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GLOBAL EXISTENCE AND UNIFORM DECAY OF COUPLED WAVE EQUATION OF KIRCHHOFF TYPE IN A NONCYLINDRICAL DOMAIN

Tae Gab Ha

Abstract. In this paper, we consider coupled wave equation of Kirchhoff type in a noncylindrical domain. This work is devoted to prove the existence and uniqueness of global solutions and decay for the energy of solutions.

1. Introduction

Let Ω be an open bounded domain of $\mathbb{R}^n$ containing the origin and having C^2 boundary. Let $\gamma : [0,\infty[\to \mathbb{R}$ be a continously differentiable function. Consider the family of subdomains $\{\Omega_t\}_{0\le t<\infty}$ of $\mathbb{R}^n$ given by $\Omega_t = T(\Omega)$, $T : y \in \Omega \mapsto x = \gamma(t)y$, whose boundaries are denoted by Γ_t, and let $\hat{Q}$ be the noncylindrical domain of $\mathbb{R}^{n+1}$ given by

$$\hat{Q} = \bigcup_{0\le t<\infty} \Omega_t \times \{t\}$$

with boundary

$$\hat{\Sigma} = \bigcup_{0\le t<\infty} \Gamma_t \times \{t\}.$$

In this paper, we are concerned with global existence and uniform decay of the energy to coupled wave equation of Kirchhoff type with memory given by

$$u'' - (a + b||\nabla u||^2_{2,t} + b||\nabla v||^2_{2,t})\Delta u + \delta u' + f(u) = 0 \quad \text{in } \hat{Q}, \tag{1.1}$$

$$v'' - (a + b||\nabla u||^2_{2,t} + b||\nabla v||^2_{2,t})\Delta v + \delta v' + h(v) = 0 \quad \text{in } \hat{Q}, \tag{1.2}$$

$$u = v = 0 \quad \text{on } \hat{\Sigma}, \tag{1.3}$$

$$u(x,0) = u_0, \quad u'(x,0) = u_1 \quad \text{in } \Omega_0, \tag{1.4}$$

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$$v(x,0) = v_0, \quad v'(x,0) = v_1 \quad \text{in } \Omega_0, \tag{1.5}$$

where u and v are the transverse displacement and $a \geq 1$, b and δ are positive constants. Δ and ∇ stand for the Laplacian and gradient with respect to the spatial variables respectively, $'$ denotes the derivative with respect to time t.

The problem of proving existence of solutions and uniform decay rates of the solution has recently attracted a lot of attention and various results are available (see [1,8–10,12] and a list of references therein). In the case of Liu and Wang [8], they studied the problem (1.1)-(1.5) in the cylindrical domain. They proved the global existence and blow-up of solutions in this system. However, the case of noncylindrical domain problem has been studied less than cylindrical domain problem (cf. [2,3,5,6,11,13]). For example, Benabidallah and Ferreira [2] and Clark et al. [3] studied existence of solutions in noncylindrical domain with $n = 2$ and proved exponential decay to the energy. Ferreira et al. [5] studied existence of solutions of Kirchhoff type wave equation in noncylindrical domain and Santos et al. [13] considered the coupled viscoelastic wave equation. Moreover [5] and [13] proved the exponential decay in the case $n > 2$.

In this paper, we prove the existence and uniqueness of solutions and uniform decay of coupled wave equation of Kirchhoff type with memory in noncylindrical domain and allow to apply the method developed in [4] and [6]. Ferreira and Lar'kin [4] studied existence and uniqueness of solutions for hyperbolic-parabolic equation in noncylindrical domains having an assumption

$$\sum_{i,j=1}^{n} \alpha_{ij}(y,t)\xi_i\xi_j \geq \beta|\xi|^2,$$

where β is a positive constant and α_{ij} is defined in $A(t)v = \sum_{i,j=1}^{n} \partial_{y_i}(\alpha_{ij}\partial_{y_j} v)$. Thanks to this assumption, they were able to apply Faedo-Galerkin's method and prove existence and uniqueness of solutions. In this paper, we do not consider this assumption. In order to obtain existence and uniqueness of solutions we use a Gronwall's lemma, but we can not apply a Gronwall's lemma directly because of terms $A(t)\varphi$ and $A(t)\psi$. To overcome this point we give a control of absolute value of $\gamma'(t)$ (see (2.12)) which was used to Ha and Park [6]. We only obtained the exponential decay of solution for our problem for the case $n > 4$. Because $E(t)$, which is the energy of the problem (1.1)-(1.5), is nonincreasing function for the case $n > 4$ but for the case $n \leq 4$, $E(t)$ is not necessary decreasing function (see (4.3)). The main difficulty to obtain the decay for the case $n \leq 4$ is due to the influence of the geometry of the noncylindrical domain and it is an open problem.

This paper is organized as follows: In Section 2, we recall a notation and hypotheses and introduce our main results. In Section 3, we prove the existence and uniqueness of strong solution employing Faedo-Galerkin's method. In Section 4, we prove the exponential decay rate for the solution.

2. Hypotheses and main results

We begin this section introducing some hypotheses and our main results. Throughout this paper we use standard functional spaces and denote that $||\cdot||_p$, $||\cdot||_{p,t}$ are $L^p(\Omega)$ norm and $L^p(\Omega_t)$ norm. Also we define $(u,v) = \int_\Omega u(x)v(x)dx$ and $(u,v)_t = \int_{\Omega_t} u(x)v(x)dx$.

The method that we use to prove the result of existence and uniqueness is based on the transformation of our problem into another initial boundary value problem defined over a cylindrical domain whose sections are not time dependent. This is done using a suitable change of variable. Our existence result on noncylindrical domain will follow using the inverse transformation. That is, using the diffeomorphism $\tau : \hat{Q} \to Q = \Omega \times [0,\infty)$ defined by

$$\tau : \hat{Q} \to Q, \quad (x,t) \in \Omega_t \times \{t\} \mapsto (y,t) = \Big(\frac{x}{\gamma(t)}, t\Big) \tag{2.1}$$

and $\tau^{-1} : Q \to \hat{Q}$ defined by

$$\tau^{-1}(y,t) = (x,t) = (\gamma(t)y, t). \tag{2.2}$$

Denoting by φ and ψ functions

$$\varphi(y,t) = u \circ \tau^{-1}(y,t) = u(\gamma(t)y,t) \text{ and } \psi(y,t) = v \circ \tau^{-1}(y,t) = v(\gamma(t)y,t) \tag{2.3}$$

the initial boundary value problem (1.1)-(1.5) becomes

$$\begin{aligned} &\varphi'' - \gamma^{-2}(a + b\gamma^{n-2}||\nabla\varphi||_2^2 + b\gamma^{n-2}||\nabla\psi||_2^2)\Delta\varphi + \delta\varphi' \\ &+ f(\varphi) + A(t)\varphi + \alpha_1 \cdot \nabla\varphi' + \alpha_2 \cdot \nabla\varphi = 0 \quad \text{in} \quad Q, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\psi'' - \gamma^{-2}(a + b\gamma^{n-2}||\nabla\varphi||_2^2 + b\gamma^{n-2}||\nabla\psi||_2^2)\Delta\psi + \delta\psi' \\ &+ h(\psi) + A(t)\psi + \alpha_1 \cdot \nabla\psi' + \alpha_2 \cdot \nabla\psi = 0 \quad \text{in} \quad Q, \end{aligned} \tag{2.5}$$

$$\varphi|_\Gamma = \psi|_\Gamma = 0, \tag{2.6}$$

$$\varphi(y,0) = \varphi_0, \quad \varphi'(y,0) = \varphi_1 \quad \text{in} \quad \Omega, \tag{2.7}$$

$$\psi(y,0) = \psi_0, \quad \psi'(y,0) = \psi_1 \quad \text{in} \quad \Omega, \tag{2.8}$$

where

$$\begin{aligned} &A(t)v = \sum_{i,j=1}^{n} \partial_{y_i}(\alpha_{ij}\partial_{y_j} v), \\ &\alpha_{ij}(y,t) = (\gamma'\gamma^{-1})^2 y_i y_j, \\ &\alpha_1(y,t) = -2\gamma'\gamma^{-1} y, \\ &\alpha_2(y,t) = -\gamma^{-2}(\gamma''\gamma + \gamma'(\delta\gamma + (n-1)\gamma'))y. \end{aligned}$$

The above method was introduced by Del Passo and Ughi [11] to study a certain class of parabolic equations in noncylindrical domains.

