

Block Sparse Signals Recovery via Block Backtracking-Based Matching Pursuit Method

Rui Qi^{*,***}, Yujie Zhang^{*}, and Hongwei Li^{*,***}

Abstract

In this paper, a new iterative algorithm for reconstructing block sparse signals, called block backtracking-based adaptive orthogonal matching pursuit (BBAOMP) method, is proposed. Compared with existing methods, the BBAOMP method can bring some flexibility between computational complexity and reconstruction property by using the backtracking step. Another outstanding advantage of BBAOMP algorithm is that it can be done without another information of signal sparsity. Several experiments illustrate that the BBAOMP algorithm occupies certain superiority in terms of probability of exact reconstruction and running time.

Keywords

Block Sparse Signal, Compressed Sensing, Sparse Signal Reconstruction

1. Introduction

The last several decades have witnessed a large development of sparse signal recovery problems, which can be formulated as a linear underdetermined system. The compressed sensing (CS) involves recovery of the unknown sparse signal from this linear system [1]: $\mathbf{y} = \Phi \mathbf{x}$, where \mathbf{x} is an unknown signal of length N , $\Phi \in \mathbf{R}^{M \times N}$ ($M < N$) is the measurement matrix, \mathbf{y} denotes the observation vector of length M . This problem has been widely applied in sparse channel estimation [2] and remote spectral sensing [3]. Since $M < N$, it is ill-posed to reconstruct \mathbf{x} given \mathbf{y} . Therefore, some extra conditions should be added to insure recovering \mathbf{x} uniquely. Candes and his colleagues [4,5] stated that if \mathbf{x} is a K -sparse signal ($K \ll N$), that means sparse signal \mathbf{x} has at most K non-zero elements, then exact recovery is possible by using basis pursuit (BP) method [6,7].

Nowadays, many greedy algorithms have been proposed, such as orthogonal pursuit (OMP) [8], stagewise orthogonal matching pursuit (StOMP) [9], regularized orthogonal matching pursuit (ROMP) [10], compressive sampling matching pursuit (CoSaMP) [11], subspace pursuit (SP) [12], etc. Unlike above algorithms that the sparsity K should to be known in advance, backtracking-based adaptive OMP (BAOMP) [13] can adaptively estimate the sparsity K of original signal. Moreover, convex algorithm like smoothed ℓ_0 norm (SL0) [14] has also been tested to be effective.

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The unknown sparse signals \mathbf{x} with block sparsity are considered in this paper, the structures of \mathbf{x} can be written as follows:

$$\mathbf{x} = \left[\underbrace{x_1, \dots, x_d}_{\mathbf{x}^T[1]}, \underbrace{x_{d+1}, \dots, x_{2d}}_{\mathbf{x}^T[2]}, \dots, \underbrace{x_{N-d+1}, \dots, x_N}_{\mathbf{x}^T[L]} \right], \quad (1)$$

where $\mathbf{x}^T[\ell]$ denotes the ℓ -th block of \mathbf{x} with length d .

A vector \mathbf{x} is called block K -sparse if $\mathbf{x}[\ell]$ has nonzero Euclidean norm for at most K -indices ℓ [15]. Specially, when $d=1$, block sparse degenerates into a general case. Block sparse signal has been found in multi-band signals [16] and equalization of sparse communication channels [17], etc.

Denote

$$\|\mathbf{x}\|_{2,0} = \sum_{\ell=1}^L I(\|\mathbf{x}[\ell]\|_2 > 0), \quad (2)$$

where $I(\|\mathbf{x}[\ell]\|_2 > 0)$ is an indicator function. In this case, a block K -sparse vector \mathbf{x} is defined as $\|\mathbf{x}\|_{2,0} \leq K$ [15]. It is known that we can recover the unique sparse signal \mathbf{x} by solving ℓ_2/ℓ_0 -norm problem [18]:

$$\min_{\mathbf{x}} \|\mathbf{x}\|_{2,0} \quad \text{s.t. } \mathbf{y} = \Phi \mathbf{x}. \quad (3)$$

Unfortunately, finding the optimal solution of (3) is a NP-hard problem. One natural idea is to replace the ℓ_2/ℓ_0 -norm by ℓ_2/ℓ_1 -norm, that is:

$$\min_{\mathbf{x}} \|\mathbf{x}\|_{2,1} \quad \text{s.t. } \mathbf{y} = \Phi \mathbf{x}, \quad (4)$$

where $\|\mathbf{x}\|_{2,1} = \sum_{i=1}^L \|\mathbf{x}[i]\|_2$.

