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THE CONE PROPERTY FOR A CLASS OF PARABOLIC EQUATIONS

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ABSTRACT. In this note, we show that the cone property is satisfied for a class of dissipative equations of the form $u_t = \Delta u + f(x, u, \nabla u)$ in a domain $\Omega \subset \mathbb{R}^2$ under the so called exactness condition for the nonlinear term. From this, we see that the global attractor is represented as a Lipshitz graph over a finite dimensional eigenspace.

1. INTRODUCTION

An inertial manifold is a positively invariant finite dimensional Lipshitz manifold which attracts all solutions with an exponential rate. Thus, it contains a global attractor and the existence of an inertial manifold can explain the long time behavior of the solutions of evolutionary equations. Moreover, it allows for the reduction of the dynamics to a finite dimensional ordinary differential equation, which is called an inertial form. Inertial manifolds have been constructed for a wide class of partial differential equations. We refer to [2] and [5] for a more detailed exposition of this theory. However, the theory stands incomplete since there are important equations, including the Navier-Stokes equation, for which the inertial manifolds are not known to exist. The main reason for this is the failure of the spectral gap condition for the eigenvalues of the leading partial differential operator.

In this note, we give a new observation that leads to the existence of an inertial manifold for a class of equations of the form

$$u_t = \Delta u + f(x, u, \nabla u), \; x \in \Omega \subset \mathbb{R}^2 \tag{1.1}$$

with the Dirichlet boundary condition

$$u|_{\partial\Omega} = 0.$$

Here, $u = u(t, x)$ is a scalar function and Ω is a rectangular domain, for which the spectral gap condition is satisfied. Until now, it is known that the gap condition holds only for special

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bounded domains in $\mathbb{R}^2$ (see [5]). Furthermore, we impose a so called exactness condition on the nonlinear term:

$$f(x_1, x_2, u, p_1, p_2)_{p_1 x_2} = f(x_1, x_2, u, p_1, p_2)_{p_2 x_1}. \tag{1.2}$$

The condition (1.2) is restrictive, but it allows f to be a combination of functions $g(x, u)$, $|\nabla u|^2$ and $\nabla\phi\cdot\nabla u$ for some differentiable functions g and ϕ in each variable. In the subsequent sections, we give the basic notations and the main results.

2. Notation and the main result

Let the space H be a Hilbert space with an inner product $< \cdot, \cdot >$ and a norm $\|\cdot\|$. Let $\{\lambda_j\}$ be the eigenvalues of the operator $A = -\Delta +$ boundary condition such that

$$\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N < \lambda_{N+1} \ldots,$$

and $\{\phi_j\}$ be the corresponding eigenvectors . Denote by P the orthogonal projection operator from H to a finite dimensional space spanned by $\{\phi_1, \phi_2, \ldots, \phi_N\}$. If $Q = I - P$, then H is orthogonally decomposed as $H = PH \oplus QH$. Let $\mathcal{A}$ be a global attractor for the solutions of the equation (1.1). Then we are interested in the question whether the projection operator P restricted to $\mathcal{A}$ is injective, i.e.,

$$P : \mathcal{A} \to PH \text{ is injective?} \tag{2.1}$$

Equivalently, if $u_1(t) = p_1(t) + q_1(t)$ and $u_2(t) = p_2(t) + q_2(t)$ are two solutions with $p_1(t), p_2(t) \in PH$ and $q_1(t), q_2(t) \in QH$, then the question (2.1) can be rephrased as

$$p_1(0) = p_2(0) \Rightarrow u_1(t) = u_2(t), \text{ for all } t? \tag{2.2}$$

To state the main result, we recall the cone property ([4]). Let

$$u_1(t) = p_1(t) + q_1(t),\ u_2(t) = p_2(t) + q_2(t),$$
$$\rho(t) = p_1(t) - p_2(t),\ \sigma(t) = q_1(t) - q_2(t).$$

The cone C_k is defined as a subset of H by

$$C_k = \{(\rho, \sigma) \in H : \|\sigma\| \leq k\|\rho\|\} \tag{2.3}$$

for some $k > 0$. Then the cone property is stated as

(i) If $u_2(0) \in u_1(0) + C_k$, then $u_2(t) \in u_1(t) + C_k$ for all $t > 0$.
(ii) If $u_2(0) \notin u_1(0) + C_k$, then either $u_2(t_0) \in u_1(t_0) + C_k$ for some t_0 and remains there for all $t > t_0$ or $u_2(t) \notin u_1(t) + C_k$ and $\|u_1(t) - u_2(t)\| \to 0$ exponentially as $t \to \infty$.