Now we give hypotheses for the main result.

($\mathbf{H_1}$) Hypotheses on $\boldsymbol{\gamma}$.

$$\gamma' \leq 0 \quad \text{if} \quad n > 4, \qquad \gamma' \geq 0 \quad \text{if} \quad n \leq 4, \tag{2.9}$$

$$\gamma(\cdot) \in L^\infty(0,\infty), \qquad \inf_{0 \leq t < \infty} \gamma(t) = \gamma_0 > 0, \tag{2.10}$$

$$\gamma' \in W^{2,\infty}(0,\infty) \cap W^{2,1}(0,\infty), \tag{2.11}$$

$$0 < \max_{0 \leq t < \infty} |\gamma'(t)| \leq \frac{1}{\sqrt{2\jmath} d}, \text{ where } d = diam(\Omega),\ \jmath = meas(\Omega). \tag{2.12}$$

($\mathbf{H_2}$) Hypotheses on $\boldsymbol{f}$ and $\boldsymbol{h}$.

$$f(s), h(s) \in C^1(\mathbb{R}) \quad \text{satisfies} \quad f(s)s \geq 0, \quad h(s)s \geq 0 \quad \forall s \in \mathbb{R}.$$

Suppose that f and h are superlinear, that is

$$f(s)s \geq (2+\beta_1)F(s),\ h(s)s \geq (2+\beta_2)H(s) \ \text{ for some } \ \beta_1, \beta_2 > 0, \tag{2.13}$$

where

$$F(z) = \int_0^z f(s)ds \quad \text{and} \quad H(z) = \int_0^z h(s)ds.$$

Additionally, assume that

$$|f'(s)| \leq |s|^p \quad \text{and} \quad |h'(s)| \leq |s|^q, \tag{2.14}$$

where

$$0 < p, q \leq \frac{2}{n-2} \ \text{ if } \ n \geq 3 \ \text{ and } \ p, q > 0 \ \text{ if } \ n = 1, 2.$$

Moreover, $a \geq 1$, b and δ are positive constants.

Now we are in a position to state our main results.

Theorem 2.1. *Let $\varphi_0, \psi_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $\varphi_1, \psi_1 \in H_0^1(\Omega)$ and* (H_1)-(H_2) *hold. Then there exists a unique solution φ and ψ of the problem* (2.4)-(2.8) *satisfying*

$$\varphi, \psi \in L^\infty(0,\infty; H_0^1(\Omega) \cap H^2(\Omega)),$$
$$\varphi', \psi' \in L^\infty(0,\infty; H_0^1(\Omega)),$$
$$\varphi'', \psi'' \in L^\infty(0,\infty; L^2(\Omega)).$$

As a consequence of the above theorem and using the change variable given in (2.1), we obtain the next result.

Theorem 2.2. *Under the hypotheses of Theorem 2.1, let $u_0, v_0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$ and $u_1, v_1 \in H_0^1(\Omega_0)$. Then there exists a unique solution u, v of the problem* (1.1)-(1.5) *satisfying*

$$u, v \in L^\infty(0,\infty; H_0^1(\Omega_t)),$$
$$u', v' \in L^\infty(0,\infty; H_0^1(\Omega_t)),$$
$$u'', v'' \in L^\infty(0,\infty; L^2(\Omega_t)).$$

In order to state another main result, we define the energy of the problem (1.1)-(1.5) by

$$\begin{aligned} E(t) = &\frac{1}{2}||u'||^2_{2,t} + \frac{1}{2}||v'||^2_{2,t} + \frac{a}{2}||\nabla u||^2_{2,t} + \frac{a}{2}||\nabla v||^2_{2,t} \\ &+ \frac{b}{4}(||\nabla u||^2_{2,t} + ||\nabla v||^2_{2,t})^2 + \int_{\Omega_t}(F(u)+H(v))dx. \end{aligned} \tag{2.15}$$

Theorem 2.3. *Let us take $u_0, v_0 \in H^1_0(\Omega_0) \cap H^2(\Omega_0)$, $u_1, v_1 \in H^1_0(\Omega_0)$ and assume that hypotheses* (H$_1$)-(H$_2$) *hold* (*except the case $n \leq 4$*). *Then there exist positive constants ω and C such that*

$$E(t) \leq CE(0)e^{-\omega t}$$

for all $t > 0$.

3. Existence of solutions

3.1. Proof of Theorem 2.1

In this section we prove the existence and uniqueness of regular solutions to problem (2.4)-(2.8). Let us denote by B the operator

$$Bw = -\Delta w, \qquad D(B) = H^1_0(\Omega) \cap H^2(\Omega).$$

It is well known that B is a positive self-adjoint operator in the Hilbert space $L^2(\Omega)$ for which there exist sequences $\{w_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ of eigenfunctions and eigenvalues of B such that the set of linear combinations of $\{w_n\}_{n\in\mathbb{N}}$ is dense in $D(B)$ and $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \to \infty$ as $n \to \infty$. Let us denote by

$$\varphi_{0m} = \sum_{j=1}^m (\varphi_0, w_j)w_j, \qquad \psi_{0m} = \sum_{j=1}^m (\psi_0, w_j)w_j,$$

$$\varphi_{1m} = \sum_{j=1}^m (\varphi_1, w_j)w_j, \qquad \psi_{1m} = \sum_{j=1}^m (\psi_1, w_j)w_j.$$

Note that for any $\{(\varphi_0, \psi_0), (\varphi_1, \psi_1)\} \in (D(B))^2 \times (H^1_0(\Omega))^2$, we have $(\varphi_{0m}, \psi_{0m}) \to (\varphi_0, \psi_0)$ strongly in $(D(B))^2$ and $(\varphi_{1m}, \psi_{1m}) \to (\varphi_1, \psi_1)$ strongly in $(H^1_0(\Omega))^2$.

