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DIFFERENTIALS OF THE BICOMPLEX FUNCTIONS FOR EACH CONJUGATIONS BY THE NAIVE APPROACH

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Abstract. In this paper, we aim to compare the differentials with the regularity of the hypercomplex valued functions in Clifford analysis. For three kinds of conjugation of the bicomplex numbers, we define the differentials of the bicomplex number functions by the naive approach. And we investigate some relations of the corresponding Cauchy-Riemann system and the conditions of the differentiable functions in the bicomplex number system.

1. Introduction

The bicomplex number system $\mathcal{B}$ is identified with $\mathbb{C}^2$, where $\mathbb{C}$ is denoted by the complex number system. The bicomplex number system $\mathcal{B}$ is a commutative real 4 dimensional (skew) field. And $\mathcal{B}$ is considered by the extension of $\mathbb{C}$.

Many authors have studied the properties of the hypercomplex functions and applied to various Clifford algebra. In 1971, Naser [10] studied the hyperholomorphy of the hypercomplex functions, and provided several properties of the hyperholomorphic functions in the quaternion field. And Naser [10] obtained the corresponding Cauchy-Riemann equation, the corresponding Cauchy theorem, etc.

Recently, Kim et al. [5, 6] have investigated the regularity of the hypercomplex functions valued the ternary number and the reduced quaternion. Jung and Shon [1] have given the properties of regular functions on the dual ternary number system. And Jung et al. [2] have researched the structures of the dual quaternionic regular functions.

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Lim and Shon [7] developed the properties of the split hypercomplex functions, and Kim and Shon [4] investigated the regularities of functions valued dual split quaternion in Clifford analysis.

Luna-Elizarrarás and Shapiro [8, 9] provided a survey on the hypercomplex number system in Clifford analysis. They described the bicomplex number functions and fundamental properties of the hypercomplex functions. Kang et al. [3] studied properties of regular functions for 3 kinds of the conjugation in the bicomplex number system. They provided the corresponding Cauchy-Riemann system and the corresponding Cauchy theorem for each conjugations of the bicomplex functions.

In this papaer, we aim to describe the relations of the differentials and the regularities of the bicomplex functions in Clifford analysis. We show the condition of the differentiable functions and the derivatives of the bicomplex functions. And we find some properties for the bicomplex functions for each conjugation in Clifford analysis.

2. Preliminary

The bicomplex number system $\mathcal{B}$ over the real field $\mathbb{R}$ is denoted by

$$\mathcal{B} = \{z \mid z = z_1 + e_2 z_2,\ z_j \in \mathbb{C}\,(j = 1, 2)\} = \{z \mid z = \sum_{j=0}^{3} e_j x_j,\ x_j \in \mathbb{R}\,(j = 1, 2)\}.$$

By the direct computation, we obtain the notation of the bicomplex number z as follows:

$$\begin{aligned} z &= z_1 + e_2 z_2 \\ &= (x_0 + e_1 x_1) + e_2(x_2 + e_1 x_3) \\ &= x_0 + e_1 x_1 + e_2 x_2 + e_3 x_3, \end{aligned}$$

where $e_3 := e_2 e_1$, z_j $(j = 1, 2)$ are usual complex numbers and x_j $(j = 0, 1, 2, 3)$ are usual real numbers. By putting $e_3 = e_2 e_1$ with $e_3{}^2 = 1$, the basis $\{e_0, e_1, e_2, e_3\}$ satisfies the following properties :

$$e_0 = id.,\ e_1{}^2 = e_2{}^2 = -1,\ e_1 e_2 = e_2 e_1$$

and

$$e_1 e_3 = e_3 e_1 = -e_2,\ e_2 e_3 = e_3 e_2 = -e_1.$$

There are three kinds of conjugation of bicomplex number z. The 1st kind of conjugation z^* with respect to e_2 is determined by

