J. Chosun Natural Sci.
Vol. 10, No. 2 (2017) pp. 110 – 114
https://doi.org/10.13160/ricns.2017.10.2.110

The Geometry Descriptions of Crystallographic Groups of Sol_1^4

Won Sok Yoo†

Abstract

The connected and simply connected four-dimensional matrix solvable Lie group Sol_1^4 is the four-dimensional geometry. A crystallographic group of Sol_1^4 is a discrete cocompact subgroup of $\mathrm{Sol}_1^4 \rtimes D(4)$. In this paper, we geometrically describe the crystallographic groups of Sol_1^4.

Keywords: Matrix Lie Group Sol_1^4, Crystallographic Group of Sol_1^4

1. Introduction

The connected and simply connected four-dimensional matrix Lie group

$$\mathrm{Sol}_1^4 = \left\{ \begin{bmatrix} 1 & y & x \\ 0 & e^{\theta} & z \\ 0 & 0 & 1 \end{bmatrix} : x, y, z, \theta \in \mathbb{R} \right\}$$

is one of the four-dimensional geometries which were classified by Filipkiewicz[1], see also literature[2].

Let G be a connected, simply connected Lie group. Then $\mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$ is called the affine group of G, where the group operation is given by

$$(g, \alpha)(h, \beta) = (g \cdot \alpha(h), \alpha\beta)$$

and $\mathrm{Aff}(G)$ acts on G by

$$(g, \alpha)z = g \cdot \alpha(z).$$

Let G be a connected, simply connected nilpotent Lie group and let C be any maximal compact subgroup of $\mathrm{Aff}(G)$. Then a discrete cocompact subgroup Π of $G \rtimes C$ is called a crystallographic group.

In this paper, we will consider the 4-dimensional connected and simply connected solvable Lie group Sol_1^4, and we geometrically describe the crystallographic groups of Sol_1^4. These crystallographic groups Π naturally project onto crystallographic groups $\overline{\Pi}$ of Sol^3 with kernel $\mathbb{Z}$.

Department of Applied Mathematics, Kumoh National Institute of Technology, Kumi 730-701, Korea

†Corresponding author : wsyoo@kumoh.ac.kr
(Received : April 9, 2017, Accepted : June 25, 2017)

2. Sol_1^4-geometry

The group Sol_1^4 has the three-dimensional Heisenberg group Nil ($\theta = 0$) as its nilradical. Indeed, the derived group of Sol_1^4 is Nil^3. On the other hand, Sol_1^4 has the center $Z(\mathrm{Sol}_1^4) = \mathbb{R}$ $(y = z = \theta = 0)$, and the quotient turns out to be isomorphic to the three-dimensional solvable Lie group Sol^3. Therefore, we have the following results.

Theorem 2.1 Sol_1^4 fits in the following commutative diagram between short exact sequences:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathbb{R}^+ & \xrightarrow{\log} & \mathbb{R} & & \\
 & & & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathrm{Sol}_1^4 & \longrightarrow & \mathrm{Sol}^3 & \longrightarrow & 1 \\
 & & \uparrow = & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathrm{Nil}^3 & \longrightarrow & \mathbb{R}^2 & \longrightarrow & 1 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 1 & & 1 & &
\end{array} \tag{2-1}$$

The following results describe the lattices of Sol_1^4 and the automorphisms on any lattice. These are obtained by using the description of lattices and endomorphisms of both Nil^3 and Sol^3 as follows:

Theorem 2.2 (Theorem 3.1[3]). Every lattice Γ of Sol_1^4 can be generated by $\gamma_0, \gamma_1, \gamma_2$ and γ_3 with relations

