

REGULAR VARIATION AND STABILITY OF RANDOM MEASURES

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ABSTRACT. The paper presents a characterization of stable random measures, giving a canonical form of their Laplace transform. Domain of attraction of stable random measures is concerned in a theorem showing that a random measure belongs to domain of attraction of any stable random measures if and only if it varies regularly at infinity.

Introduction

The concept of *stability* of probability distributions on the real line was introduced by Paul Lévy in 1937 [10]. Gnedenko and Kolmogorov issued a well-known book [5] with a complete description of one-dimensional stable distributions. Stability of a probability distribution is strongly related to its *domain of attraction*. It is shown that a probability distribution is stable if and only if its domain of attraction is not empty. In 2000, J. L. Geluk and L. de Haan [4] gave a nice description of one-dimensional domain of attraction proving that a probability distribution belongs domain of attraction of any one-dimensional stable distribution if and only if it is *regularly varying*.

Stable distributions arise as solutions to central limit problems and have attracted very much attention, both in theoretical research [4, 7, 9, 15, 16, 17] as well as in applied research [1, 11, 13]. However, most of researches limited to investigate stable distributions on linear structures like Euclidean, Hilbert or Banach spaces, very few of articles published with topic related to more abstract cases of convex cone or random measures. The aim of this paper is to study stability of random measures and their domain of attraction.

1. Preliminaries and notation

For a locally compact second countable Hausdorff topological space \mathbf{S} , let denote by $\Phi(\mathbf{S})$ the class of all functions $g : \mathbf{S} \rightarrow [0; \infty]$ which have compact

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support; by $m(\mathbf{S})$ the class of all non negative Radon measures on \mathbf{S} and by $m_b(\mathbf{S})$ the subclass of $m(\mathbf{S})$ containing all totally bounded measures. The family $m(\mathbf{S})$ is an abelian semigroup with respect to the addition of measures, its neutral element $\mathbf{0}$ is the measure that is identically zero. Moreover, $m(\mathbf{S})$ is a separable metric space with the vague topology. By *random measure* (r.m.) we mean a probability measure on the σ -algebra of all Borel subsets of $m(\mathbf{S})$. By $\Pi(\mathbf{S})$ we denote the class of all r.m.'s on $m(\mathbf{S})$.

The addition of measures in $m(\mathbf{S})$ leads to a *convolution* in $\Pi(\mathbf{S})$. Namely, let $\kappa : m(\mathbf{S}) \times m(\mathbf{S}) \rightarrow m(\mathbf{S})$ be the continuous mapping given by $\kappa(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y}$, for $\mathbf{p}, \mathbf{q} \in \Pi(\mathbf{S})$ the convolution $\mathbf{p} * \mathbf{q}$ is by definition the image measure of the Radon product measure $\mathbf{p} \otimes \mathbf{q}$ under the mapping κ ,

$$\mathbf{p} * \mathbf{q}(\mathbf{B}) = \mathbf{p} \otimes \mathbf{q}(\kappa^{-1}(\mathbf{B}))$$

for any Borel subset \mathbf{B} of $m(\mathbf{S})$. Then $\Pi(\mathbf{S})$ with the weak convergence (denoted by \rightarrow_w) is a separable metrizable abelian semigroup with neutral element $\delta(\mathbf{0})$, the r.m. concentrated at the neutral element $\mathbf{0}$ of $m(\mathbf{S})$.

$\Pi(\mathbf{S})$ is endowed with continuous automorphisms $T_a : \Pi(\mathbf{S}) \rightarrow \Pi(\mathbf{S})$, called scaling operations by positive real numbers $a > 0$, defined by

$$T_a \mathbf{p}(\mathbf{B}) = \mathbf{p}(a^{-1}\mathbf{B})$$

for any $\mathbf{p} \in \Pi(\mathbf{S})$, where $a^{-1}\mathbf{B} = \{\mathbf{x} \in m(\mathbf{S}) : a\mathbf{x} \in \mathbf{B}\}$ for each Borel subset \mathbf{B} of $m(\mathbf{S})$.