In [18], as an extension of BP method, the mixed ℓ_2/ℓ_1 -norm algorithm is proposed. Subsequently, many of iterative algorithms have been carried out for recovering block sparse signals [19], which solves a weighted ℓ_2/ℓ_1 minimization in each iteration. The above two methods have good quality, but they are slow. Another convex method called BSLO [20] is also tested to be effective. The second family of approaches are greedy algorithms, such as block OMP (BOMP) [15], block CoSaMP [21], block StOMP [22], they are very fast, but the sparsity should be known as a prior. The third kind of approaches are non-convex methods [23,24], it is shown that the non-convex methods surpass the mixed ℓ_2/ℓ_1 -norm algorithm. As shown in these papers, we find that it can provide better reconstruction performance by making use of block sparsity than regarding the block sparse signal as a general sparse case. Inspired by the technique of extension of OMP to BOMP, a block BAOMP (BBAOMP) method is introduced in this paper.

The remainder of the paper is organized as follows. Section 2 depicts with reviewing BAOMP, and then describes the block version of BAOMP. Simulation results are given in Section 3 to compare the proposed algorithm with conventional OMP, SP, BOMP, BSLO, and BAOMP methods. Finally a conclusion is given in Section 4.

2. A Block Version of BAOMP

In this section, we first review the algorithm of BAOMP, and a block version of BAOMP will be investigated in the next.

As an extension of OMP algorithm, BAOMP was proposed by Huang and Makur [13]. The BAOMP algorithm first finds one or several atoms which corresponding to the much larger correlation between measurement vectors and the residual. To be mentioned, the atoms selected in the previous stage may be wrong. Subsequently, BAOMP method identifies the atoms which are wrongly chosen. By using this backtracking procedure, it refines the estimated support set. In the next, BAOMP method produces a new residual by using the least-square fit. The BAOMP method does not need the information of signal sparsity, it is repeated until the norm of residual is smaller than a threshold or the iteration count n reaches the number of maximum iteration.

Because of the backtracking step, BAOMP algorithm gives double checks about how to choose the atom reliably, which yields much better sparse reconstruction performance. Moreover, the BAOMP method brings some flexibility between reconstruction property and computational complexity by adaptively adjusting parameters μ_1 and μ_2 .

Generalizing the BAOMP algorithm to the block sparse signals, we obtain the BBAOMP algorithm. It first chooses several block indices, and then subtracts some wrong block indices, the final estimated support set will be identified after several iterations. The details about the BBAOMP algorithm are shown in Table 1.

Table 1. BBAOMP algorithm

Input:	measurement matrix: $\Phi \in \mathbf{R}^{M \times N}$, measurement vector: \mathbf{y} atom-adding threshold in $[0, 1]$: μ_1 atom-deleting threshold in $[0, 1]$: μ_2 convergence threshold: ε maximum iteration number: n_{\max}
Initialize:	initial solution: $\mathbf{x}^0 = \mathbf{0}$, initial residual: $\mathbf{r}^0 = \mathbf{y}$ estimated support set: $\Lambda = \emptyset$, iteration count: $n = 1$ candidate set: $C^0 = \emptyset$, delete set: $\Gamma^0 = \emptyset$.