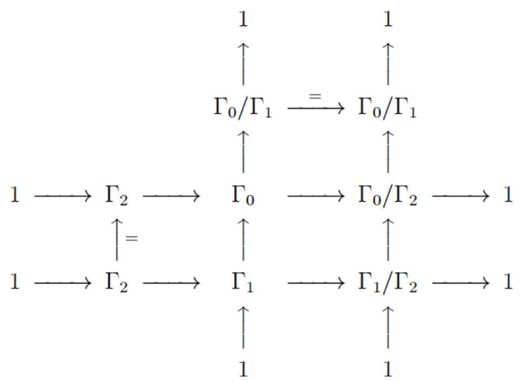
$$[\gamma_1,\gamma_2]=\gamma_3^k,\ [\gamma_1,\gamma_3]=[\gamma_2,\gamma_3]=1,$$
$$\gamma_0\gamma_1\gamma_0^{-1}=\gamma_1^{n_{11}}\gamma_2^{n_{21}}\gamma_3^{p_1},\ \gamma_0\gamma_2\gamma_0^{-1}=$$
$$\gamma_1^{n_{12}}\gamma_2^{n_{22}}\gamma_0^{p_2},\ \gamma_0\gamma_3\gamma_0^{-1}=\gamma_3$$

for some integers k, p_1, p_2 with $k\neq 0$ and $N=[n_{ij}]\in \mathrm{SL}(2,\mathbb{Z})$ with trace > 2.

Proof. Consider the derived series of Sol_1^4: $\mathrm{Sol}_1^4 \supset \mathrm{Nil}^3 \supset \mathrm{Z}(\mathrm{Nil}^3)$. Taking intersections with Γ, we obtain

$$\Gamma=\Gamma_0\supset\Gamma_1\supset\Gamma_2$$

where Γ_1 is a lattice of Nil^3. From the commutative diagram (2-1), we obtain a commutative diagram between lattices



Remark that the bottom exact sequence comes from the short exact sequence $1\to\mathbb{R}\to\mathrm{Nil}^3\to\mathbb{R}^2\to 1$. Then it is well-known for example[4] that such Γ_1 is generated by $\gamma_1, \gamma_2, \gamma_4$ satisfying the relations

$$[\gamma_1,\gamma_4]=[\gamma_2,\gamma_4]=1,\quad [\gamma_1,\gamma_2]=\gamma_4^k$$

for some nonzero integer k. In particular, γ_4 is a generator of $\Gamma_2\cong\mathbb{Z}$ and $\tilde\gamma_1, \tilde\gamma_2$ generate $\Gamma_1/\Gamma_2\cong\mathbb{Z}^2$. Now from the middle vertical, we can choose $\gamma_3\in\Gamma_0$ so that $\{\gamma_1,\cdots,\gamma_4\}$ generates Γ_0. We denote by $\tilde\gamma_3$ and $\tilde{\tilde\gamma}_3$ the images of γ_3 under the projections $\Gamma_0\to\Gamma_0/\Gamma_2$ and $\Gamma_0\to\Gamma_0/\Gamma_1$ respectively. Remark also that $\{\tilde\gamma_1,\tilde\gamma_2,\tilde\gamma_3\}$ is a generator set of Γ_0/Γ_2, which is a lattice of Sol^3. Because $\{\tilde\gamma_1,\tilde\gamma_2\}$ generates $\Gamma_1/\Gamma_2\cong\mathbb{Z}^2$ and $\tilde{\tilde\gamma}_3$ generates $\Gamma_0/\Gamma_1\cong\mathbb{Z}$, we must have

$$[\tilde\gamma_1,\tilde\gamma_2]=1,\ \tilde\gamma_3\tilde\gamma_i\tilde\gamma_3^{-1}=\tilde\gamma_1^{\ell_{1i}}\tilde\gamma_2^{\ell_{2i}}$$

for some integers ℓ_{ij}. Let $A=[\ell_{ij}]$. Then it can be seen that $A\in\mathrm{SL}(2,\mathbb{Z})$ with trace > 2. For details about lattices of Sol^3, we refer to references[5,6]. On the other hand, the conjugation by γ_3 induces an automorphism on Γ_1. Because this automorphism must preserve the relation $[\gamma_1,\gamma_2]=\gamma_3^k$, it follows that $\gamma_3\gamma_4\gamma_3^{-1}=\gamma_4^{\det(A)}=\gamma_4$. Consequently, the theorem is proved.