The *Laplace transform* of a r.m. $\mathbf{p} \in \Pi(\mathbf{S})$ defined by

$$L(\mathbf{p}, g) = \int_{m(\mathbf{S})} e^{-\mathbf{x}.g} \mathbf{p}(d\mathbf{x})$$

for every function $g \in \Phi(\mathbf{S})$, where $\mathbf{x}.g = \int_{\mathbf{S}} g(s)\mathbf{x}(ds)$. Then, by virtue of Theorem 3.1 [8], the r.m. \mathbf{p} is uniquely determined by its Laplace transform. Besides,

$$L(\mathbf{p} * \mathbf{q}, g) = L(\mathbf{p}, g) \cdot L(\mathbf{q}, g)$$

for any r.m.'s $\mathbf{p}, \mathbf{q} \in \Pi(\mathbf{S})$.

A r.m. $\mathbf{p} \in \Pi(\mathbf{S})$ is called *infinitely divisible* (inf.div.) if for every positive integer n there exists r.m. $\mathbf{p}_n \in \Pi(\mathbf{S})$ such that

$$\mathbf{p} = \mathbf{p}_n^{*n} := \underbrace{\mathbf{p}_n * \dots * \mathbf{p}_n}_n.$$

Then for any inf.div. r.m. \mathbf{p} , by virtue of Theorem 6.1 [8], we get the formula

$$(1.1) \quad -\log L(\mathbf{p}, g) = \mathbf{a}.g + \mathbf{\Gamma}.(1 - e^{-\pi_g})$$

for all $g \in \Phi(\mathbf{S})$, where π_g is the projection from $m(\mathbf{S})$ on $[0; +\infty]$ defined by $\pi_g(\mathbf{x}) := \mathbf{x}.g$ for every measure $\mathbf{x} \in m(\mathbf{S})$; $\mathbf{a} \in m(\mathbf{S})$; and $\mathbf{\Gamma}$ is a measure on $m(\mathbf{S}) \setminus \{\mathbf{0}\}$ satisfying

$$(1.2) \quad (\mathbf{\Gamma}\pi_{\mathbf{B}}^{-1}) \cdot h < \infty$$

for all relatively compact Borel subset \mathbf{B} of \mathbf{S} ; $\pi_{\mathbf{B}}(\mathbf{x}) = \mathbf{x}(\mathbf{B})$ for all $\mathbf{x} \in m(\mathbf{S})$; $h(t) := 1 - e^{-t}$. Moreover, for any given $\mathbf{a} \in m(\mathbf{S})$ and measure $\mathbf{\Gamma}$ with the

mentioned property, the formula (1.1) always exactly defines an inf.div. r.m. \mathbf{p} . The formula (1.1) is called *spectral representation*, the measure $\mathbf{\Gamma}$ is called *spectral measure* of \mathbf{p} . Then it is convenient to write $\mathbf{p} = \mathbf{I}(\mathbf{a}, \mathbf{\Gamma})$.

It is clear that if $\mathbf{\Gamma}$ is spectral measure of \mathbf{p} with the spectral representation (1.1), then $n \cdot \mathbf{\Gamma}$ is spectral measure of \mathbf{p}^{*n} , for every natural n , and

$$-\log L(\mathbf{p}^{*n}, g) = n\mathbf{a} \cdot g + n\mathbf{\Gamma} \cdot (1 - e^{-\pi g}).$$

Moreover, for every positive number r , the measure $r \cdot \mathbf{\Gamma}$ is spectral measure of some inf.div. r.m., which can be denoted by \mathbf{p}^{*r} , and

$$-\log L(\mathbf{p}^{*r}, g) = r\mathbf{a} \cdot g + r\mathbf{\Gamma} \cdot (1 - e^{-\pi g}).$$

2. Characterization of stable random measures

In this section we concern with the concepts of stability of r.m.'s and give a characterization of the stability.

Definition. A r.m. $\mu \in \Pi(\mathbf{S})$ is called α -stable, $\alpha \neq 0$, if

$$(2.1) \quad \mu^{*n} = \mathbf{T}_{n^{1/\alpha}} \mu$$

for every natural number $n > 1$.

Lemma 2.1. *Suppose that $\mu \neq \delta(\mathbf{0})$; $\alpha \neq 0$; and (2.1) is true for all natural numbers n . Then α is a positive number.*

Proof. Suppose that $\alpha < 0$. Then because $n^{1/\alpha} \rightarrow 0$ and the scaling operation \mathbf{T} is continuous, $\mathbf{T}_{n^{1/\alpha}} \mu \rightarrow_w \mathbf{T}_0 \mu = \delta(\mathbf{0})$ as $n \rightarrow \infty$, from (2.1) we get

$$\mu^{*n} = \mathbf{T}_{n^{1/\alpha}} \mu \rightarrow_w \delta(\mathbf{0})$$

as $n \rightarrow \infty$, that yield $\mu * \delta(\mathbf{0}) = \delta(\mathbf{0})$. Consequently, we get $\mu = \delta(\mathbf{0})$. □

The parameter α is called the *stability exponent* of μ . For convenience, sometimes we omit the term α and say that a given r.m. μ is *stable*. From (2.1) it is evident that every stable r.m. is inf.div. and has spectral representation of the form (1.1) with some special spectral measure $\mathbf{\Gamma}$. In the follows we will give a representation of the stable r.m. μ .