Let us denote by V_m the space generated by $\{w_1, w_2, \ldots, w_m\}$. We search functions

$$\varphi_m(t) = \sum_{j=1}^m g_{jm}(t)w_j, \qquad \psi_m(t) = \sum_{j=1}^m k_{jm}(t)w_j$$

satisfying the approximate equation

$$\begin{aligned} &\int_\Omega \varphi''_m w dy - a\gamma^{-2}\int_\Omega \Delta\varphi_m w dy - b\gamma^{n-4}||\nabla\varphi_m||^2_2 \int_\Omega \Delta\varphi_m w dy \\ &- b\gamma^{n-4}||\nabla\psi_m||^2_2 \int_\Omega \Delta\varphi_m w dy + \delta\int_\Omega \varphi'_m w dy + \int_\Omega f(\varphi_m) w dy \end{aligned} \tag{3.1}$$

$$+\int_\Omega A(t)\varphi_m w dy+\int_\Omega \alpha_1\cdot\nabla\varphi_m' w dy+\int_\Omega \alpha_2\cdot\nabla\varphi_m w dy=0$$

and

$$\begin{aligned}(3.2)\qquad &\int_\Omega \psi_m'' w dy-a\gamma^{-2}\int_\Omega \Delta\psi_m w dy-b\gamma^{n-4}||\nabla\varphi_m||_2^2\int_\Omega \Delta\psi_m w dy\\ &-b\gamma^{n-4}||\nabla\psi_m||_2^2\int_\Omega \Delta\psi_m w dy+\delta\int_\Omega \psi_m' w dy+\int_\Omega h(\psi_m) w dy\\ &+\int_\Omega A(t)\psi_m w dy+\int_\Omega \alpha_1\cdot\nabla\psi_m' w dy+\int_\Omega \alpha_2\cdot\nabla\psi_m w dy=0,\end{aligned}$$

for all $w\in V_m$ and with initial data

$$(3.3)\qquad \begin{aligned}&(\varphi_{0m},\psi_{0m})\to(\varphi_0,\psi_0)\ \text{ in }\ (D(B))^2 \text{ and}\\ &(\varphi_{1m},\psi_{1m})\to(\varphi_1,\psi_1)\ \text{ in }\ (H_0^1(\Omega))^2.\end{aligned}$$

By standard methods in differential equation, we prove the existence of solutions to the approximate equation (3.1)-(3.2) on some interval $[0,t_m)$. Then, this solution can be extended to the whole interval $[0,T]$, for all $T>0$, by using the following first estimate.

In this section the symbol C_i, $i\in\mathbb{N}$ indicates positive constants, which may be different.

3.1.1. *The first estimate.* Replacing $w=\varphi_m'(t)$ and $w=\psi_m'(t)$ in (3.1) and (3.2) we obtain

$$(3.4)\qquad\begin{aligned}&\frac{d}{dt}\mathcal{E}_{1m}(t)+\delta||\varphi_m'||_2^2+\delta||\psi_m'||_2^2+a\gamma^{-3}\gamma'||\nabla\varphi_m||_2^2+a\gamma^{-3}\gamma'||\nabla\psi_m||_2^2\\ &-\frac{b(n-4)}{4}\gamma^{n-5}\gamma'\Big(||\nabla\varphi_m||_2^2+||\nabla\psi_m||_2^2\Big)^2\\ =\ &-\int_\Omega A(t)\varphi_m\varphi_m' dy-\int_\Omega A(t)\psi_m\psi_m' dy-\int_\Omega \alpha_1\cdot\nabla\varphi_m'\varphi_m' dy\\ &-\int_\Omega \alpha_1\cdot\nabla\psi_m'\psi_m' dy-\int_\Omega \alpha_2\cdot\nabla\varphi_m\varphi_m' dy-\int_\Omega \alpha_2\cdot\nabla\psi_m\psi_m' dy,\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}_{1m}(t)=&\frac{1}{2}||\varphi_m'||_2^2+\frac{1}{2}||\psi_m'||_2^2+\frac{a}{2}\gamma^{-2}||\nabla\varphi_m||_2^2+\frac{a}{2}\gamma^{-2}||\nabla\psi_m||_2^2\\ &+\frac{b}{4}\gamma^{n-4}\Big(||\nabla\varphi_m||_2^2+||\nabla\psi_m||_2^2\Big)^2+\int_\Omega(F(\varphi_m)+H(\psi_m))dy.\end{aligned}$$

Now we will estimate terms of the right-hand side of (3.4). From hypotheses on γ and Green's formula, we get

$$\begin{aligned}-\int_\Omega A(t)\varphi_m\varphi_m' dy=&\ -\int_\Omega\sum_{i,j=1}^n\partial_{y_i}((\gamma'\gamma^{-1})^2 y_i y_j\partial_{y_j}\varphi_m)\varphi_m' dy\\ =&\ -(\gamma'\gamma^{-1})^2\int_\Omega\Big(\nabla(\nabla\varphi_m\cdot y)\cdot y+n\nabla\varphi_m\cdot y\Big)\varphi_m' dy\end{aligned}$$

$$= \frac{d}{dt}\left[\frac{1}{2}(\gamma'\gamma^{-1})^2||\nabla\varphi_m \cdot y||_2^2\right] - (\gamma'\gamma^{-1})\{\gamma''\gamma^{-1} - (\gamma')^2\gamma^{-2}\}||\nabla\varphi_m \cdot y||_2^2,$$

$$\begin{aligned}\int_\Omega \alpha_1 \cdot \nabla\varphi_m'\varphi_m' dy &= \int_\Omega -2\gamma'\gamma^{-1} y \cdot \nabla\varphi_m'\varphi_m' dy \\ &= \int_\Omega -\gamma'\gamma^{-1} y \cdot \nabla|\varphi_m'|^2 dy \\ &= n\gamma'\gamma^{-1}||\varphi_m'||_2^2\end{aligned}$$

and

$$\begin{aligned}&-\int_\Omega \alpha_2 \cdot \nabla\varphi_m\varphi_m' dy \\ &= \gamma''\gamma^{-1}\int_\Omega y \cdot \nabla\varphi_m\varphi_m' dy + \delta\gamma'\gamma^{-1}\int_\Omega y \cdot \nabla\varphi_m\varphi_m' dy \\ &\quad + (n-1)(\gamma'\gamma^{-1})^2\int_\Omega y \cdot \nabla\varphi_m\varphi_m' dy \\ &\le \Big(\frac{|\gamma''\gamma^{-1}|}{2} + \frac{\delta|\gamma'\gamma^{-1}|}{2} + \frac{(n-1)(\gamma'\gamma^{-1})^2}{2}\Big)\int_\Omega |y \cdot \nabla\varphi_m|^2 dy \\ &\quad + \Big(\frac{|\gamma''\gamma^{-1}|}{2} + \frac{\delta|\gamma'\gamma^{-1}|}{2} + \frac{(n-1)(\gamma'\gamma^{-1})^2}{2}\Big)\int_\Omega |\varphi_m'|^2 dy \\ &\le C_1(||\varphi_m'||_2^2 + ||\nabla\varphi_m||_2^2).\end{aligned}$$

Using the same argument, it is easily check that

$$\begin{aligned}&-\int_\Omega A(t)\psi_m\psi_m' dy \\ &= \frac{d}{dt}\left[\frac{1}{2}(\gamma'\gamma^{-1})^2||\nabla\psi_m \cdot y||_2^2\right] - (\gamma'\gamma^{-1})\{\gamma''\gamma^{-1} - (\gamma')^2\gamma^{-2}\}||\nabla\psi_m \cdot y||_2^2,\end{aligned}$$

$$\int_\Omega \alpha_1 \cdot \nabla\psi_m'\psi_m' dy = n\gamma'\gamma^{-1}||\psi_m'||_2^2$$

and

$$-\int_\Omega \alpha_2 \cdot \nabla\psi_m\psi_m' dy \le C_1(||\psi_m'||_2^2 + ||\nabla\psi_m||_2^2).$$

Replacing above calculations in (3.4) we have

$$\begin{aligned}&\frac{d}{dt}\left[\mathcal{E}_{1m}(t) - \frac{1}{2}(\gamma'\gamma^{-1})^2||\nabla\varphi_m \cdot y||_2^2 - \frac{1}{2}(\gamma'\gamma^{-1})^2||\nabla\psi_m \cdot y||_2^2\right] \\ &+ \delta(||\varphi_m'||_2^2 + ||\psi_m'||_2^2) \le C_2\mathcal{E}_{1m}(t).\end{aligned} \tag{3.5}$$