We denote by $\Gamma_{k,N,\mathrm{P}}$ a lattice of Sol_1^4 with presentation in Theorem 2.2. As it can be observed easily, the canonical projection $\mathrm{Sol}_1^4\to\mathrm{Sol}_1^3$ sends the lattice $\Gamma_{k,N,\mathrm{P}}$ Sol_1^4 to a lattice of Sol^3 with presentation

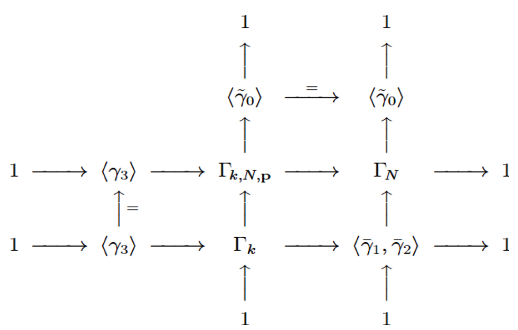
$$<\bar\gamma_0,\bar\gamma_1,\bar\gamma_2\mid[\bar\gamma_1,\bar\gamma_2]=1,\ \bar\gamma_0\bar\gamma_i\bar\gamma_0^{-1}=\bar\gamma_1^{n_{1i}}\bar\gamma_2^{n_{2i}},\ i=1,2>.$$

We will denote this lattice of Sol^3 by Γ_N. Moreover, $\Gamma_{k,N,\mathrm{P}}\cap\mathrm{Nil}^3$ is a lattice of Nil^3 with presentation

$$\Gamma_k=\langle\gamma_1,\gamma_2,\gamma_3\mid[\gamma_1,\gamma_3]=[\gamma_2,\gamma_3]=1,\ \ [\gamma_1,\gamma_2]=\gamma_3^k\rangle.$$

Therefore, we have the following results.

Theorem 2.3 The following diagram is commutative.



Theorem 2.4 (Theorem 3.2[3]). Let

$$\Gamma_{k,N,\mathrm{P}} = \left\langle \gamma_0,\gamma_1,\gamma_2,\gamma_3 \;\middle|\; \begin{array}{c} [\gamma_1,\gamma_2]=\gamma_3^k, \quad [\gamma_0,\gamma_3]=[\gamma_1,\gamma_3]=[\gamma_2,\gamma_3]=1, \\ \gamma_0\gamma_1\gamma_0^{-1}=\gamma_1^{n_{11}}\gamma_2^{n_{21}}\gamma_3^{p_1}, \quad \gamma_0\gamma_2\gamma_0^{-1}=\gamma_1^{n_{12}}\gamma_2^{n_{22}}\gamma_3^{p_2} \end{array} \right\rangle$$

be a lattice of Sol_1^4. Then any endomorphism ϕ on $\Gamma_{k,N,\mathrm{P}}$ is either one of the following forms:

Type (I)

$$\phi(\gamma_0)=\gamma_0\gamma_1^{r_1}\gamma_2^{r_2}\gamma_3^{q_0},$$
$$\phi(\gamma_1)=\gamma_1^{u}\gamma_2^{\frac{n_{21}}{n_{12}}\nu}\gamma_3^{q_1}, \quad \phi(\gamma_2)=\gamma_1^{\nu}\gamma_2^{\mu+\frac{n_{22}-n_{11}}{n_{12}}\nu}\gamma_3^{q_2},$$
$$\phi(\gamma_3)=\gamma_3^{\mu(\mu+\frac{n_{22}-n_{11}}{n_{12}}\nu)-\frac{n_{21}}{n_{12}}\nu^2};$$

Type (II)

$$\phi(\gamma_0)=\gamma_0^{-1}\gamma_1^{r_1}\gamma_2^{r_2}\gamma_3^{q_0},$$
$$\phi(\gamma_1)=\gamma_1^{-\mu}\gamma_2^{\nu}\gamma_3^{q_1}, \quad \phi(\gamma_2)=\gamma_1^{\frac{n_{11}-n_{22}}{n_{21}}\mu-\frac{n_{12}}{n_{21}}\nu}\gamma_2^{\mu}\gamma_3^{q_2},$$
$$\phi(\gamma_3)=\gamma_3^{-\mu^2-(\frac{n_{11}-n_{22}}{n_{21}}\mu-\frac{n_{12}}{n_{21}}\nu)\nu};$$