It is well-known that the cone property is satisfied for the case of global Lipschitz nonlinearity under the gap condition (2.6) below. More precisely, we consider the equation

$$\frac{du}{dt} = -Au + F(u), \tag{2.4}$$

and assume the nonlinear term is global Lipschitz continuous with the constant K:

$$\|F(u) - F(v)\| \leq K\|u - v\|, \text{ for all } u, v \in H. \tag{2.5}$$

First, we prove the cone property in an equivalent but concise form.

Proposition 1. *Let $u_1(t)$ and $u_2(t)$ be any two orbits in the global attractor of the equation* (2.4). *Under the assumption of the spectral gap condition*

$$\lambda_{N+1} - \lambda_N > \frac{(1+k)^2}{k} K, \tag{2.6}$$

we have $\|\sigma\| \leq k\|\rho\|$ for all $t \in \mathbb{R}$ and, therefore, the projection P is injective.

Remark 1. The proof is standard, however we provide a new simpler proof here.

Proof. Let $u_1(t) = p_1(t) + q_1(t)$ and $u_2(t) = p_2(t) + q_2(t)$. Then, for $\rho(t) = p_1(t) - p_2(t)$ and $\sigma(t) = q_1(t) - q_2(t)$, we have

$$\begin{aligned} \rho_t &= -A\rho + PF(u_1) - PF(u_2), \\ \sigma_t &= -A\sigma + QF(u_1) - QF(u_2). \end{aligned} \tag{2.7}$$

The standard estimates for ρ and σ give

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\rho\|^2 = < \rho_t, \rho > &= -\|A^{1/2}\rho\|^2 + < PF(u_1) - PF(u_2), \rho > \\ &\geq -\lambda_N\|\rho\|^2 - K(\|\rho\| + \|\sigma\|)\|\rho\|, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\sigma\|^2 = < \sigma_t, \sigma > &= -\|A^{1/2}\sigma\|^2 + < QF(u_1) - QF(u_2), \sigma > \\ &\leq -\lambda_{N+1}\|\sigma\|^2 + K(\|\rho\| + \|\sigma\|)\|\sigma\|. \end{aligned} \tag{2.9}$$

From (2.8) and (2.9), it follows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\|\sigma\|^2 - k^2\|\rho\|^2) \leq -\lambda_{N+1}\|\sigma\|^2 + \lambda_N k^2\|\rho\|^2 + K(\|\rho\| + \|\sigma\|)\|\sigma\| \\ + k^2 K(\|\rho\| + \|\sigma\|)\|\rho\|. \end{aligned} \tag{2.10}$$

On the boundary of the cone $\|\sigma\| = k\|\rho\|$, we find that

$$\frac{1}{2}\frac{d}{dt}(\|\sigma\|^2 - k^2\|\rho\|^2) \leq -(\lambda_{N+1} - \lambda_N)k^2\|\rho\|^2 + k(1+k)^2 K\|\rho\|^2 < 0, \tag{2.11}$$

under the condition

$$\lambda_{N+1} - \lambda_N > \frac{(1+k)^2}{k} K. \tag{2.12}$$

This implies, once $u_1(t) - u_2(t)$ is in C_k, it will never leave it through the boundary $\|\sigma\| = k\|\rho\|$ and stays in the cone for all time.

Furthermore, if two orbits sit outside of the cone for some time t_0, then they must stay there for all $t \leq t_0$. That is, if $(\|\sigma\|^2 - k^2\|\rho\|^2)(t_0) > 0$ for some $t_0 \in \mathbb{R}$, then

$$(\|\sigma\|^2 - k^2\|\rho\|^2)(t) > 0$$

for all $t \le t_0$. From (2.9),

$$\frac{1}{2}\frac{d}{dt}\|\sigma\|^2 \le -\lambda_{N+1}\|\sigma\|^2 + (1+\frac{1}{k})K\|\sigma\|^2 = -a_N\|\sigma\|^2, \tag{2.13}$$

with

$$a_N = \lambda_{N+1} - \frac{k+1}{k}K > 0.$$

The inequality (2.13) gives

$$\frac{d}{dt}(e^{2a_N t}\|\sigma\|^2) \le 0 \tag{2.14}$$

or

$$e^{2a_N t_0}\|\sigma(t_0)\|^2 \le e^{2a_N t}\|\sigma(t)\|^2, \text{ for all } t \le t_0. \tag{2.15}$$