$$z^* := z_1 - e_2 z_2\ (z_1,\ z_2 \in \mathbb{C}).$$

The 2nd kind of conjugation $z^{\#}$ with respect to e_1 is determined by

$$z^{\#} := \overline{z_1} + e_2\overline{z_2} \ (z_1,\ z_2 \in \mathbb{C})$$

and the 3rd kind of conjugation $z^{\dagger}$ is determined by

$$z^{\dagger} := ((z_1 + e_2 z_2)^*)^{\#} = ((z_1 + e_2 z_2)^{\#})^* = \overline{z_1} - e_2\overline{z_2} \ (z_1,\ z_2 \in \mathbb{C})$$

as the composition of the above two conjugations z^* and $z^{\#}$. The modulus of each kind of conjugations is defined by

$$\begin{aligned} |z|_*^2 := zz^* &= (z_1 + e_2 z_2)(z_1 - e_2 z_2) \\ &= z_1^2 + z_2^2, \\ |z|_{\#}^2 := zz^{\#} &= (z_1 + e_2 z_2)(\overline{z_1} + e_2\overline{z_2}) \\ &= z_1\overline{z_1} + e_2(z_1\overline{z_2} + z_2\overline{z_1}) - z_2\overline{z_2} \\ &= |z_1|^2 - |z_2|^2 + 2e_2 Re(z_1\overline{z_2}) \end{aligned}$$

and

$$\begin{aligned} |z|_{\dagger}^2 := zz^{\dagger} &= (z_1 + e_2 z_2)(\overline{z_1} - e_2\overline{z_1}) \\ &= z_1\overline{z_1} - e_2(z_1\overline{z_2} - z_2\overline{z_1}) + z_2\overline{z_2} \\ &= |z_1|^2 + |z_2|^2 - 2e_2 Im(z_1\overline{z_2}),\ \text{respectively.} \end{aligned}$$

The bicomplex number z has the unique inverse value for each kind of conjugation as follows:

$$z_*^{-1} = \frac{z^*}{|z|_*^2},\ z_{\#}^{-1} = \frac{z^{\#}}{|z|_{\#}^2} \quad \text{and} \quad z_{\dagger}^{-1} = \frac{z^{\dagger}}{|z|_{\dagger}^2}$$

for nonzero modulus $|z|_*^2$, $|z|_{\#}^2$ and $|z|_{\dagger}^2$ of z, respectively.

Let Ω be a bounded open set in $\mathcal{B}$. A complex function $f : \Omega \to \mathcal{B}$ is defined by

$$\begin{aligned} f(z) &= f_1 + e_2 f_2 \\ &= (u_0 + e_1 u_1) + e_2(u_2 + e_1 u_3) \\ &= u_0 + e_1 u_1 + e_2 u_2 + e_3 u_3, \end{aligned}$$

where $u_j = u_j(x_0, x_1, x_2, x_3)$ $(j = 0, 1, 2, 3)$ are real valued functions and $f_1,\ f_2$ are complex valued functions of two complex variables z_1 and z_2.

We consider bicomplex differential operators as follows :

$$D := \frac{1}{2}(\frac{\partial}{\partial z_1} - e_2 \frac{\partial}{\partial z_2}),$$
$$D^* = \frac{1}{2}(\frac{\partial}{\partial z_1} + e_2 \frac{\partial}{\partial z_2}),$$
$$D^{\#} = \frac{1}{2}(\frac{\partial}{\partial \overline{z_1}} - e_2 \frac{\partial}{\partial \overline{z_2}}),$$
$$D^{\dagger} = \frac{1}{2}(\frac{\partial}{\partial \overline{z_1}} + e_2 \frac{\partial}{\partial \overline{z_2}}).$$

Definition 2.1. *Let Ω be a bounded open set in $\mathcal{B}$. A function f is said to be the 1st(2nd, 3rd)-regular function in Ω if*
(a) $f \in C^1(\Omega)$,
(b) $D^* f = 0 (D^{\#} f = 0,\ D^{\dagger} f = 0)$ *in* Ω.