Type (III)

$$\phi(\gamma_0)=\gamma_0^{m}\gamma_1^{r_1}\gamma_2^{r_2}\gamma_3^{q_0} \ \mathit{with}\ m\neq\pm1,$$
$$\phi(\gamma_1)=\gamma_3^{q_1}, \quad \phi(\gamma_2)=\gamma_3^{q_2}, \quad \phi(\gamma_3)=1.$$

Remark from the above theorem that the type of ϕ is determined by the exponent of γ_0 in the image $\phi(\gamma_0)$. If ϕ is of type (II), then ϕ^2 is of type (I). When ϕ is an automorphism, the type (III) cannot occur.

Let ϕ be an endomorphism on the lattice $\Gamma_{k,N,\mathrm{P}} \subset \mathrm{Sol}_1^4$. Since Sol_1^4 is of type $(\mathbb{R})$, ϕ extends uniquely to a Lie group endomorphism of Sol_1^4, and then induces a Lie group endomorphism of $\mathrm{Sol}^3 \cong \mathrm{Sol}_1^4/Z(\mathrm{Sol}_1^4)$ so that these endomorphisms commute with the canonical projection $\mathrm{Sol}_1^4 \to \mathrm{Sol}^3$. This implies that $\phi : \Gamma_{k,N,\mathrm{P}} \to \Gamma_{k,N,\mathrm{P}}$ induces $\overline{\phi} : \Gamma_N \to \Gamma_N$. Therefore, we have the following results.

Theorem 2.5 The following diagram is commutative.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \langle\gamma_3\rangle & \longrightarrow & \Gamma_{k,N,\mathbf{P}} & \longrightarrow & \Gamma_N & \longrightarrow & 1 \\ & & \downarrow \phi' & & \downarrow \phi & & \downarrow \bar{\phi} & & \\ 1 & \longrightarrow & \langle\gamma_3\rangle & \longrightarrow & \Gamma_{k,N,\mathbf{P}} & \longrightarrow & \Gamma_N & \longrightarrow & 1 \end{array}$$

Since the type of ϕ is determined by the exponent of γ_0 in the image $\phi(\gamma_0)$, it follows that ϕ and $\overline{\phi}$ have the same type.

Remark 2.6 From the diagram above, we have that $\phi(\gamma_3) = \phi'(\gamma_3) = \gamma_3^d$ for some integer d. By Theorem 2.4, d is completely determined by the images $\phi(\gamma_1)$ and $\phi(\gamma_2)$ and hence by the images $\overline{\phi}(\overline{\gamma}_1)$ and $\overline{\phi}(\overline{\gamma}_2)$. Namely, if $\phi(\gamma_1)=\gamma_1^{d_{11}}\gamma_2^{d_{21}}\gamma_3^{*}$ and $\phi(\gamma_2)=\gamma_1^{d_{12}}\gamma_2^{d_{22}}\gamma_3^{*}$ then $\overline{\phi}(\overline{\gamma}_1)=\overline{\gamma}_1^{d_{11}}\overline{\gamma}_2^{d_{21}}$, $\overline{\phi}(\overline{\gamma}_2)=\overline{\gamma}_1^{d_{12}}\overline{\gamma}_2^{d_{22}}$ and $d=d_{11}d_{22}-d_{12}d_{21}$. We will denote d by $n(\phi)=n(\overline{\phi})$.

Recall that $\mathrm{Aut}(\mathrm{Sol}_1^4)$ has a maximal compact subgroup which is isomorphic to the dihedral group $D(4)$ of order 8, see literatures[7,8]. A crystallographic group of Sol_1^4 is a discrete cocompact subgroup Π of $\mathrm{Sol}_1^4 \rtimes D(4)$. In this case, $\Gamma = \Pi \cap \mathrm{Sol}_1^4$ is a lattice of Sol_1^4, and Γ has finite index in Π. The finite group $\Phi = \Pi/\Gamma$ is called the holonomy group Π. A torsion-free crystallographic of Sol_1^4 is a Bieberbach group of Sol_1^4. Therefore, we have the following results.

