0\in P$ such that $r^2n_0\notin\phi(P)$. Then $r^2(m+n_0)\in P\setminus\phi(P)$. Thus $r(m+n_0)\in P$ and so $rm\in P$. Hence we can assume that $r^2P\subseteq\phi(P)$.

Next, suppose that $(r+r_0)^2m\notin\phi(P)$ for some $r_0\in(P:M)$. Therefore, $(r+r_0)^2m\in P\setminus\phi(P)$ and so $(r+r_0)m\in P$. Hence $rm\in P$. So we can assume that $(r+(P:M))^2m\subseteq\phi(P)$. Since $(r+(P:M))^2P\nsubseteq\phi(P)$ there exists $k\in(P:M)$ and $n\in P$ with $(r+k)^2n\notin\phi(P)$. Then $(r+k)^2(n+m)\in P\setminus\phi(P)$. So $(r+k)(n+m)\in P$. Hence $rm\in P$. Therefore P is a semiprime submodule of M. □

Theorem 2.2. *Let R be a commutative ring with characteristic 2 and M be an R-module. Let $\phi: S(M)\to S(M)\cup\{\emptyset\}$ be a function and P be a ϕ-semiprime submodule of M. If $r_0^2P\nsubseteq\phi(P)$ for some $r_0\in(P:M)$, then P is a semiprime submodule of M.*

Proof. Let $r^2m \in P$, where $r \in R$ and $m \in M$. Similar to the proof of Theorem 2.1, we can assume that $r^2P \subseteq \phi(P)$. If $r^2m \notin \phi(P)$, then $r^2m \in P \setminus \phi(P)$ and so $rm \in P$. Because P is ϕ-semiprime.

Next, suppose that $r_0^2m \notin \phi(P)$. Therefore, $(r + r_0)^2m = (r^2 + r_0^2)m \in P \setminus \phi(P)$ and so $(r + r_0)m \in P$. Hence $rm \in P$. So we can assume that $r_0^2m \in \phi(P)$. Since $r_0^2P \not\subseteq \phi(P)$, there exists $n \in P$ with $r_0^2n \notin \phi(P)$. Then $(r + r_0)^2(m + n) = (r^2 + r_0^2)(m + n) \in p \setminus \phi(P)$. Hence $(r + r_0)(m + n) \in P$ and so $rm \in P$. Therefore, P is a semiprime submodule of M. □

Corollary 2.3. *Let R be a commutative ring and M be an R-module. Let P be a weakly semiprime submodule of M which is not semiprime. Then $r^2P = (0)$ for some $r \in R \setminus (P : M)$.*

Proof. In Theorem 2.1, set $\phi = \phi_0$. □

Corollary 2.4. *Let R be a commutative ring with characteristic 2 and M be an R-module. Let P be a weakly semiprime submodule of M that is not semiprime. Then $r^2P = (0)$ for all $r \in (P : M)$.*

Proof. In Theorem 2.2, set $\phi = \phi_0$. □

In Theorem 2.5, we give several characterizations of ϕ-semiprime submodules.

Theorem 2.5. *Let R be a commutative ring, M be an R-module, P be a proper submodule of M and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Then the following statements are equivalent*:

1) *P is a ϕ-semiprime submodule of M.*
2) *$(P :_M (r^2)) = (\phi(P) :_M (r^2)) \cup (P :_M (r))$ for all $r \in R$.*
3) *$(P :_M (r^2)) = (\phi(P) :_M (r^2))$ or $(P :_M (r^2)) = (P :_M (r))$ for all $r \in R$.*

Proof. (1) $\Rightarrow$ (2) Let $x \in (P :_M (r^2))$. Then $r^2x \in P$. If $r^2x \notin \phi(P)$, then $rx \in P$, because P is ϕ-semiprime and so $x \in (P :_M (r))$. Now, let $r^2x \in \phi(P)$. Then $x \in (\phi(P) :_M (r^2))$. Hence $(P :_M (r^2)) \subseteq (\phi(P) :_M (r^2)) \cup (P :_M (r))$. Since $\phi(P) \subseteq P$ we have $(\phi(P) :_M (r^2)) \cup (P :_M (r)) \subseteq (P :_M (r^2))$. Therefore, $(P :_M (r^2)) = (\phi(P) :_M (r^2)) \cup (P :_M (r))$.