3. Differentials of bicomplex number functions

Let Ω be a bounded open set in $\mathcal{B}$, $z \in \Omega$ and let $f : \Omega \to \mathcal{B}$ defined by $f(z) = e_0 u_0 + e_1 u_1 + e_2 u_2 + e_3 u_3$ a bicomplex function. For increment of the argument at the point z, we put a bicomplex number $h \neq 0$ satisfying $z + h \in \Omega$. Then we can consider

$$f(z+h) - f(z) = \sum_{j=0}^{3} e_j u_j(z+h) - \sum_{j=0}^{3} e_j u_j(z) = \sum_{j=0}^{3} e_j (u_j(z+h) - u_j(z)).$$

Since h is an arbitrary element of $\mathcal{B}$, h can be expressed by $h = a + e_1 b + e_2 c + e_3 d$ for $a,\ b,\ c,\ d \in \mathbb{R}$. Then the 1st-inverse of h is

$$h_*^{-1} = \frac{a + e_1 b - e_2 c - e_3 d}{a^2 + b^2 + c^2 + d^2}.$$

And the 2nd-inverse and 3rd-inverse are

$$h_{\#}^{-1} = \frac{a - e_1 b + e_2 c - e_3 d}{a^2 + b^2 + c^2 + d^2} \text{ and } \mathrm{h}_{\dagger}^{-1} = \frac{\mathrm{a} - \mathrm{e}_1 \mathrm{b} - \mathrm{e}_2 \mathrm{c} + \mathrm{e}_3 \mathrm{d}}{\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2 + \mathrm{d}^2}.$$

Definition 3.1. *Let Ω be a bounded open set in $\mathcal{B}$ and $z \in \Omega$. A function f is defined on a bounded open set Ω. If $h_*^{-1}\{f(z+h) - f(z)\}$ has a limit as $h \to 0$, then we say that f is* the 1st-differentiable function at $z \in \Omega$. *And the limit is said to be* the 1st-derivative of f *at z and denoted by*

(1) $$f'_*(z) := h_*^{-1} \lim_{h \to 0} \{f(z+h) - f(z)\}.$$

Definition 3.2. *Let* Ω *be a bounded open set in* $\mathcal{B}$ *and* $z \in \Omega$. *A function* f *is defined on a bounded open set* Ω. *If* $h_{\#}^{-1}\{f(z+h) - f(z)\}\Big(h_{\dagger}^{-1}\{f(z+h) - f(z)\}\Big)$ *has a limit as* $h \to 0$, *then we say that* f *is* the 2nd-differentiable (3rd-differentiable) function at $z \in \Omega$. *And the limit is said to be* the 2nd-derivative of (3rd-derivative) f *at* z *and denoted by*

$$f'_{\#}(z) := h_{\#}^{-1} \lim_{h\to 0}\{f(z+h) - f(z)\} \quad \left(f'_{\dagger}(z) := h_{\dagger}^{-1} \lim_{h\to 0}\{f(z+h) - f(z)\}\right).$$

Proposition 3.3. *Let* Ω *be a bounded open set in* $\mathcal{B}$. *If the function* f *is the 1st-differentiable function at* $z \in \Omega$, *then* f *satisfies*

$$f'_{*}(z) = \frac{\partial f}{\partial x_0} = e_1\frac{\partial f}{\partial x_1} = -e_2\frac{\partial f}{\partial x_2} = -e_3\frac{\partial f}{\partial x_3}. \tag{2}$$

In cases f *is the 2nd (3rd)-differentiable function in* $z \in \Omega$, f *satisfies*

$$f'_{\#}(z) = \frac{\partial f}{\partial x_0} = -e_1\frac{\partial f}{\partial x_1} = e_2\frac{\partial f}{\partial x_2} = -e_3\frac{\partial f}{\partial x_3}$$

$$\left(f'_{\dagger}(z) = \frac{\partial f}{\partial x_0} = -e_1\frac{\partial f}{\partial x_1} = -e_2\frac{\partial f}{\partial x_2} = e_3\frac{\partial f}{\partial x_3}\right).$$