While $\|\mathbf{r}^n\|_2 > \varepsilon$ and $n < n_{\max}$ do

Step 1: Find the set C^n that satisfying

$$\left\| \langle \mathbf{r}^{n-1}, \Phi[C^n] \rangle \right\|_2 \geq \mu_1 \cdot \max_{j \in \Omega^t} \left\| \langle \mathbf{r}^{n-1}, \Phi[j] \rangle \right\|_2, \quad \Omega^t = \{1, 2, \dots, L\};$$

Step 2: Calculate $\mathbf{x}_{\Lambda \cup C^n}^n = \Phi_{\Lambda \cup C^n}^+ \cdot \mathbf{y}$. Define $\tilde{\mathbf{x}}^n \in \mathbf{R}^{Ld}$, where $\tilde{\mathbf{x}}^n|_{\Lambda \cup C^n} = \mathbf{x}_{\Lambda \cup C^n}^n$, $\tilde{\mathbf{x}}^n|_{(\Lambda \cup C^n)^c} = \mathbf{0}$.

Find the delete set Γ^n in which the block indices i satisfy

$$0 < \left\| \tilde{\mathbf{x}}^n[i] \right\|_2 < \mu_2 \cdot \max_{j \in C^n} \left\| \tilde{\mathbf{x}}^n[j] \right\|_2;$$

Step 3: Update support set: $\Lambda = \{\Lambda \cup C^n\} \setminus \Gamma^n$;

Step 4: Projection and pursuit: $\mathbf{x}_{\Lambda}^n = \Phi_{\Lambda}^+ \cdot \mathbf{y}$;

Step 5: Update the residual: $\mathbf{r}^n = \mathbf{y} - \Phi_{\Lambda} \mathbf{x}_{\Lambda}^n$; $n = n + 1$;

Output: The estimate signal $\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}|_{\Omega \setminus \Lambda} = \mathbf{0}$ and $\hat{\mathbf{x}}|_{\Lambda} = \Phi_{\Lambda} \cdot \mathbf{y}$.

At the n^{th} iteration, the BBAOMP algorithm first finds the candidate set C^n whose norms of correlations between $\Phi[j](j \in \Omega')$ and the residual $\mathbf{r}^{n-1} \geq \mu_1 \cdot \max_{j \in \Omega'} \|\langle \mathbf{r}^{n-1}, \Phi[j] \rangle\|_2$, where $\Omega' = \{1, 2, \dots, L\}$ denotes the whole block indices. μ_1 is a constant which determines the number of block indices chosen at each time. When $\mu_1 = 1$, it is same as BOMP which selects only one block index corresponding to the maximal norm of correlations. When μ_1 becomes smaller, the BBAOMP algorithm can select more than one block index at each time, smaller μ_1 results in much more block indices and speeds up the algorithm. Unfortunately, the block indices selected at the above process may be wrong. At the next step, several block indices will be deleted. We first calculate $\mathbf{x}_{\Lambda \cup C^n}^n$ by using $\Phi_{\Lambda \cup C^n}^+ \cdot \mathbf{y}$, and define $\tilde{\mathbf{x}}^n \in \mathbf{R}^{Ld}$, where $\tilde{\mathbf{x}}^n|_{\Lambda \cup C^n} = \mathbf{x}_{\Lambda \cup C^n}^n$, $\tilde{\mathbf{x}}^n|_{(\Lambda \cup C^n)^c} = \mathbf{0}$. Then the block indices whose norms of $\tilde{\mathbf{x}}^n[i]$ are smaller than μ_2 times the maximal norm of $\tilde{\mathbf{x}}^n[j](j \in \Omega')$ will be removed, where μ_2 is a parameter determining the number of deleted block indices at each time. Similarly, bigger μ_2 results in smaller deleted block indices and slows down the algorithm. After updating the support set Λ , we produce a new residual by using the least-square fit. Due to the block sparsity K is not known in advance, the BBAOMP algorithm is repeated until the norm of residual \mathbf{r}^n is smaller than a threshold ε or the iteration count n reaches the number of maximum iterations n_{\max} .