Integrating (3.5) over $(0,t)$ with $t \in (0, t_m)$ we have

$$\mathcal{E}_{1m}(t) - \frac{1}{2}(\gamma'\gamma^{-1})^2||\nabla\varphi_m \cdot y||_2^2 - \frac{1}{2}(\gamma'\gamma^{-1})^2||\nabla\psi_m \cdot y||_2^2 \tag{3.6}$$

$$+\delta\int_0^t(||\varphi'(s)_m||_2^2+||\psi'(s)_m||_2^2)ds\le\mathcal{E}_{1m}(0)+C_3\int_0^t\mathcal{E}_{1m}(s)ds.$$

From the fact $a\ge 1$ and (2.12), we have

$$\frac{1}{2}(\gamma'\gamma^{-1})^2||\nabla\varphi_m\cdot y||_2^2\le\frac{1}{2}\gamma^{-2}(\gamma')^2||\nabla\varphi_m||_2^2||y||_2^2\le\frac{a}{4}\gamma^{-2}||\nabla\varphi_m||_2^2. \tag{3.7}$$

Similarly,

$$\frac{1}{2}(\gamma'\gamma^{-1})^2||\nabla\psi_m\cdot y||_2^2\le\frac{a}{4}\gamma^{-2}||\nabla\psi_m||_2^2. \tag{3.8}$$

Hence, by the definition of $\mathcal{E}_{1m}(t)$, we have

$$\frac{1}{2}(\gamma'\gamma^{-1})^2||\nabla\varphi_m\cdot y||_2^2+\frac{1}{2}(\gamma'\gamma^{-1})^2||\nabla\psi_m\cdot y||_2^2\le 2\mathcal{E}_{1m}(t).$$

Therefore, we can rewrite (3.6) as

$$\mathcal{E}_{1m}(t)+2\delta\int_0^t(||\varphi'(s)_m||_2^2+||\psi'(s)_m||_2^2)ds\le 2C_3\int_0^t\mathcal{E}_{1m}(s)ds+2\mathcal{E}_{1m}(0).$$

Therefore, by Gronwall's lemma we get

$$\mathcal{E}_{1m}(t)+\int_0^t(||\varphi'(s)_m||_2^2+||\psi'(s)_m||_2^2)ds\le C_4, \tag{3.9}$$

where C_4 is a positive constant which is independent m and t.

3.1.2. *The second estimate.* Replacing $w=-\Delta\varphi'_m(t)$ and $w=-\Delta\psi'_m(t)$ in (3.1) and (3.2) we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[||\nabla\varphi'_m||_2^2+||\nabla\psi'_m||_2^2\\
&\quad+(a\gamma^{-2}+b\gamma^{n-4}(||\nabla\varphi_m||_2^2+||\nabla\psi_m||_2^2))(||\Delta\varphi_m||_2^2+||\Delta\psi_m||_2^2)\Big]\\
&\quad+\delta(||\nabla\varphi'_m||_2^2+||\nabla\psi'_m||_2^2)\\
=&-a\gamma^{-3}\gamma'(||\Delta\varphi_m||_2^2+||\Delta\psi_m||_2^2)\\
&+\frac{b(n-4)}{2}\gamma^{n-5}\gamma'(||\nabla\varphi_m||_2^2||\Delta\varphi_m||_2^2+||\nabla\psi_m||_2^2||\Delta\varphi_m||_2^2\\
&+||\nabla\varphi_m||_2^2||\Delta\psi_m||_2^2+||\nabla\psi_m||_2^2||\Delta\psi_m||_2^2)\\
&+b\gamma^{n-4}\big(||\Delta\varphi_m||_2^2+||\Delta\psi_m||_2^2\big)\big((\nabla\varphi_m,\nabla\varphi'_m)+(\nabla\psi_m,\nabla\psi'_m)\big)\\
&+\int_\Omega f(\varphi_m)\Delta\varphi'_m dy+\int_\Omega h(\psi_m)\Delta\psi'_m dy+\int_\Omega A(t)\varphi_m\Delta\varphi'_m dy\\
&+\int_\Omega A(t)\psi_m\Delta\psi'_m dy+\int_\Omega\alpha_1\cdot\nabla\varphi'_m\Delta\varphi'_m dy+\int_\Omega\alpha_1\cdot\nabla\psi'_m\Delta\psi'_m dy\\
&+\int_\Omega\alpha_2\cdot\nabla\varphi_m\Delta\varphi'_m dy+\int_\Omega\alpha_2\cdot\nabla\psi_m\Delta\psi'_m dy.
\end{aligned} \tag{3.10}$$

From the Hölder inequality and (3.9), we can easily check that

$$
\begin{aligned}
(3.11)\qquad & b\gamma^{n-4}\big(||\Delta\varphi_m||_2^2+||\Delta\psi_m||_2^2\big)\big((\nabla\varphi_m,\nabla\varphi_m')+(\nabla\psi_m,\nabla\psi_m')\big)\\
&\le bC_4^{\frac{1}{2}}\gamma^{n-4}(||\Delta\varphi_m||_2^2+||\Delta\psi_m||_2^2)(||\nabla\varphi_m'||_2+||\nabla\psi_m'||_2).
\end{aligned}
$$

Observing that $\frac{p}{2(p+1)}+\frac{1}{2(p+1)}+\frac{1}{2}=1$, then from (2.14), (3.9), Hölder inequality and Sobolev imbedding theorem, it holds that

$$
\begin{aligned}
\int_\Omega f(\varphi_m)\Delta\varphi_m' dy &= -\int_\Omega f'(\varphi_m)\nabla\varphi_m\nabla\varphi_m' dy\\
&\le \int_\Omega |\varphi_m|^p\ |\nabla\varphi_m|\ |\nabla\varphi'|dy\\
&\le ||\varphi_m||_{2(p+1)}^p||\nabla\varphi_m||_{2(p+1)}||\nabla\varphi_m'||_2\\
&\le \frac{1}{2}\tilde{c}^{2(p+1)}C_4^p||\Delta\varphi_m||_2^2+\frac{1}{2}||\nabla\varphi_m'||_2^2,
\end{aligned}\tag{3.12}
$$

where $\tilde{c}$ is a $H_0^1(\Omega)\hookrightarrow L^{2(p+1)}(\Omega)$ imbedding constant. Similarly,

$$
\int_\Omega h(\psi_m)\Delta\psi_m' dy \le \frac{1}{2}\hat{c}^{2(q+1)}C_4^q||\Delta\psi_m||_2^2+\frac{1}{2}||\nabla\psi_m'||_2^2,\tag{3.13}
$$

where $\hat{c}$ is a $H_0^1(\Omega)\hookrightarrow L^{2(q+1)}(\Omega)$ imbedding constant.