Proof. By direct computation, we know

$$\begin{aligned}
&h_{*}^{-1}\{f(z+h) - f(z)\}\\
=&\frac{a + e_1 b - e_2 c - e_3 d}{a^2+b^2+c^2+d^2}\{(u_0(z+h) - u_0(z))\\
&+ e_1(u_1(z+h) - u_1(z)) + e_2(u_2(z+h) - u_2(z)) + e_3(u_3(z+h) - u_3(z))\}\\
=&\frac{a + e_1 b - e_2 c - e_3 d}{a^2+b^2+c^2+d^2}[u_0(x_0+h, x_1+h, x_2+h, x_3+h) - u_0(x_0, x_1, x_2, x_3)\\
&+ e_1\{u_1(x_0+h, x_1+h, x_2+h, x_3+h) - u_1(x_0, x_1, x_2, x_3)\}\\
&+ e_2\{u_2(x_0+h, x_1+h, x_2+h, x_3+h) - u_2(x_0, x_1, x_2, x_3)\}\\
&+ e_3\{u_3(x_0+h, x_1+h, x_2+h, x_3+h) - u_3(x_0, x_1, x_2, x_3)\}].
\end{aligned}$$

Considering the cases of that h forms $h = a,\ e_1b,\ e_2c$ and e_3d.
At first, if $h = a$, then

$$\begin{aligned} h_*^{-1}\{f(z+h)-f(z)\} =& h_*^{-1}\{f(z+a)-f(z)\} \\ =& \frac{1}{a}[u_0(x_0+a,x_1,x_2,x_3)-u_0(x_0,x_1,x_2,x_3) \\ &+ e_1\{u_1(x_0+a,x_1,x_2,x_3)-u_1(x_0,x_1,x_2,x_3)\} \\ &+ e_2\{u_2(x_0+a,x_1,x_2,x_3)-u_2(x_0,x_1,x_2,x_3)\} \\ &+ e_3\{u_3(x_0+a,x_1,x_2,x_3)-u_3(x_0,x_1,x_2,x_3)\}]. \end{aligned}$$

So we have

$$\begin{aligned} \lim_{h\to 0} h_*^{-1}\{f(z+h)-f(z)\} &= \frac{\partial u_0}{\partial x_0} + e_1\frac{\partial u_1}{\partial x_0} + e_2\frac{\partial u_2}{\partial x_0} + e_3\frac{\partial u_3}{\partial x_0} \\ &= \frac{\partial f}{\partial x_0}. \end{aligned} \tag{3}$$

In case of $h = e_1 b$,

$$\begin{aligned} h_*^{-1}\{f(z+h)-f(z)\} =& h_*^{-1}\{f(z+e_1b)-f(z)\} \\ =& \frac{e_1}{b}[u_0(x_0,x_1+b,x_2,x_3)-u_0(x_0,x_1,x_2,x_3) \\ &+ e_1\{u_1(x_0,x_1+b,x_2,x_3)-u_1(x_0,x_1,x_2,x_3)\} \\ &+ e_2\{u_2(x_0,x_1+b,x_2,x_3)-u_2(x_0,x_1,x_2,x_3)\} \\ &+ e_3\{u_3(x_0,x_1+b,x_2,x_3)-u_3(x_0,x_1,x_2,x_3)\}]. \end{aligned}$$

So we have

$$\begin{aligned} \lim_{h\to 0} h_*^{-1}\{f(z+h)-f(z)\} &= e_1\frac{\partial u_0}{\partial x_1} - \frac{\partial u_1}{\partial x_1} + e_3\frac{\partial u_2}{\partial x_1} - e_2\frac{\partial u_3}{\partial x_1} \\ &= e_1\left(\frac{\partial u_0}{\partial x_1} + e_1\frac{\partial u_1}{\partial x_1} + e_2\frac{\partial u_2}{\partial x_1} + e_3\frac{\partial u_3}{\partial x_1}\right) \\ &= e_1\frac{\partial f}{\partial x_1}. \end{aligned} \tag{4}$$