The most important difference between BOMP algorithm and the proposed algorithm is that the BBAOMP algorithm involves a backtracking step. Owing to the backtracking step, the BBAOMP algorithm gives double checks of the chosen atom's accuracy, which yields much better reconstruction property. What's more, there are two parameters μ_1 and μ_2 in the proposed method. Smaller μ_1 and bigger μ_2 result in smaller estimated support set at each time, which leads to better reconstruction performance and longer running time. On the contrary, bigger μ_1 and smaller μ_2 result in bigger estimated support set at each time, which leads to worse reconstruction performance and shorter running time.

3. Experimental Simulations

In this section, the reconstruction property of BBAOMP method is experimentally studied and is compared with OMP, BOMP, SP, BSL0, and BAOMP.

3.1 Experimental Settings

In each trial, the block sparse signal \mathbf{x} is artificially generated as follows: for a fixed sparsity K , the nonzero blocks are randomly chosen. Each element in the nonzero blocks is drawn from standard Gaussian distribution $N(0,1)$ and the elements of other blocks are zero. The observation vector is $\mathbf{y} = \Phi \mathbf{x}$, where the entries of sensing matrix $\Phi \in \mathbf{R}^{M \times N}$ are generated from standard Gaussian distribution $N(0,1)$ independently. In all the simulations, $M=128$, $N=256$, and the block size $d=2$.

To evaluate the estimation quality, we use a measure named signal-to-noise ratio SNR (in dB) defined as [20]:

$$\text{SNR} = 10 \log \left(\frac{\|\mathbf{x}\|}{\|\mathbf{x} - \hat{\mathbf{x}}\|} \right), \quad (5)$$

where \mathbf{x} and $\hat{\mathbf{x}}$ denote the original and reconstructed signal, respectively.

To be mentioned, each test is repeated 200 times, the values of the probability of exact reconstruction and running time are averaged. The reconstruction is viewed to be successful if $\|\mathbf{x} - \hat{\mathbf{x}}\|_2 < 10^{-3}$ in each trial. In the following experiments, the proposed method uses $\mu_1=0.4, \mu_2=0.6, n_{\max} = M, \varepsilon=10^{-6}$ as the input parameters. OMP and SP use the default setting given in [8,12], the parameters of BAOMP, BOMP and BSL0 are same to the paper [13,15,20].

3.2 Recovery Property versus Block Sparsity

Fig. 1 shows each algorithm's recovery probability under different block sparsity level. As shown in Fig. 1, the performance of BBAOMP algorithm surpasses another methods. When sparsity level $K > 25$, all the algorithms start to fail besides BBAOMP method. Even if the block sparsity level $K = 40$, the probability of reconstruction of BBAOMP method is almost 50%.

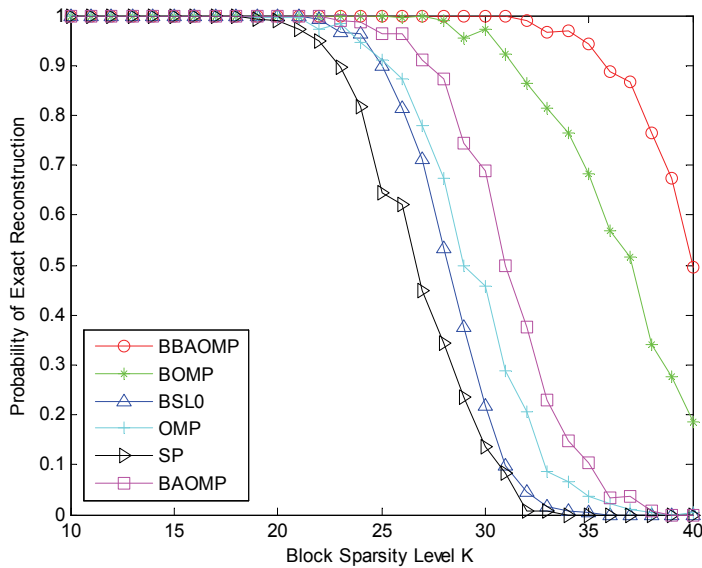


Fig. 1. The recovery probability versus block sparsity.