Now, we will analyze the term $\int_\Omega A(t)\varphi_m\Delta\varphi_m' dy$ and $\int_\Omega A(t)\psi_m\Delta\psi_m' dy$. From the definition of $A(t)v$ and the first estimate, we get

$$
\begin{aligned}
&\int_\Omega A(t)\varphi_m\Delta\varphi_m' dy\\
&= \int_\Omega \sum_{i,j=1}^n \partial_{y_i}((\gamma'\gamma^{-1})^2 y_i y_j\partial_{y_j}\varphi_m)\Delta\varphi_m' dy\\
&= (\gamma'\gamma^{-1})^2\int_\Omega \sum_{i,j=1}^n \Big[y_j\partial_{y_j}\varphi_m+y_i\partial_{y_i}y_j\partial_{y_j}\varphi_m+y_iy_j\partial_{y_i}\partial_{y_j}\varphi_m\Big]\Delta\varphi_m' dy\\
&\le (\gamma'\gamma^{-1})^2\int_\Omega |y|^2|\Delta\varphi_m\Delta\varphi_m'|dy-(\gamma'\gamma^{-1})^2\int_\Omega \nabla\Big(\sum_{\substack{i,j=1\\ i\ne j}}^n y_iy_j\partial_{y_i}\partial_{y_j}\varphi_m\Big)\cdot\nabla\varphi_m' dy\\
&\quad -(\gamma'\gamma^{-1})^2\int_\Omega \nabla\Big(\sum_{i,j=1}^n (y_i\partial_{y_j}\varphi_m+y_i\partial_{y_i}y_j\partial_{y_j}\varphi_m)\Big)\cdot\nabla\varphi_m' dy\\
&\le \frac{1}{2}(\gamma'\gamma^{-1})^2||y||_2^2\int_\Omega\left|\frac{d}{dt}|\Delta\varphi_m|^2\right|dy+C_5||\Delta\varphi_m||_2^2+C_6||\nabla\varphi_m'||_2^2.
\end{aligned}
$$

We now will consider the first term of the right-hand side of above inequality dividing two cases.

Case 1 : $\frac{d}{dt}|\Delta\varphi_m|^2>0$.

We can rewrite above inequality as

$$\begin{aligned}
&\int_\Omega A(t)\varphi_m \Delta\varphi_m' dy \\
\le\ & \frac{1}{2}(\gamma'\gamma^{-1})^2 ||y||_2^2 \frac{d}{dt}||\Delta\varphi_m||_2^2 + C_5||\Delta\varphi_m||_2^2 + C_6||\nabla\varphi_m'||_2^2 \\
=\ & \frac{1}{2}\frac{d}{dt}\left[(\gamma'\gamma^{-1})^2||y||_2^2||\Delta\varphi_m||_2^2\right] \\
&- (\gamma'\gamma^{-1})(\gamma''\gamma^{-1} - (\gamma')^2\gamma^{-2})||y||_2^2||\Delta\varphi_m||_2^2 + C_5||\Delta\varphi_m||_2^2 + C_6||\nabla\varphi_m'||_2^2 \\
\le\ & \frac{1}{2}\frac{d}{dt}\left[(\gamma'\gamma^{-1})^2||y||_2^2||\Delta\varphi_m||_2^2\right] + C_7||\Delta\varphi_m||_2^2 + C_6||\nabla\varphi_m'||_2^2.
\end{aligned}$$

Case 2 : $\frac{d}{dt}|\Delta\varphi_m|^2 \le 0$.

Similarly above computation, we have

$$\begin{aligned}
&\int_\Omega A(t)\varphi_m \Delta\varphi_m' dy \\
\le\ & -\frac{1}{2}(\gamma'\gamma^{-1})^2||y||_2^2\frac{d}{dt}||\Delta\varphi_m||_2^2 + C_5||\Delta\varphi_m||_2^2 + C_6||\nabla\varphi_m'||_2^2 \\
\le\ & -\frac{1}{2}\frac{d}{dt}\left[(\gamma'\gamma^{-1})^2||y||_2^2||\Delta\varphi_m||_2^2\right] + C_7||\Delta\varphi_m||_2^2 + C_6||\nabla\varphi_m'||_2^2.
\end{aligned}$$

Therefore,

(3.14)

$$\begin{aligned}
&\int_\Omega A(t)\varphi_m\Delta\varphi_m' dy \\
\le &\begin{cases} \frac{1}{2}\frac{d}{dt}\left[(\gamma'\gamma^{-1})^2||y||_2^2||\Delta\varphi_m||_2^2\right] + C_7||\Delta\varphi_m||_2^2 + C_6||\nabla\varphi_m'||_2^2 & \text{if } \frac{d}{dt}|\Delta\varphi_m|^2 > 0, \\ -\frac{1}{2}\frac{d}{dt}\left[(\gamma'\gamma^{-1})^2||y||_2^2||\Delta\varphi_m||_2^2\right] + C_7||\Delta\varphi_m||_2^2 + C_6||\nabla\varphi_m'||_2^2 & \text{if } \frac{d}{dt}|\Delta\varphi_m|^2 \le 0. \end{cases}
\end{aligned}$$

Similarly,

(3.15)

$$\begin{aligned}
&\int_\Omega A(t)\psi_m\Delta\psi_m' dy \\
\le &\begin{cases} \frac{1}{2}\frac{d}{dt}\left[(\gamma'\gamma^{-1})^2||y||_2^2||\Delta\psi_m||_2^2\right] + C_8||\Delta\psi_m||_2^2 + C_9||\nabla\psi_m'||_2^2 & \text{if } \frac{d}{dt}|\Delta\psi_m|^2 > 0, \\ -\frac{1}{2}\frac{d}{dt}\left[(\gamma'\gamma^{-1})^2||y||_2^2||\Delta\psi_m||_2^2\right] + C_8||\Delta\psi_m||_2^2 + C_9||\nabla\psi_m'||_2^2 & \text{if } \frac{d}{dt}|\Delta\psi_m|^2 \le 0. \end{cases}
\end{aligned}$$

Using similar arguments as the first estimate we obtain

(3.16)

$$\int_\Omega \alpha_1\cdot\nabla\varphi_m'\Delta\varphi_m' dy = n\gamma'\gamma^{-1}||\nabla\varphi_m'||_2^2, \quad \int_\Omega \alpha_1\cdot\nabla\psi_m'\Delta\psi_m' dy = n\gamma'\gamma^{-1}||\nabla\psi_m'||_2^2,$$

$$\int_\Omega \alpha_2\cdot\nabla\varphi_m\Delta\varphi_m' dy \le C_{10}(||\Delta\varphi_m||_2^2 + ||\nabla\varphi_m'||_2^2 + ||\nabla\varphi_m||_2^2) \tag{3.17}$$

and

$$\int_\Omega \alpha_2 \cdot \nabla\psi_m \Delta\psi'_m dy \le C_{11}(||\Delta\psi_m||_2^2 + ||\nabla\psi'_m||_2^2 + ||\nabla\psi_m||_2^2). \tag{3.18}$$

Replacing (3.11)-(3.18) in (3.10), and then integrating over $(0,t)$ with $t \in (0,t_m)$ we have

$$\begin{aligned} &\mathcal{E}_{2m}(t) + \Phi_m(t) + \Psi_m(t) + \delta\int_0^t (||\nabla\varphi'(s)_m||_2^2 + ||\nabla\psi'(s)_m||_2^2)ds \\ &\le C_{12}\int_0^t \mathcal{E}_{2m}(s)ds + C_{13}, \end{aligned} \tag{3.19}$$

where

$$\Phi_m(t) = \begin{cases} -\frac{1}{2}(\gamma'\gamma^{-1})^2||y||_2^2||\Delta\varphi_m||_2^2 & \text{if} \quad \frac{d}{dt}|\Delta\varphi_m|^2 > 0 \\ \frac{1}{2}(\gamma'\gamma^{-1})^2||y||_2^2||\Delta\varphi_m||_2^2 & \text{if} \quad \frac{d}{dt}|\Delta\varphi_m|^2 \le 0, \end{cases}$$

$$\Psi_m(t) = \begin{cases} -\frac{1}{2}(\gamma'\gamma^{-1})^2||y||_2^2||\Delta\psi_m||_2^2 & \text{if} \quad \frac{d}{dt}|\Delta\psi_m|^2 > 0 \\ \frac{1}{2}(\gamma'\gamma^{-1})^2||y||_2^2||\Delta\psi_m||_2^2 & \text{if} \quad \frac{d}{dt}|\Delta\psi_m|^2 \le 0 \end{cases}$$

and

$$\begin{aligned} \mathcal{E}_{2m}(t) = &\frac{1}{2}||\nabla\varphi'_m||_2^2 + \frac{1}{2}||\nabla\psi'_m||_2^2 \\ &+ \frac{1}{2}(a\gamma^{-2} + b\gamma^{n-4}(||\nabla\varphi_m||_2^2 + ||\nabla\psi_m||_2^2))(||\Delta\varphi_m||_2^2 + ||\Delta\psi_m||_2^2). \end{aligned}$$