Theorem 2.7 The following diagram is commutative.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Sol}_1^4 & \longrightarrow & \mathrm{Sol}_1^4 \rtimes D(4) & \longrightarrow & D(4) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & \Phi & \longrightarrow & 1 \end{array}$$

Note that the canonical projection $\mathrm{Sol}_1^4 \to \mathrm{Sol}^3 \cong \mathrm{Sol}_1^4/Z(\mathrm{Sol}_1^4)$ induces a homomorphism $\mathrm{Aut}(\mathrm{Sol}_1^4) \to \mathrm{Aut}(\mathrm{Sol}^3)$, which maps isomorphically a maximal compact subgroup of $\mathrm{Aut}(\mathrm{Sol}_1^4)$ onto a maximal compact subgroup of $\mathrm{Aut}(\mathrm{Sol}^3)$ (see Theorem 2.4). Therefore, we have the following results.

Theorem 2.8 The following diagram between short exact sequences is commutative.

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
1 & \longrightarrow & \mathrm{Sol}^3 & \longrightarrow & \mathrm{Sol}^3 \rtimes D(4) & \longrightarrow & D(4) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow\cong & & \\
1 & \longrightarrow & \mathrm{Sol}_1^4 & \longrightarrow & \mathrm{Sol}_1^4 \rtimes D(4) & \longrightarrow & D(4) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathbb{R} & \xrightarrow{=} & \mathbb{R} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Theorem 2.9 Let $\Pi \subset \mathrm{Sol}_1^4 \rtimes D(4)$ be a crystallographic group. Then it fits in the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
1 & \longrightarrow & \bar{\Gamma} & \longrightarrow & \bar{\Pi} & \longrightarrow & \bar{\Phi} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow\cong & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathbb{Z} & \xrightarrow{=} & \mathbb{Z} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \qquad (2\text{-}2)$$

Here $\overline{\Pi}$ is a crystallographic group of Sol^3.

It is known from Theorem 8.2[6] that there are 9 kinds of crystallographic groups of Sol^3: Γ_N, $\Pi_1(\mathrm{k})$, $\Pi_2^{\pm}$, $\Pi_3(\mathrm{k},\mathrm{k}')$, $\Pi_4(\mathrm{k})$, $\Pi_5(\mathrm{m},\mathrm{k},\mathrm{k}',\mathrm{n})$, $\Pi_6(\mathrm{k},\mathrm{k}')$, $\Pi_7(\mathrm{k})$ and $\Pi_8(\mathrm{k},\mathrm{m})$. Here $N \in \mathrm{SL}(2, \mathbb{N})$ of trace > 2. There are 4 kinds of Bieberbach groups of Sol^3. We recall from Corollary 8.3[6,8] that Γ_A and $\Pi_2^{\pm}$ are Bieberbach groups, and the crystallographic groups $\Pi_1(\mathrm{k})$, $\Pi_4(\mathrm{k})$, $\Pi_5(\mathrm{m},\mathrm{k},\mathrm{k}',\mathrm{n})$, $\Pi_7(\mathrm{k})$ and $\Pi_8(\mathrm{k},\mathrm{m})$ are not Bieberbach groups. The crystallographic groups $\Pi_3(\mathrm{k},\mathrm{k}')$ and $\Pi_6(\mathrm{k},\mathrm{k}')$ become Bieberbach groups for a particular choice of k and k′. In fact, we may assume

$$M = \begin{bmatrix} -1 & m \\ 0 & 1 \end{bmatrix}$$

where m=0 or 1. If m=0, then $\ell_{11} = \ell_{22}$ and $\ker(I-M)/\mathrm{im}(I+M) \cong \mathbb{Z}_2$ is generated by $\mathrm{e}_2 = (0,1)^t$. If m=1, then $\ell_{11} - \ell_{22} = \ell_{21}$ and $\ker(I-M)/\mathrm{im}(I+M)$ is a trivial group and hence $\mathrm{k}=0$. It is known in Sect. 3[8] that they are Bieberbach groups if and only if m=0, $\mathrm{k} = \mathrm{e}_2$ and $\mathrm{k}' - \mathrm{k} \neq 0$. Thus they are not Bieberbach groups if and only if

1. $m = 1$,
2. $m = 0$ and $\mathrm{k} = 0$, or
3. $m = 0$ and $\mathrm{k} = \mathrm{k}' = \mathrm{e}_2$.