Since any orbit in the global attractor is bounded for all time, taking $t \to -\infty$, we get $\|\sigma(t_0)\|^2 = 0$, which is a contradiction. Thus, we conclude that

$$\|\sigma\|^2 \le k^2\|\rho\|^2, \text{ for all } t \in \mathbb{R}. \tag{2.16}$$

□

Now, we state the main results of this note.

Theorem 1. *Let $u_1(t)$ and $u_2(t)$ satisfy the cone property:*

$$\|\sigma\| \le k\|\rho\|, \textit{ for all } t \in \mathbb{R} \tag{2.17}$$

with $v = u_1 - u_2 = \rho + \sigma$. Let us consider a nonlinear change of variable given by

$$V(x,t) = v(x,t)e^{-\gamma(x,t)}. \tag{2.18}$$

If $\gamma = \gamma(x,t)$ is any bounded smooth function for $x \in \Omega$ and $t \in \mathbb{R}$, then $V(x,t)$ also satisfies the cone property with a different constant.

Remark 2. The change of variable (2.18) was first used for one dimensional dissipative equations from a different point of view in [3].

Proof. Denote $\widetilde{\rho} = PV$ and $\widetilde{\sigma} = QV$. Recalling $v = Ve^{\gamma}$, we obtain

$$\begin{aligned} \|\rho\|\|\widetilde{\rho}\| &\ge <\rho,\widetilde{\rho}> = <\rho, PV> = <\rho, Pve^{-\gamma}> = <\rho,(\rho+\sigma)e^{-\gamma}> \\ &= \int_\Omega \rho^2 e^{-\gamma}dx + \int_\Omega \rho\sigma e^{-\gamma}dx \ge m\|\rho\|^2 - M\|\rho\|\|\sigma\|, \end{aligned} \tag{2.19}$$

where $m = \min e^{-|\gamma(x,t)|}$ and $M = \max e^{|\gamma(x,t)|}$.
Thus

$$\|\rho\| \le \frac{1}{m}\|\widetilde{\rho}\| + \frac{M}{m}\|\sigma\| \tag{2.20}$$

and then from (2.17), we have

$$\|\sigma\| \le k\|\rho\| \le \frac{k}{m}\|\widetilde{\rho}\| + \frac{kM}{m}\|\sigma\|. \tag{2.21}$$

Without loss of generality, we may assume $\frac{kM}{m} \leq \frac{1}{2}$ and then (2.21) becomes

$$\frac{1}{2}\|\sigma\| \leq \frac{k}{m}\|\widetilde{\rho}\|. \tag{2.22}$$

Similarly, we have

$$\begin{aligned} \|\sigma\|\|\widetilde{\sigma}\| \geq < \sigma, \widetilde{\sigma} > = < QVe^{\gamma}, \widetilde{\sigma} > = < \widetilde{\sigma}, (\widetilde{\rho} + \widetilde{\sigma})e^{\gamma} > \\ \geq m\|\widetilde{\sigma}\|^2 - M\|\widetilde{\rho}\|\|\widetilde{\sigma}\| \end{aligned} \tag{2.23}$$

and

$$\|\sigma\| \geq m\|\widetilde{\sigma}\| - M\|\widetilde{\rho}\|. \tag{2.24}$$

Finally, combining (2.22) and (2.24), it yields

$$m\|\widetilde{\sigma}\| - M\|\widetilde{\rho}\| \leq \|\sigma\| \leq \frac{2k}{m}\|\widetilde{\rho}\|, \tag{2.25}$$

thus, we get

$$\|\widetilde{\sigma}\| \leq \frac{mM + 2k}{m^2}\|\widetilde{\rho}\|, \tag{2.26}$$

which completes the proof. □

As an application of the Theorem 1, let us consider the equation (1.1) with the condition (1.2). Let $v = u_1 - u_2$. Then, from (1.1), we can write