By the same arguments as (3.7) and (3.8), we have

$$\frac{a}{4}\gamma^{-2}||\Delta\varphi_m||_2^2 \ge \frac{1}{2}(\gamma'\gamma^{-1})^2||y||_2^2||\Delta\varphi_m||_2^2$$

and $\frac{a}{4}\gamma^{-2}||\Delta\psi_m||_2^2 \ge \frac{1}{2}(\gamma'\gamma^{-1})^2||y||_2^2||\Delta\psi_m||_2^2$ for all $t \ge 0$, therefore, by Gronwall's lemma we obtain

$$\mathcal{E}_{2m}(t) + \int_0^t (||\nabla\varphi'(s)_m||_2^2 + ||\nabla\psi'(s)_m||_2^2)ds \le C_{14}, \tag{3.20}$$

where C_{14} is a positive constant which is independent m and t.

3.1.3. *The third estimate.* First of all we are going to estimate $\varphi''_m(0)$ and $\psi''_m(0)$. Taking $t = 0$ and considering $w = \varphi''_m(0)$ and $w = \psi''_m(0)$ in (3.1) and (3.2), we get

$$||\varphi''_m(0)||_2^2 + ||\psi''_m(0)||_2^2 \le C_{15}. \tag{3.21}$$

Now differentiating (3.1) and (3.2) with respect to t and substituting $w = \varphi''_m(t)$ in (3.1) and $w = \psi''_m(t)$ in (3.2) and then adding the results, we have

(3.22)
$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[||\varphi''_m||_2^2 + ||\psi''_m||_2^2 + \ell_m(t)||\nabla\varphi'_m||_2^2 + \ell_m(t)||\nabla\psi'_m||_2^2\Big] \\
&\quad + \delta||\varphi''_m||_2^2 + \delta||\psi''_m||_2^2 \\
&= \frac{1}{2}\ell'_m(t)||\nabla\varphi'_m||_2^2 + \frac{1}{2}\ell'_m(t)||\nabla\psi'_m||_2^2 + \int_\Omega \ell'_m(t)\Delta\varphi_m\varphi''_m dy \\
&\quad + \int_\Omega \ell'_m(t)\Delta\psi_m\psi''_m dy - \int_\Omega f'(\varphi_m)\varphi'_m\varphi''_m dy \\
&\quad - \int_\Omega h'(\psi_m)\psi'_m\psi''_m dy - \int_\Omega \frac{d}{dt}\Big[A(t)\varphi_m\Big]\varphi''_m dy - \int_\Omega \frac{d}{dt}\Big[A(t)\psi_m\Big]\psi''_m dy \\
&\quad - \int_\Omega \frac{d}{dt}\Big[\alpha_1\cdot\nabla\varphi'_m\Big]\varphi''_m dy - \int_\Omega \frac{d}{dt}\Big[\alpha_1\cdot\nabla\psi'_m\Big]\psi''_m dy \\
&\quad - \int_\Omega \frac{d}{dt}\Big[\alpha_2\cdot\nabla\varphi_m\Big]\varphi''_m dy - \int_\Omega \frac{d}{dt}\Big[\alpha_2\cdot\nabla\psi_m\Big]\psi''_m dy,
\end{aligned}$$

where $\ell_m(t) = a\gamma^{-2} + b\gamma^{n-4}(||\nabla\varphi_m||_2^2 + ||\nabla\psi_m||_2^2)$.

From the first and second estimate, $\ell_m(t)$ is bounded. And using similar arguments as the first estimate and the second estimate we obtain

$$\begin{aligned}
&||\varphi''_m||_2^2 + ||\psi''_m||_2^2 + \ell_m(t)||\nabla\varphi'_m||_2^2 + \ell_m(t)||\nabla\psi'_m||_2^2 + \int_0^t (||\varphi''_m(s)||_2^2 \\
&\quad + ||\psi''_m(s)||_2^2)ds \le C_{16},
\end{aligned} \tag{3.23}$$

where C_{16} is a positive constant which is independent m and t.

According to (3.9), (3.20) and (3.23), we get

$$\{\varphi_m\},\ \{\psi_m\} \text{ are bounded in } L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \tag{3.24}$$

$$\{\varphi'_m\},\ \{\psi'_m\} \text{ are bounded in } L^\infty(0,T;H_0^1(\Omega)), \tag{3.25}$$

$$\{\varphi''_m\},\ \{\psi''_m\} \text{ are bounded in } L^\infty(0,T;L^2(\Omega)). \tag{3.26}$$

From (3.24) to (3.26), there exist subsequences of $\{\varphi_m\}$ and $\{\psi_m\}$, which we still denote by $\{\varphi_m\}$ and $\{\psi_m\}$, such that

$$\varphi_m \to \varphi,\ \psi_m \to \psi \text{ weak star in } L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \tag{3.27}$$

$$\varphi'_m \to \varphi',\ \psi'_m \to \psi' \text{ weak star in } L^\infty(0,T;H_0^1(\Omega)), \tag{3.28}$$

$$\varphi''_m \to \varphi'',\ \psi''_m \to \psi'' \text{ weak star in } L^\infty(0,T;L^2(\Omega)). \tag{3.29}$$

Using Aubin-Lion's compactness lemma (cf. [7]), we have

$$\varphi_m \to \varphi,\ \psi_m \to \psi \text{ strongly in } L^2(0,T;H_0^1(\Omega)), \tag{3.30}$$

$$\varphi'_m \to \varphi',\ \psi'_m \to \psi' \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{3.31}$$

Now, (3.30) implies

$$\begin{aligned}&a\gamma^{-2}+b\gamma^{n-4}(||\nabla\varphi_m||_2^2+||\nabla\psi_m||_2^2)\\ &\to a\gamma^{-2}+b\gamma^{n-4}(||\nabla\varphi||_2^2+||\nabla\psi||_2^2) \text{ in } L^2(0,T).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\Big(a\gamma^{-2}+b\gamma^{n-4}(||\nabla\varphi_m||_2^2+||\nabla\psi_m||_2^2)\Big)\varphi_m\\ &\to \Big(a\gamma^{-2}+b\gamma^{n-4}(||\nabla\varphi||_2^2+||\nabla\psi||_2^2)\Big)\varphi \text{ in } L^2(0,T;H_0^1(\Omega))\end{aligned}$$

and

$$\begin{aligned}&\Big(a\gamma^{-2}+b\gamma^{n-4}(||\nabla\varphi_m||_2^2+||\nabla\psi_m||_2^2)\Big)\psi_m\\ &\to \Big(a\gamma^{-2}+b\gamma^{n-4}(||\nabla\varphi||_2^2+||\nabla\psi||_2^2)\Big)\psi \text{ in } L^2(0,T;H_0^1(\Omega)).\end{aligned}$$