In Fig. 2, compared with other methods, the running time of the above algorithms under different block sparsity level K is studied. It is clear from Fig. 2, the running time of BSL0, BOMP, BAOMP, and BBAOMP method is similar as block sparsity varies from 10 to 40. To be mentioned, the running time of BSL0 algorithm stays constant versus block sparsity, the reason is that the BSL0 is a convex optimization algorithm. In addition, as the block sparsity level varies from 10 to 40, the running time of the SP and OMP increases much faster than another algorithms. In particular, when block sparsity $K > 25$, the running time of SP seems to increase linearly over K .

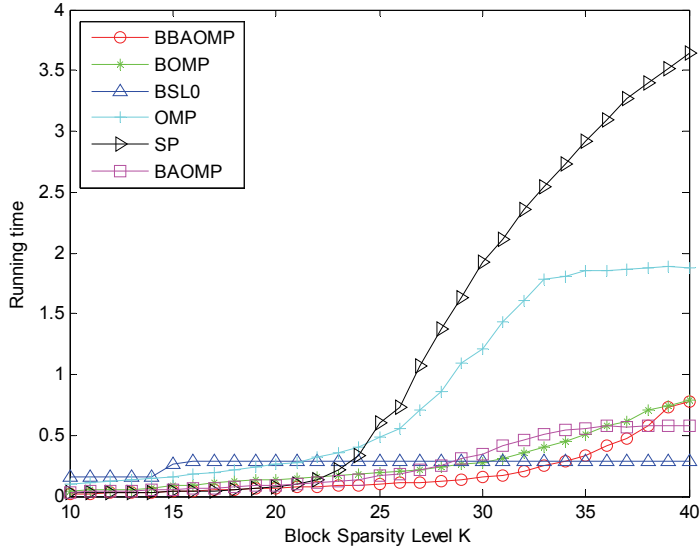


Fig. 2. The probability of running time versus block sparsity.

3.3 Recovery Property versus Number of Measurement

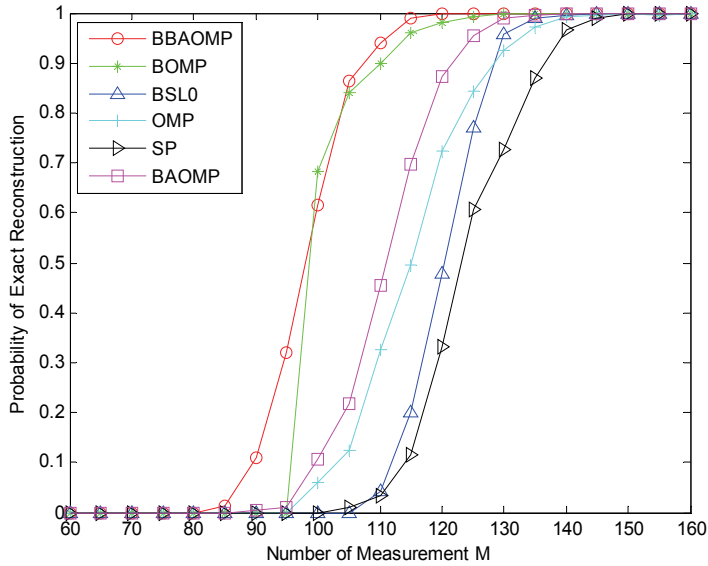


Fig. 3. The recovery probability versus number of measurement M .

In Fig. 3, the recovery probability is observed as a function of measurement M . The number of measurement M varies from 60 to 160 as the sparsity K is fixed to 25. As can be seen from Fig. 3, the recovery probability increases as number of measurement increases. We find that the proposed method surpasses another five methods. When the number of measurement is almost 115, the recovery probability of BBAOMP algorithm reaches 100%, while another five algorithms fail to recover original signal completely.