Similarly, we have

$$\lim_{h\to 0} h_*^{-1}\{f(z+h)-f(z)\} = -e_2\frac{\partial f}{\partial x_2} \text{ (when h = e}_2\text{c)} \tag{5}$$

and

$$\lim_{h\to 0} h_*^{-1}\{f(z+h)-f(z)\} = -e_3\frac{\partial f}{\partial x_3} \text{ (when h = e}_3\text{d).} \tag{6}$$

All limits of each case have to be same to clarify (1). By (3), (4), (5) and (6), we obtain (2).

Similarly in cases of the 2nd-differentiable functions and the 3rd-differentiable functions, we can obatin the results. □

Theorem 3.4. *Let* Ω *be a bounded open set in* $\mathcal{B}$. *A function* f *is regular in* Ω *if and only if* f *is the 1st (2nd, 3rd)-differentiable function at* $z \in \Omega$:

$$D^* f = 0 \ \text{ iff } \ \frac{\partial f}{\partial x_0} = e_1 \frac{\partial f}{\partial x_1} = -e_2 \frac{\partial f}{\partial x_2} = -e_3 \frac{\partial f}{\partial x_3}$$

$$\left(\begin{array}{l} D^{\#} f = 0 \ \text{ iff } \ \dfrac{\partial f}{\partial x_0} = -e_1 \dfrac{\partial f}{\partial x_1} = e_2 \dfrac{\partial f}{\partial x_2} = -e_3 \dfrac{\partial f}{\partial x_3}, \\ \\ D^{\dagger} f = 0 \ \text{ iff } \ \dfrac{\partial f}{\partial x_0} = -e_1 \dfrac{\partial f}{\partial x_1} = -e_2 \dfrac{\partial f}{\partial x_2} = e_3 \dfrac{\partial f}{\partial x_3} \end{array} \right).$$

Proof. We consider $\dfrac{\partial}{\partial z_j}$ and $\dfrac{\partial}{\partial \overline{z_j}}$ $(j = 1, 2)$ are usual complex differential operators. Then, we denote

$$(7) \quad \begin{aligned} &\frac{\partial}{\partial x_0} = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial \overline{z_1}}, \quad \frac{\partial}{\partial x_1} = -e_1 \left(\frac{\partial}{\partial \overline{z_1}} - \frac{\partial}{\partial z_1} \right), \\ &\frac{\partial}{\partial x_2} = \frac{\partial}{\partial z_2} + \frac{\partial}{\partial \overline{z_2}} \ \text{ and } \ \frac{\partial}{\partial x_3} = -e_1 \left(\frac{\partial}{\partial \overline{z_2}} - \frac{\partial}{\partial z_2} \right). \end{aligned}$$

Since the function f is the 1st-regular function, f satisfies (2). So, we have the following equations by (7):

$$\frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial \overline{z_1}} = -e_1^2 \left(\frac{\partial f}{\partial \overline{z_1}} - \frac{\partial f}{\partial z_1} \right) = -e_2 \left(\frac{\partial f}{\partial z_2} + \frac{\partial f}{\partial \overline{z_2}} \right) = e_3 e_1 \left(\frac{\partial f}{\partial \overline{z_2}} - \frac{\partial f}{\partial z_2} \right).$$

Then,

$$\frac{\partial f}{\partial z_1} = -e_2 \frac{\partial f}{\partial z_2} \ \text{ and } \ \frac{\partial f}{\partial z_1} + e_2 \frac{\partial f}{\partial z_2} = 0.$$

Thus, we obtain

$$D^* f = 0.$$

Conversely, we obtain the result clearly because the above conditions are equivalent. Similarly in cases of the 2nd-regular functions and the 3rd-regular functions, we can obatin the results. □

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