On the other hand, hypotheses on f and h, we have

$$f(\varphi_m)\to f(\varphi),\ h(\psi_m)\to h(\psi) \text{ a.e. in } Q.$$

Thus, we can pass to the limit in (3.1) and (3.2) to obtain

$$\begin{aligned}&\varphi''-\gamma^{-2}(a+b\gamma^{n-2}||\nabla\varphi||_2^2+b\gamma^{n-2}||\nabla\psi||_2^2)\Delta\varphi+\delta\varphi'\\ &+f(\varphi)+A(t)\varphi+\alpha_1\cdot\nabla\varphi'+\alpha_2\cdot\nabla\varphi=0 \text{ in } \mathcal{D}'(0,\infty;L^2(\Omega)),\\ &\psi''-\gamma^{-2}(a+b\gamma^{n-2}||\nabla\varphi||_2^2+b\gamma^{n-2}||\nabla\psi||_2^2)\Delta\psi+\delta\psi'\\ &+h(\psi)+A(t)\psi+\alpha_1\cdot\nabla\psi'+\alpha_2\cdot\nabla\psi=0 \text{ in } \mathcal{D}'(0,\infty;L^2(\Omega)).\end{aligned}$$

Since φ', ψ', φ'' and $\psi''\in L^2_{loc}(0,\infty;L^2(\Omega))$,

$$\begin{aligned}&\varphi''-\gamma^{-2}(a+b\gamma^{n-2}||\nabla\varphi||_2^2+b\gamma^{n-2}||\nabla\psi||_2^2)\Delta\varphi+\delta\varphi'\\ &+f(\varphi)+A(t)\varphi+\alpha_1\cdot\nabla\varphi'+\alpha_2\cdot\nabla\varphi=0 \text{ in } L^2_{loc}(0,\infty;L^2(\Omega)),\\ &\psi''-\gamma^{-2}(a+b\gamma^{n-2}||\nabla\varphi||_2^2+b\gamma^{n-2}||\nabla\psi||_2^2)\Delta\psi+\delta\psi'\\ &+h(\psi)+A(t)\psi+\alpha_1\cdot\nabla\psi'+\alpha_2\cdot\nabla\psi=0 \text{ in } L^2_{loc}(0,\infty;L^2(\Omega)).\end{aligned}$$

3.1.4. *Uniqueness.* Let (φ,ψ) and $(\hat\varphi,\hat\psi)$ be two solutions of (2.4)-(2.8) satisfying Theorem 2.1. Define $(\phi,\theta)=(\varphi-\hat\varphi,\psi-\hat\psi)$, we have

$$\begin{aligned}&\phi''-\gamma^{-2}(a+b\gamma^{n-2}||\nabla\varphi||_2^2+b\gamma^{n-2}||\nabla\psi||_2^2)\Delta\phi\\ &-\Big(b\gamma^{n-4}(||\nabla\varphi||_2^2+||\nabla\psi||_2^2)-b\gamma^{n-4}(||\nabla\hat\varphi||_2^2+||\nabla\hat\psi||_2^2)\Big)\Delta\hat\varphi\\ &+\delta\phi'+f(\varphi)-f(\hat\varphi)\\ &+A(t)\phi+\alpha_1\cdot\nabla\phi'+\alpha_2\cdot\nabla\phi=0\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}&\theta'' - \gamma^{-2}(a + b\gamma^{n-2}||\nabla\varphi||_2^2 + b\gamma^{n-2}||\nabla\psi||_2^2)\Delta\theta \\ &- \Big(b\gamma^{n-4}(||\nabla\varphi||_2^2 + ||\nabla\psi||_2^2) - b\gamma^{n-4}(||\nabla\hat{\varphi}||_2^2 + ||\nabla\hat{\psi}||_2^2)\Big)\Delta\hat{\psi} \\ &+ \delta\theta' + h(\psi) - h(\hat{\psi}) \\ &+ A(t)\theta + \alpha_1 \cdot \nabla\theta' + \alpha_2 \cdot \nabla\theta = 0.\end{aligned} \tag{3.33}$$

Multiplying (3.32) and (3.33) by ϕ' and θ', respectively, summing up the product result we obtain

(3.34)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[||\phi'||_2^2 + ||\theta'||_2^2 + \ell(t)||\nabla\phi||_2^2 + \ell(t)||\nabla\theta||_2^2\Big] + \delta(||\phi'||_2^2 + ||\theta'||_2^2) \\ &= \frac{1}{2}\ell'(t)||\nabla\phi||_2^2 + \frac{1}{2}\ell'(t)||\nabla\theta||_2^2 \\ &\quad + \Big[b\gamma^{n-4}(||\nabla\varphi||_2^2 + ||\nabla\psi||_2^2 - ||\nabla\hat{\varphi}||_2^2 - ||\nabla\hat{\psi}||_2^2)\Big]\ \Big[(\Delta\hat{\varphi}, \phi') + (\Delta\hat{\psi}, \theta')\Big] \\ &\quad - (f(\varphi) - f(\hat{\varphi}), \phi') - (h(\psi) - h(\hat{\psi}), \theta') - (A(t)\phi, \phi') - (A(t)\theta, \theta') \\ &\quad - (\alpha_1 \cdot \nabla\phi', \phi') - (\alpha_1 \cdot \nabla\theta', \theta') - (\alpha_2 \cdot \nabla\phi, \phi') - (\alpha_2 \cdot \nabla\theta, \theta'),\end{aligned}$$

where $\ell(t) = a\gamma^{-2} + b\gamma^{n-4}||\nabla\varphi||_2^2 + b\gamma^{n-4}||\nabla\psi||_2^2$.

Note that $\Big|(f(\varphi) - f(\hat{\varphi}), \phi')\Big| \le C_{17}||\phi||_{2(p+1)}||\phi'||_2$ and $\Big|(h(\psi) - h(\hat{\psi}), \theta')\Big| \le C_{18}||\theta||_{2(q+1)}||\theta'||_2$. And using the similar arguments as the first and second estimate and integrating (3.34) from 0 to t, we obtain

$$\begin{aligned}&||\phi'||_2^2 + ||\theta'||_2^2 + ||\nabla\phi||_2^2 + ||\nabla\theta||_2^2 \\ &\le C_{19}\int_0^t (||\phi'(s)||_2^2 + ||\theta'(s)||_2^2 + ||\nabla\phi(s)||_2^2 + ||\nabla\theta(s)||_2^2)ds,\end{aligned}$$

which implies $||\phi'||_2^2 = ||\theta'||_2^2 = ||\nabla\phi||_2^2 = ||\nabla\theta||_2^2 = 0$ by using Gronwall's lemma. This completes the proof of uniqueness.

3.2. Proof of Theorem 2.2

To show the existence in noncylindrical domain, we return to our original problem in the noncylindrical domain by using the change of variable given in (2.1) by $(y, t) = \tau(x, t)$, $(x, t) \in \hat{Q}$. Let (φ, ψ) be the solution obtained from Theorem 2.1 and (u, v) defined by (2.3), then (u, v) belongs to the class

$$\begin{aligned}&u, v \in L^\infty(0, \infty; H_0^1(\Omega_t)), \\ &u', v' \in L^\infty(0, \infty; H_0^1(\Omega_t)), \\ &u'', v'' \in L^\infty(0, \infty; L^2(\Omega_t)).\end{aligned}$$

Denoting by $u(x, t) = \varphi(y, t) = (\varphi \circ \tau)(x, t)$ and $v(x, t) = \psi(y, t) = (\psi \circ \tau)(x, t)$ then from (2.2) it is easy to see that (u, v) satisfies equations (1.1)-(1.5) in the sense of $L^\infty(0, \infty; L^2(\Omega_t))$. Let (u_1, v_1) and (u_2, v_2) be two solutions to

(1.1)-(1.5) and (φ_1, ψ_1) and (φ_2, ψ_2) be the functions obtained through the diffeomorphism τ given by (2.1). Then (φ_1, ψ_1) and (φ_2, ψ_2) are solutions to (2.4)-(2.8). By the uniqueness result of Theorem 2.1, we have $(\varphi_1, \psi_1) = (\varphi_2, \psi_2)$, so $(u_1, v_1) = (u_2, v_2)$. Thus the proof of Theorem 2.2 is completed.