(2) $\Rightarrow$ (3) It is straightforward.

(3) $\Rightarrow$ (1) Let $r^2x \in P \setminus \phi(P)$, where $r \in R$ and $x \in M$. Hence $x \in (P :_M (r^2))$ and $x \notin (\phi(P) :_M (r^2))$. So $x \in (P :_M (r))$, by assumption. Thus $rx \in P$ and P is ϕ-semiprime. □

Theorem 2.6. *Let R be a commutative ring, M be an R-module, P be a proper submodule of M and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. If P is a ϕ-semiprime submodule of M, then $\sqrt{(P : x)} = \sqrt{(\phi(P) : x)}$ or $\sqrt{(P : x)} = (p : x)$ for all $x \in M \setminus P$.*

Proof. Let $x \in M \setminus P$ and $a \in \sqrt{(P : x)} \setminus \sqrt{(\phi(P) : x)}$. Thus $a^sx \in P \setminus \phi(P)$ for some $s \in \mathbf{N}$. Hence $a \in (P : x)$. Because P is a ϕ-semiprime submodule of M. So $\sqrt{(P:x)} \subseteq \sqrt{(\phi(P):x)} \cup (P:x)$. Since $\phi(P) \subseteq P$ we have $\sqrt{(\phi(P):x)} \cup (P:x) \subseteq \sqrt{(P:x)}$ and so $\sqrt{(P:x)} = \sqrt{(\phi(P):x)} \cup (P:x)$. Therefore, $\sqrt{(P:x)} = \sqrt{(\phi(P):x)}$ or $\sqrt{(P:x)} = (p:x)$. □

2.2. m-almost semiprime submodules

Corollary 2.7. *Let R be a commutative ring, M be an R-module and P be a proper submodule of M. Then P is an m-almost semiprime submodule of M if and only if for any submodule N of M and $a \in R$ with $(a^2)N \subseteq P$ and $(a^2)N \nsubseteq (P:M)^{m-1}P$, one has $(a)N \subseteq P$.*

Theorem 2.8. *Let R be a commutative ring, M be an R-module and $0 \neq Rx$ be a proper submodule of M such that $(0:x) = 0$ and $\sqrt{(Rx:M)} = (Rx:M)$. If Rx is not a semiprime submodule of M, then Rx is not an m-almost semiprime submodule of M, $(m \geq 2)$.*

Proof. Since Rx is not a semiprime submodule of M, there exist $a \in R$ and $y \in M$ such that $a^2y \in Rx$ and $ay \notin Rx$. If $a^2y \notin (Rx:M)^{m-1}x$, then Rx is not m-almost semiprime, by definition. So we can assume that $a^2y \in (Rx:M)^{m-1}x$. We have $a(x+y) \notin Rx$ and $a^2(x+y) \in Rx$. If $a^2(x+y) \notin (Rx:M)^{m-1}x$, then again Rx is not m-almost semiprime. So let $a^2(x+y) \in (Rx:M)^{m-1}x$. Then $a^2x \in (Rx:M)^{m-1}x$. Which gives that $a^2x = rx$ for some $r \in (Rx:M)^{m-1}$. Then $(0:x) = 0$ gives that $a^2 = r \in (Rx:M)^{m-1} \subseteq (Rx:M)$. So $a \in \sqrt{(Rx:M)} = (Rx:M)$. Thus $ay \in Rx$, which is a contradiction. □

Corollary 2.9. *Let R be a commutative ring, M be an R-module and $0 \neq Rx$ be a proper submodule of M such that $(0:x) = 0$ and $\sqrt{(Rx:M)} = (Rx:M)$. Then Rx is a semiprime submodule of M if and only if Rx is an m-almost semiprime submodule of M, $(m \geq 2)$.*

Corollary 2.10. *Let the assumptions be as in Corollary 2.9. Then Rx is a semiprime submodule of M if and only if Rx is an m-almost semiprime submodule of M, $(m \geq 2)$.*

Proof. Let Rx be m-almost semiprime. This is clear that Rx is almost semiprime. Conversely, let Rx be almost semiprime. So Rx is semiprime, by Corollary 2.9. So again Rx is m-almost semiprime, by Corollary 2.9. □

Lemma 2.11. *Let R be a commutative ring, I be an ideal of R, M be a finitely generated faithful multiplication R-module and N be a submodule of M. Then $(IN:M) = I(N:M)$.*

Proof. See [16, Lemma 3.4]. □

Theorem 2.12. *Let $m \geq 2$ be a positive integer, R be a commutative ring, M be a finitely generated faithful multiplication R-module and P be a proper submodule of M. Then the following conditions are equivalent:*

1) *P is an m-almost semiprime submodule of M.*

2) $(P : M)$ *is an* m*-almost semiprime ideal of* R.
3) $P = QM$ *for some* m*-almost semiprime ideal* Q *of* R.