Consequently, given $\overline{\Pi}$ with $\overline{\Gamma} = \Gamma_N$ and given $\Gamma = \Gamma_{k,N,\mathrm{P}}$, any abstract group Π fitting the diagram (2-2) is a crystallographic group of Sol_1^4. We can describe the crystallographic groups of Sol_1^4 as follows: Since $\Gamma_{k,N,\mathrm{P}} \subset \Pi$ and Φ has at most two generators, say α, β, we can choose a set of generators $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \alpha, \beta\}$ of Π so that Π has relations: the relations for $\Gamma_{k,N,\mathrm{P}}$ plus new relations

$$\begin{aligned}
&\alpha\gamma_i\alpha^{-1} = v_i(\gamma_0, \gamma_1, \gamma_2, \gamma_3), \\
&\beta\gamma_i\beta^{-1} = w_i(\gamma_0, \gamma_1, \gamma_2, \gamma_3) \quad (i = 0, 1, 2, 3), \\
&r_j = u_j(\gamma_0, \gamma_1, \gamma_2, \gamma_3), \text{ a relation } r_j \text{ of } \Phi.
\end{aligned}$$

Here the words v_i and w_i are of the forms given in Theorem 2.4. In particular,

$$\begin{aligned}
\alpha\gamma_0\alpha^{-1} &= \gamma_0^{\pm 1}\gamma_2^{*}\gamma_3^{*}\gamma_4^{*}, & \alpha\gamma_3\alpha^{-1} &= \gamma_3^{\pm 1}, \\
\beta\gamma_0\beta^{-1} &= \gamma_0^{\pm 1}\gamma_2^{*}\gamma_3^{*}\gamma_4^{*}, & \beta\gamma_3\beta^{-1} &= \gamma_3^{\pm 1}.
\end{aligned}$$

But we cannot choose the integers * in the above relations completely freely. For the details, we refer to reference[7,9,10].

Acknowledgments

This research was supported by Kumoh National Institute of Technology.

References

[1] R. Filipkiewicz, “Four dimensional geometries”, PhD diss, University of Warick, 1983.
[2] C. T. C. Wall, “Geometric structures on compact

complex analytic surfaces", Topology, Vol. 25, pp. 119-153, 1986.

[3] J. H. Jo and J. B. Lee, "Nielsen type numbers and homotopy minimal periods for maps on solvmanifolds with Sol_1^4-geometry", Fixed Point Theory and Applications, Vol. 175, pp. 1-15, 2015.

[4] J. B. Lee and X. Zhao, "Nielsen type numbers and homotopy minimal periods for all continuous maps on the 3-nilmanifolds", Sci. China Ser. A Math., Vol. 51, pp. 351-360, 2008.

[5] J. B. Lee and X. Zhao, "Nielsen type numbers and homotopy minimal periods for maps on the 3-solvmanifolds", Algebraic & Geometric Topology, Vol. 8, pp. 563-580, 2008.

[6] K. Y. Ha and J. B. Lee, "Crystallographic groups of Sol", Mathematische Nachrichten, Vol. 286, pp. 1614-1667, 2013.

[7] K. B. Lee and S. Thuong, "Infra-solvmanifolds of Sol_1^4", J. Korean Math. Soc., Vol. 52, pp. 1209-1251, 2015.

[8] K. Y. Ha and J. B. Lee, "The R_∞ property for crystallographic groups of Sol", Topol. Appl., Vol. 181, pp. 112-133, 2015.

[9] K. Y. Ha, J. B. Lee, and P. penninckx, "Formulas for the reidemeister, lefschetz and nielsen coincidence number of maps between infra-nilmanifolds", Fixed Point Theory Applications, Vol. 39, pp. 1-23, 2012.

[10] P. R. Health, "Groupoids and relations among reidemeister and among nielsen numbers", Topol. Appl., Vol. 181, pp. 3-33, 2015.