4. Uniform decay

In this section we will prove Theorem 2.3. First of all, we introduce the useful lemma in noncylindrical domain.

Lemma 4.1 (cf. [5])**.** *Let $G(\cdot,\cdot)$ be the smooth function defined in $\Omega_t \times [0,\infty[$ $(t \in [0,\infty[)$. Then*

$$\frac{d}{dt}\int_{\Omega_t} G(x,t)dx = \int_{\Omega_t} \frac{d}{dt}G(x,t)dx + \frac{\gamma'}{\gamma}\int_{\Gamma_t} G(x,t)(x\cdot\bar{\nu})d\Gamma_t, \tag{4.1}$$

where $\bar{\nu}$ is the outward unit normal vector with respect to Γ_t, that is, the x-component of the unit normal exterior ν.

Proof. By a change variable $x = \gamma(t)y$, $y \in \Omega$, we have

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} G(x,t)dx &= \frac{d}{dt}\int_{\Omega} G(\gamma(t)y,t)\gamma^n(t)dy \\ &= \int_\Omega \frac{\partial G}{\partial t}\gamma^n(t)dy + \frac{\gamma'}{\gamma}\int_\Omega \sum_{i=1}^n x_i \frac{\partial G}{\partial x_i}\gamma^n(t)dy \\ &\quad + n\int_\Omega \gamma'(t)\gamma^{n-1}(t)G(\gamma(t)y,t)dy.\end{aligned}$$

If we return to the variable x, we get

$$\frac{d}{dt}\int_{\Omega_t} G(x,t)dx = \int_{\Omega_t}\frac{\partial G}{\partial t}dx + \frac{\gamma'}{\gamma}\int_{\Omega_t} x\cdot\nabla G(x,t)dx + n\frac{\gamma'}{\gamma}\int_{\Omega_t} G(x,t)dx.$$

Integrating by part in the last equality, we obtain (4.1). □

Now, we define the energy of the problem (1.1)-(1.5) by

$$\begin{aligned}E(t) = {}& \frac{1}{2}||u'||_{2,t}^2 + \frac{1}{2}||v'||_{2,t}^2 + \frac{a}{2}||\nabla u||_{2,t}^2 + \frac{a}{2}||\nabla v||_{2,t}^2 + \frac{b}{4}(||\nabla u||_{2,t}^2 \\ & + ||\nabla v||_{2,t}^2)^2 + \int_{\Omega_t}(F(u)+H(v))dx.\end{aligned} \tag{4.2}$$

We observe that $E(t)$ is a positive function. Using Lemma 4.1, we have

$$\begin{aligned}2E'(t) = {}& -2\delta(||u'||_{2,t}^2 + ||v'||_{2,t}^2) \\ & + \frac{\gamma'}{\gamma}\int_{\Gamma_t}(|u'|^2 + |v'|^2 + a|\nabla u|^2 + a|\nabla v|^2)(x\cdot\bar{\nu})d\Gamma_t \\ & + \frac{b\gamma'}{\gamma}(||\nabla u||_{2,t}^2 + ||\nabla v||_{2,t}^2)\int_{\Gamma_t}(|\nabla u|^2 + |\nabla v|^2)(x\cdot\bar{\nu})d\Gamma_t\end{aligned} \tag{4.3}$$

$$+\frac{2\gamma'}{\gamma}\int_{\Gamma_t}(F(u)+H(v))(x\cdot\bar{\nu})d\Gamma_t.$$

From (2.9) and (4.3), $E(t)$ is nonincreasing function for the case $n>4$ but for the case $n\leq 4$, $E(t)$ is not necessary decreasing function. The main difficulty to obtain the decay for the case $n\leq 4$ is due to the influence of the geometry of the noncylindrical domain and it is an open problem. So, we are going to show the exponential decay for the case $n>4$.

Let us consider the following functional

$$\rho(t)=2\int_{\Omega_t}(u'u+v'v)dx+\delta(||u||_{2,t}^2+||v||_{2,t}^2). \tag{4.4}$$

Multiplying (1.1) and (1.2) by u and v, and then integrating over Ω_t we obtain

$$\begin{aligned}\frac{1}{2}\rho'(t)=&\ ||u'||_{2,t}^2+||v'||_{2,t}^2-a(||\nabla u||_{2,t}^2+||\nabla v||_{2,t}^2)-b(||\nabla u||_{2,t}^2+||\nabla v||_{2,t}^2))^2\\&-\int_{\Omega_t}f(u)udx-\int_{\Omega_t}h(v)vdx+\frac{\gamma'}{\gamma}\int_{\Gamma_t}(u'u+v'v)(x\cdot\bar{\nu})d\Gamma_t\\&+\frac{\delta\gamma'}{2\gamma}\int_{\Gamma_t}(|u|+|v|)(x\cdot\bar{\nu})d\Gamma_t.\end{aligned}$$

From (2.13) and the fact $\gamma'\leq 0$, we have

$$\begin{aligned}\rho'(t)\leq&\ 2||u'||_{2,t}^2+2||v'||_{2,t}^2-2a(||\nabla u||_{2,t}^2+||\nabla v||_{2,t}^2)\\&-2b(||\nabla u||_{2,t}^2+||\nabla v||_{2,t}^2))^2\\&-2(2+\beta)\int_{\Omega_t}(F(u)+H(v))dx-\frac{1}{\delta}\frac{\gamma'}{\gamma}\int_{\Gamma_t}(|u'|^2+|v'|^2)(x\cdot\bar{\nu})d\Gamma_t,\end{aligned} \tag{4.5}$$

where $\beta=\min\{\beta_1,\beta_2\}$.

Now, let us introduce the functional

$$\Xi(t)=NE(t)+\rho(t), \tag{4.6}$$

with N is a positive constant. It is not difficult to see that $\Xi(t)$ verifies

$$k_0E(t)\leq\Xi(t)\leq k_1E(t) \tag{4.7}$$

for k_0 and k_1 are positive constants.

Differentiating (4.6) with respect to the time t, from (4.3) and (4.5) we get

$$\begin{aligned}\Xi'(t)=&\ NE'(t)+\rho'(t)\\\leq&\ -(N\delta-2)(||u'||_{2,t}^2+||v'||_{2,t}^2)-2a(||\nabla u||_{2,t}^2+||\nabla v||_{2,t}^2)\\&-2b(||\nabla u||_{2,t}^2+||\nabla v||_{2,t}^2)^2\\&-2(2+\beta)\int_{\Omega_t}(F(u)+H(v))dx\\&+\Big(\frac{N}{2}-\frac{1}{\delta}\Big)\frac{\gamma'}{\gamma}\int_{\Gamma_t}(|u'|^2+|v'|^2)(x\cdot\bar{\nu})d\Gamma_t.\end{aligned}$$

Choosing $N \geq \frac{2}{\delta}$ we obtain

$$\Xi'(t) \leq -\lambda E(t),$$

where λ is a positive constant independent of t. Therefore, from (4.7) we have

$$\Xi(t) \leq \Xi(0)e^{-\frac{\lambda}{k_1}t}$$

for all t. From equivalence relation (4.7), the proof of Theorem 2.3 is completed.

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Tae Gab Ha
Department of Mathematics
Institute of Pure and Applied Mathematics
Chonbuk National University
Jeonju 54896, Korea
E-mail address: tgha@jbnu.ac.kr