Proof. (1) $\Rightarrow$ (2) Let $a, b \in R$ and $a^2b \in (P : M)\backslash(P : M)^m$. Then $(a^2)bM \subseteq P$ and $(a^2)bM \nsubseteq (P : M)^{m-1}P$, by Lemma 2.11. So $(a)bM \subseteq P$, by Corollary 2.7. Hence $ab \in (P : M)$ and $(P : M)$ is an m-almost semiprime ideal of R.

(2) $\Rightarrow$ (1) Let $r^2x \in P \setminus (P : M)^{m-1}P$ where $r \in R$ and $x \in M$. Then $(r^2)((x) : M) \subseteq ((r^2x) : M) \subseteq (P : M)$. If $(r^2)((x) : M) \subseteq (P : M)^m$, then

$$(r^2)((x) : M) \subseteq (P : M)^m \subseteq ((P : M)^{m-1}P : M).$$

So we have $(r^2)(x) = (r^2)((x) : M)M \subseteq (P : M)^{m-1}P$, a contradiction. Thus $(r)((x) : M) \subseteq (P : M)$, because $(P : M)$ is an m-almost semiprime ideal of R. Therefore, $(r)(x) = (r)((x) : M)M \subseteq (P : M)M = P$ and so $rx \in P$. Hence P is an m-almost semiprime submodule of M.

(2) $\Leftrightarrow$ (3) We have $Q = (P : M)$, by [15, Theorem 3.1]. □

2.3. ϕ-semiprime submodules of some well-known modules

Let S be a multiplicatively closed subset of the commutative ring R. We know by [20, Theorem 9.11], that each submodule of $S^{-1}M$ is of the form $S^{-1}N$, for some submodule N of M. It is easy to show that if P is a weakly semiprime submodule of M with $S^{-1}M \neq S^{-1}P$, then $S^{-1}P$ is a weakly semiprime submodule of $S^{-1}M$. In Theorem 2.13, we want to generalize this fact for ϕ-semiprime submodules.

Let $N(S) = \{x \in M \mid sx \in N; \exists s \in S\}$. It is clear that $N(S)$ is a submodule of M containing N and $S^{-1}(N(S)) = S^{-1}N$. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and define $S^{-1}\phi : S(S^{-1}M) \to S(S^{-1}M) \cup \{\emptyset\}$ by $S^{-1}\phi(S^{-1}N) = S^{-1}(\phi(N(S)))$ if $\phi(N(S)) \neq \emptyset$ and $S^{-1}\phi(S^{-1}N) = \emptyset$ if $\phi(N(S)) = \emptyset$ for every $N \in S(M)$. Since $\phi(N) \subseteq N$ we have $S^{-1}\phi(S^{-1}N) \subseteq S^{-1}N$.

Next, we show that if $S^{-1}(\phi(P)) \subseteq S^{-1}\phi(S^{-1}P)$, then ϕ-semiprimeness of P together with $S^{-1}P \neq S^{-1}M$ implies that $S^{-1}P$ is $S^{-1}\phi$-semiprime.

For a submodule L of M, define $\phi_L : S(\frac{M}{L}) \to S(\frac{M}{L}) \cup \{\emptyset\}$ by $\phi_L(\frac{N}{L}) = \frac{\phi(N)+L}{L}$ if $\phi(N) \neq \emptyset$ and $\phi_L(\frac{N}{L}) = \emptyset$ if $\phi(N) = \emptyset$ for $N \in S(M)$ with $L \subseteq N$.

Theorem 2.13. *Let* R *be a commutative ring,* M *be an* R*-module,* $\phi : S(M) \to S(M) \cup \{\emptyset\}$ *be a function and* P *be a* ϕ*-semiprime submodule of* M*. Then*

1) *If* $L \subseteq P$ *is a submodule of* M*, then* $\frac{P}{L}$ *is a* ϕ_L*-semiprime submodule of* $\frac{M}{L}$.