$$v_t = \Delta v + \alpha_1(x,t)v_{x_1} + \alpha_2(x,t)v_{x_2} + \beta(x,t)v, \tag{2.27}$$

with

$$\alpha_i(x,t) = \int_0^1 \frac{\partial f}{\partial p_i}(x, u_2 + \tau(u_1 - u_2), \nabla u_2 + \tau(\nabla u_1 - \nabla u_2))d\tau, \tag{2.28}$$

for $i = 1, 2$, and

$$\beta(x,t) = \int_0^1 \frac{\partial f}{\partial z}(x, u_2 + \tau(u_1 - u_2), \nabla u_2 + \tau(\nabla u_1 - \nabla u_2))d\tau. \tag{2.29}$$

Now the nonlinear change of variable $V(x,t) = v(x,t)e^{-\gamma(x,t)}$ in (2.28) yields

$$V_t = \Delta V + \sum_{i=1}^{2}(2\gamma_{x_i} + \alpha_i)V_{x_i} + (\Delta\gamma + |\nabla\gamma|^2 + \sum_{i=1}^{2}\alpha_i\gamma_{x_i} + \beta - \gamma_t)V. \tag{2.30}$$

We see that the exactness condition

$$f(x_1, x_2, u, p_1, p_2)_{p_1 x_2} = f(x_1, x_2, u, p_1, p_2)_{p_2 x_1}$$

implies that the system

$$\begin{aligned} 2\gamma_{x_1} + \alpha_1 = 0 \\ 2\gamma_{x_2} + \alpha_2 = 0 \end{aligned} \tag{2.31}$$

can be solved for γ. Then the new equation becomes

$$V_t = \Delta V + \eta(x,t)V, \tag{2.32}$$

where

$$\eta(x,t) = \Delta\gamma + |\nabla\gamma|^2 - 2\sum_{i=1}^{2}\gamma_{x_i}^2 + \beta - \gamma_t$$

and $\eta(x,t)$ and its derivatives are bounded functions. Now we apply the Proposition 1 to the equation (2.32) and obtain the cone property under the spectral gap condition (2.6). Finally, the Theorem 1 gives the cone property for the equation (1.1). This implies the global attractor is indeed Lipschitz manifold with a finite dimension.

Since the cone condition (2.26) is satisfied, we can construct an N-dimensional inertial manifold for (2.32) considering the negatively bounded solutions in [1].

Moreover, noting that

$$V_{x_i} = v_{x_i}e^{-\gamma} - ve^{-\gamma}\gamma_{x_i} = v_{x_i}e^{-\gamma} - \gamma_{x_i}V \tag{2.33}$$

(2.30) is rewritten as

$$V_t = \Delta V + \sum_{i=1}^{2}(2\gamma_{x_i} + \alpha_i)v_{x_i}e^{-\gamma} + (\Delta\gamma + |\nabla\gamma|^2 - \sum_{i=1}^{2}\gamma_{x_i}^2 + \beta - \gamma_t)V. \tag{2.34}$$

If we can solve the following linear first order partial differential equation for $z = \gamma(x,y,t)$ on a rectangular domain:

$$2v_{x_1}\gamma_{x_1} + 2v_{x_2}\gamma_{x_2} = -v_{x_1}\alpha_1 - v_{x_2}\alpha_2, \tag{2.35}$$

then we obtain a similar equation in the form (2.32).

The coefficients in (2.35) are smooth in every variables x_1, x_2 and t. Thus, we may write (2.35) in the form

$$a(x,y,t)u_x + b(x,y,t)u_y = c(x,y,t). \tag{2.36}$$

We look for a bounded smooth solution $z = u(x,y,t)$ at least C^2 in a space variable and C^1 in a time variable. The standard method is solving the characteristic system (a gradient flow):

$$\begin{aligned} \frac{dx}{ds} &= a(x(s),y(s),t(s)), \\ \frac{dy}{ds} &= b(x(s),y(s),t(s)), \\ \frac{dt}{ds} &= 0 \end{aligned} \tag{2.37}$$

and integrating along the characteristic:

$$\frac{du(s)}{ds} = c(x(s),y(s),t(s)). \tag{2.38}$$

However, the geometry of the flow is unclear at this moment and it will be investigated in future works.

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