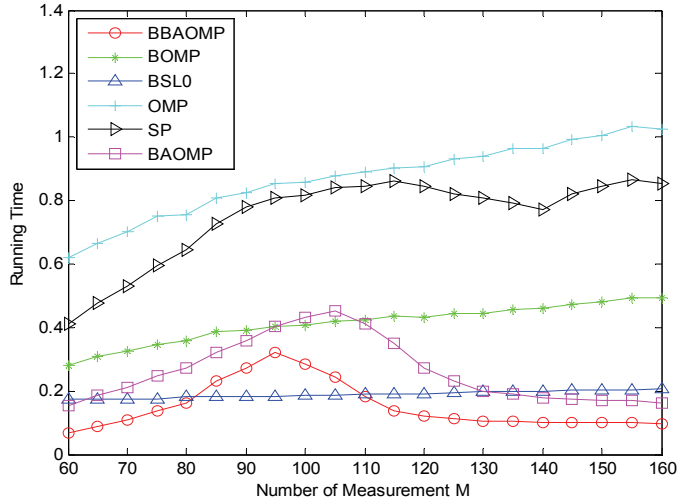


Fig. 4. The probability of running time versus number of measurement M .

Fig. 4 describes the running time versus number of measurement M . Similarly, the running time of BSL0, BOMP, BAOMP, and BBAOMP methods is more or less similar as number of measurement varies from 60 to 160. The running time of BBAOMP algorithm first increases and then decreases, it reaches the maximum when number of measurement M is 95. Broadly speaking, the proposed method is computationally effective compared with another algorithms.

3.4 Discussion on Number of Iterations

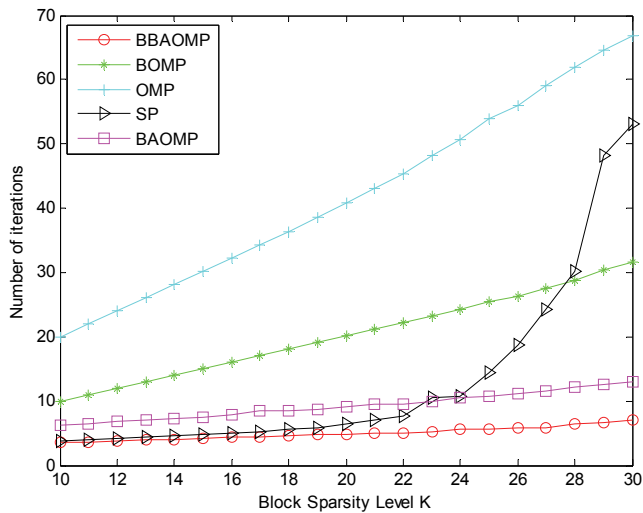


Fig. 5. Number of iterations versus block sparsity.

Fig. 5 depicts the number of iterations in terms of block sparsity. The number of iterations of BBAOMP algorithm is compared with another four algorithms (Note that BSL0 algorithm does not exist in this experiment, since it is a convex optimization algorithm). As shown in Fig. 5, we can see that

the numbers of iterations of the BBAOMP and BAOMP algorithms nearly stay constant as block sparsity level K increases, which outperform all other algorithms. In particular, while the numbers of iterations of the BOMP and OMP algorithms increase linearly over K , the SP algorithm seems to increase quadratically when $K > 20$.

4. Conclusions

In this article, we introduce a new sparse recovery algorithm which solving block sparse signal reconstruction problems. This method first chooses atoms adaptively and then removes some atoms that are wrongly chosen at the previous step by using backtracking procedure, which promotes the reconstruction property. In addition, the BBAOMP algorithm does not need the sparsity level as a prior. Simulation results demonstrate that our method produces much better reconstruction property compared with many existing algorithms.

Thanks to the two parameters μ_1 and μ_2 , the BBAOMP method brings some flexibility between computational complexity and reconstruction property. However, we do not have theoretical support to determine how to select μ_1 and μ_2 . Future works include theoretical demonstration about the choice of parameters and numerical experiments by using actual data.

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