2) *Suppose that* S *is a multiplicatively closed subset of* R *such that* $S^{-1}P \neq S^{-1}M$ *and* $S^{-1}(\phi(P)) \subseteq S^{-1}\phi(S^{-1}P)$*. Then* $S^{-1}P$ *is an* $S^{-1}\phi$*-semiprime submodule of* $S^{-1}M$.

Proof. (1) Let $a \in R$ and $\overline{x} \in \frac{M}{L}$ with $a^2\overline{x} \in \frac{P}{L} \setminus \phi_L(\frac{P}{L})$, where $\overline{x} = x + L$, for some $x \in M$. So we have $a^2x \in P \setminus \phi(P)$. Thus $ax \in P$, because P is ϕ-semiprime. Therefore, $a\overline{x} \in \frac{P}{L}$ and so $\frac{P}{L}$ is a ϕ_L-semiprime submodule of $\frac{M}{L}$.

(2) Let $\frac{a}{s} \in S^{-1}R$ and $\frac{x}{t} \in S^{-1}M$ with $(\frac{a}{s})^2\frac{x}{t} \in S^{-1}P \setminus S^{-1}\phi(S^{-1}P)$. Then $\frac{a^2x}{s^2t} \in S^{-1}P \setminus S^{-1}(\phi(P))$, by assumption. So there exists $u \in S$ such that $ua^2x \in P \setminus \phi(P)$ (note that $va^2x \notin \phi(P)$, for each $v \in S$). Thus $uax \in P$. Therefore, $\frac{a}{s}\frac{x}{t} \in S^{-1}P$ and $S^{-1}P$ is an $S^{-1}\phi$-semiprime submodule of $S^{-1}M$. □

In the semiprime submodules case, P is a semiprime submodule of M if and only if $\frac{P}{K}$ is a semiprime submodule of $\frac{M}{K}$ for any submodule $K \subseteq P$. But the converse part may not be true in the case of ϕ-semiprime submodules. For example, consider the ring $R = K[X,Y]$, where K is a field and $\phi = \phi_2$. Also, let $P = (X, Y^2)$ and $L = (X,Y)^2$. Then $\frac{P}{L}$ is an almost semiprime submodule of $\frac{R}{L}$, while P is not so in R. But we have the following theorem.

Theorem 2.14. *Let R be a commutative ring, M be an R-module and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Let P and K be submodules of M such that $K \subseteq \phi(P)$. Then P is a ϕ-semiprime submodule of M if and only if $\frac{P}{L}$ is a ϕ_L-semiprime submodule of $\frac{M}{L}$.*

Proof. $\Rightarrow$) This is clear, by Theorem 2.13(1).

$\Leftarrow$) Let $\frac{P}{L}$ be a ϕ_L-semiprime submodule of $\frac{M}{L}$ and assume that $a^2x \in P \setminus \phi(P)$, where $a \in R$ and $x \in M$. If $a^2(x+L) \in \phi_L(\frac{P}{L}) = \frac{\phi(P)+L}{L} = \frac{\phi(P)}{L}$, then $a^2x \in \phi(P)$, which is a contradiction. So we have

$$a^2(x+L) \in \frac{P}{L} \setminus \phi_L(\frac{P}{L}).$$

Thus $a(x+L) \in \frac{P}{L}$, because $\frac{P}{L}$ is ϕ_L-semiprime. So $ax \in P$ and P is ϕ-semiprime. □

Proposition 2.15. *Let R be a commutative ring, M be an R-module, $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and P be a proper submodule of M. Then P is a ϕ-semiprime submodule of M if and only if $\frac{P}{\phi(P)}$ is a weakly semiprime submodule of $\frac{M}{\phi(P)}$.*

Proof. $\Rightarrow$) Assume that P is a ϕ-semiprime submodule of M. Let $r \in R$ and $x + \phi(P) \in \frac{M}{\phi(P)}$ with $0 \neq r^2(x + \phi(P)) \in \frac{P}{\phi(P)}$. Hence $r^2x \in P \setminus \phi(P)$ and so $rx \in P$. Thus $r(x + \phi(P)) \in \frac{P}{\phi(P)}$. Therefore, $\frac{P}{\phi(P)}$ is a weakly semiprime submodule of $\frac{M}{\phi(P)}$.

$\Leftarrow$) Assume that $\frac{P}{\phi(P)}$ is a weakly semiprime submodule of $\frac{M}{\phi(P)}$. Let $r^2x \in P \setminus \phi(P)$, where $r \in R$ and $x \in M$. Then $0 \neq r^2(x + \phi(P)) \in \frac{P}{\phi(P)}$ and hence $r(x + \phi(P)) \in \frac{P}{\phi(P)}$. Therefore, $rx \in P$ and P is ϕ-semiprime. □

Let R_i be a commutative ring and M_i be an R_i-module for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is of the form $N_1 \times N_2$, where N_i is a submodule of M_i for $i = 1, 2$.

Let $P_1 \times M_2$ be a weakly semiprime submodule of M and $r_1 \in R_1$ and $x_1 \in M_1$ with $r_1^2 x_1 \in M_1$. Let $0 \neq x_2 \in M_2$. Then $(r_1, 1)^2(x_1, x_2) \in P_1 \times M_2 \setminus \{(0,0)\}$. By assumption, $r_1 x_1 \in P_1$. Therefore, P_1 is a semiprime submodule of M_1. If P_1 is a weakly semiprime submodule of M_1, then $P_1 \times M_2$ need not be a weakly semiprime submodule of M.

Next, we show that if P_1 is a weakly semiprime submodule of M_1, then $P_1 \times M_2$ is a ϕ-semiprime submodule of M if $\{0\} \times M_2 \subseteq \phi(P_1 \times M_2)$.

Proposition 2.16. *Let R_i be a commutative ring and M_i be an R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$, $M = M_1 \times M_2$ and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Suppose that P_1 is a weakly semiprime submodule of M_1 such that $\{0\} \times M_2 \subseteq \phi(P_1 \times M_2)$. Then $P_1 \times M_2$ is a ϕ-semiprime submodule of M*

Proof. We have $P_1 \times M_2 \setminus \phi(P_1 \times M_2) \subseteq P_1 \times M_2 \setminus \{0\} \times M_2 = (P_1 \setminus \{0\}) \times M_2$. Let $(r_1, r_2)^2(m_1, m_2) \in P_1 \times M_2 \setminus \phi(P_1 \times M_2)$, where $(r_1, r_2) \in R$ and $(m_1, m_2) \in M$. So $(r_1^2 m_1, r_2^2 m_2) \in P_1 \setminus \{0\} \times M_2$ and by assumption on P_1 we have $r_1 m_1 \in P_1$. This gives $(r_1, r_2)(m_1, m_2) \in P_1 \times M_2$. Therefore, $P_1 \times M_2$ is a ϕ-semiprime submodule of M. □

Proposition 2.17. *With the same notation as in Proposition* 2.16, *let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function such that $\phi_\omega \leq \phi$. Then for any weakly semiprime submodule P_1 of M_1, $P_1 \times M_2$ is a ϕ-semiprime submodule of $M_1 \times M_2$.*

Proof. We have

$$\{0\} \times M_2 \subseteq (P_1 \times M_2 : M_1 \times M_2)^i (P_1 \times M_2) = [(P_1 : M_1)^i P_1] \times M_2$$

for all $i \geq 1$ and hence

$$\{0\} \times M_2 \subseteq \cap_{i=1}^{\infty} (P_1 \times M_2 : M_1 \times M_2)^i (P_1 \times M_2) = \phi_\omega(P_1 \times M_2) \subseteq \phi(P_1 \times M_2).$$

So the result follows by Proposition 2.16. □

Proposition 2.18. *Let $R = R_1 \times \cdots \times R_n$ be a ring and $M = M_1 \times \cdots \times M_n$ be an R-module, where R_i is a commutative ring and M_i is an R_i-module, for $i = 1, \ldots, n$. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function, $P = P_1 \times \cdots \times P_n$ be a ϕ-semiprime submodule of M, where P_i is a submodule of M_i and let $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function and $\phi(P) = \psi_1(P_1) \times \cdots \times \psi_n(P_n)$. Then P_j is a ψ_j-semiprime submodule of M_j for each j with $P_j \neq M_j$, $(n \geq 2)$.*

Proof. Let $P_j \neq M_j$, $x_j \in M_j$ and $r_j \in R_j$ such that $r_j^2 x_j \in P_j \setminus \psi_j(P_j)$. Thus $(0, \ldots, 0, r_j, 0, \ldots, 0)^2(0, \ldots, 0, x_j, 0, \ldots, 0) \in P \setminus \phi(P)$. Therefore,

$$(0, \ldots, 0, r_j, 0, \ldots, 0)(0, \ldots, 0, x_j, 0, \ldots, 0) \in P.$$

Because P is ϕ-semiprime. So $r_j x_j \in P_j$. Hence P_j is a ψ_j-semiprime submodule of M_j. □

Corollary 2.19. *Let $R = R_1 \times \cdots \times R_n$ be a ring, $M = M_1 \times \cdots \times M_n$ be an R-module and $P = P_1 \times \cdots \times P_n$, where R_i is a commutative ring, M_i is an R_i-module and P_i is a submodule of M_i for $i = 1, \ldots, n$. Let P be an m-almost semiprime submodule of M. Then P_j is an m-almost semiprime submodule of M_j for each j with $P_j \neq M_j$, $(n, m \geq 2)$.*

Proof. We have $\phi_m(P) = (P : M)^{m-1}P = (P_1 : M_1)^{m-1}P_1 \times \cdots \times (P_n : M_n)^{m-1}P_n = \phi_m(P_1) \times \cdots \times \phi_m(P_n)$. So the result follows by Proposition 2.18. □

2.4. Weakly semiprime submodules and flat modules

A flat module over a commutative ring R is an R-module M such that taking the tensor product over R with M preserves exact sequences.

Let R be a commutative ring, M be an R-module, N be a submodule of M and $r \in R$. It is easy to show that $(N :_M r) = \{m \in M \mid rm \in N\}$ is a submodule of M containing N. In the following lemma we have a characterization of ϕ-semiprime submodules.

Lemma 2.20. *Let R be a commutative ring, M be an R-module and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Let P be a proper submodule of M. Then P is a ϕ-semiprime submodule of M if and only if $(P : r^2) = (\phi(P) : r^2)$ or $(P : r^2) = (P : r)$ for every $r \in R$.*

Proof. This is clear, by Theorem 2.5. □

Lemma 2.21. *Let R be a commutative ring, M be an R-module, P be a submodule of M and $r \in R$. Then for every flat R-module F we have $F \otimes (P : r) = (F \otimes P : r)$.*

Proof. See, [5, Lemma 3.2]. □

Theorem 2.22. *Let R be a commutative ring and M be an R-module.*

1) *If F is a flat R-module and P is a weakly semiprime submodule of M such that $F \otimes P \neq F \otimes M$, then $F \otimes P$ is a weakly semiprime submodule of $F \otimes M$.*

2) *Let F be a faithfully flat R-module. Then P is a weakly semiprime submodule of M if and only if $F \otimes P$ is a weakly semiprime submodule of $F \otimes M$.*

Proof. (1) Let $r \in R$. We have $(P : r^2) = (0 : r^2)$ or $(P : r^2) = (P : r)$, by Lemma 2.20. Therefore, $(F \otimes P : r^2) = F \otimes (P : r^2) = F \otimes (0 : r^2) = (0 : r^2)$ or $(F \otimes P : r^2) = F \otimes (P : r^2) = F \otimes (P : r) = (F \otimes P : r)$, by Lemma 2.21. Hence $F \otimes P$ is a weakly semiprime submodule of $F \otimes M$, by Lemma 2.20.

(2) Let P be a weakly semiprime submodule of M and $F \otimes P = F \otimes M$. Therefore $0 \to F \otimes P \to F \otimes M \to 0$ is an exact sequence and since F is a faithfully flat R-module we have $0 \to P \to M \to 0$ is an exact sequence. Hence $P = M$, which is a contradiction. So $F \otimes P \neq F \otimes M$. Now, $F \otimes P$ is a weakly semiprime submodule of $F \otimes M$, by part (1).

Conversely, suppose that $F \otimes P$ is a weakly semiprime submodule of $F \otimes M$. We have $F \otimes P \neq F \otimes M$ and obviously $P \neq M$. Let $r \in R$. We have $(F \otimes P : r^2) = (0 : r^2)$ or $(F \otimes P : r^2) = (F \otimes P : r)$, by Lemma 2.20. Then $F \otimes (P : r^2) = F \otimes (0 : r^2)$ or $F \otimes (P : r^2) = F \otimes (P : r)$, by Lemma 2.21.

So $0 \to F \otimes (P : r^2) \to F \otimes (0 : r^2) \to 0$ or $0 \to F \otimes (P : r^2) \to F \otimes (P : r) \to 0$ is an exact sequence. Thus $0 \to (P : r^2) \to (0 : r^2) \to 0$ or $0 \to (P : r^2) \to (P : r) \to 0$ is an exact sequence, because F is faithfully flat. Hence $(P : r^2) = (0 : r^2)$ or $(P : r^2) = (P : r)$. So P is a weakly semiprime submodule of M, by Lemma 2.20. □

References

[1] R. Ameri, *On the prime submodules of multiplication modules*, Int. J. Math. Math. Sci. **2003** (2003), no. 27, 1715–1724.

[2] D. F. Anderson and A. Badawi, *On n-absorbing ideals of commutative rings*, Comm. Algebra **39** (2011), no. 5, 1646–1676.

[3] D. D. Anderson and M. Bataineh, *Generalizations of prime ideals*, Comm. Algebra **36** (2008), no. 2, 686–696.

[4] D. D. Anderson and E. Smith, *Weakly prime ideals*, Houston J. Math. **29** (2003), no. 4, 831–840.

[5] A. Azizi, *Weakly prime submodules and prime submodules*, Glasg. Math. J. **48** (2006), no. 2, 343–346.

[6] A. Badawi and A. Yousefian Darani, *On weakly 2-absorbing ideals of commutative rings*, Houston J. Math. **39** (2013), no. 2, 441–452.

[7] M. Baziar and M. Behboodi, *Classical primary submodules and decomposition theory of modules*, J. Algebra Appl. **8** (2009), no. 3, 351–362.

[8] M. Behboodi and S. H. Shojaee, *On chains of classical prime submodules and dimension theory of modules*, Bull. Iranian Math. Soc. **36** (2010), no. 1, 149–166.

[9] S. Ebrahimi Atani and F. Farzalipour, *On weakly primary ideals*, Georgian Math. J. **12** (2005), no. 3, 423–429.

[10] ———, *On weakly prime submodules*, Tamkang J. Math. **38** (2007), no. 3, 247–252.

[11] M. Ebrahimpour, *On generalizations of prime ideals (II)*, Comm. Algebra **42** (2014), no. 9, 3861–3875.

[12] ———, *On generalisations of almost prime and weakly prime ideals*, Bull. Iranian Math. Soc. **40** (2014), no. 2, 531–540.

[13] M. Ebrahimpour and R. Nekooei, *On generalizations of prime ideals* , Comm. Algebra **40** (2012), no. 4, 1268–1279.

[14] ———, *On generalizations of prime submodules*, Bull. Iranian Math. Soc. **39** (2013), no. 5, 919–939.

[15] Z. A. El-bast and P. F. Smith, *Multiplication modules*, Comm. Algebra **16** (1988), no. 4, 755–779.

[16] H. A. Khashan, *On almost prime submodules*, Acta Math. Sci. B **32** (2012), no. 2, 645–651.

[17] C. P. Lu, *Prime submodules of modules*, Comment. Math. Univ. St. Pauli **33** (1984), no. 1, 61–69.

[18] R. Nekooei, *Weakly prime submodules*, Far East J. Math. Sci. **39** (2010), no. 2, 185–192.

[19] B. Saraç, *On semiprime submodules*, Commun. Algebra **37** (2009), no. 7, 2485–2495.

[20] R. Sharp, *Steps In Commutative Algebra*, Cambridge University Press, 191–211, Cambridge-New York-Sydney, 2000.

[21] A. Yousefian Darani and E. R. Puczylowski, *On 2-absorbing commutative semigroups and their applications to rings*, Semigroup Forum **88** (2013), no. 1, 83–91.

[22] A. Yousefian Darini and F. Soheilnia, *2-absorbing and weakly 2-absorbing submodules*, Thai J. Math. **9** (2011), no. 3, 577–584.

Mahdieh Ebrahimpour
Department of Mathematics
Vali-e-Asr University
Rafsanjan 518, Iran
E-mail address: m.ebrahimpour@vru.ac.ir

Fatemeh Mirzaee
Department of pure Mathematics
Shahid Bahonar University
Kerman 76169133, Iran
E-mail address: mirzaee0269@yahoo.com