The local compactness and the separability of \mathbf{S} ensure the existence of an increasing sequence of compact subsets $\{\mathbf{K}_n\}$ such that $\mathbf{K}_n \subset \mathbf{K}_{n+1}$; $\mathbf{K}_{n+1} \setminus \mathbf{K}_n \neq \emptyset$ for all natural n , and $\mathbf{S} = \bigcup_{n=1}^{\infty} \mathbf{K}_n$. In the sequel, let the sequence $\{\mathbf{K}_n\}$ be fixed. We define subsets of measures $\mathbf{M}_n \subset m(\mathbf{S})$ by

$$\mathbf{M}_1 = \{\mathbf{x} \in m(\mathbf{S}) : \mathbf{x}(\mathbf{K}_1) = 1\};$$

$$\mathbf{M}_n = \{\mathbf{x} \in m(\mathbf{S}) : \mathbf{x}(\mathbf{K}_1) = \dots = \mathbf{x}(\mathbf{K}_{n-1}) = 0, \mathbf{x}(\mathbf{K}_n) = 1\}; n = 2, 3, \dots,$$

and set $\mathbf{M} = \bigcup_{n=1}^{\infty} \mathbf{M}_n$. Then, it is clear that

$$(2.2) \quad m(\mathbf{S}) \setminus \{\mathbf{0}\} = \{c\mathbf{x} : c \in (0; +\infty), \mathbf{x} \in \mathbf{M}\}.$$

Theorem 2.2. *Let $\mu \in \Pi(\mathbf{S})$ be a random measure and $\alpha \neq 0$ be given. Suppose that $\mu \neq \delta(\mathbf{0})$ and μ is α -stable. Then*

i) *If $\alpha = 1$, the Laplace transform $L(\mu, g)$ of μ satisfies*

$$(2.3) \quad -\log L(\mu, g) = \mathbf{a} \cdot g$$

for all $g \in \Phi(\mathbf{S})$, with some $\mathbf{a} \in m(\mathbf{S})$;

ii) *If $\alpha \neq 1$, then it must be $0 < \alpha < 1$ and there exists a Radon measure \mathbf{A} on \mathbf{M} such that*

$$(2.4) \quad -\log L(\mu, g) = \int_{\mathbf{M}} (\mathbf{x} \cdot g)^\alpha \mathbf{A}(d\mathbf{x})$$

for all $g \in \Phi(\mathbf{S})$.

Conversely, if (2.3) or (2.4) is true, then μ is α -stable.

Proof. It is clear that μ is inf.div. and its Laplace transform has the form (1.1) with certain $\mathbf{a} \in m(\mathbf{S})$ and spectral measure $\mathbf{\Gamma}$. From (2.1), for every $k = 2, 3, \dots$, we get the equalities

$$(2.5) \quad k\mathbf{a} = k^{1/\alpha}\mathbf{a};$$

$$(2.6) \quad k\mathbf{\Gamma} = \mathbf{T}_{k^{1/\alpha}}\mathbf{\Gamma}.$$

Let \mathbf{B} be a Borel subset of \mathbf{M} such that $\mathbf{B} \subset \bigcup_{n=1}^J \mathbf{M}_n$ with some natural number J . For every positive number r we put

$$(2.7) \quad v(r, \mathbf{B}) = \mathbf{\Gamma}(\{t\mathbf{x} : t \geq r, \mathbf{x} \in \mathbf{B}\}).$$

Then it implies from (1.2) that the right hand side (2.7) is finite. Besides, due to (2.6), we have

$$kv(r, \mathbf{B}) = v(k^{-1/\alpha}r, \mathbf{B})$$

for $k = 1, 2, \dots$, that implies

$$kv(n^{1/\alpha}, \mathbf{B}) = v(k^{-1/\alpha}n^{1/\alpha}, \mathbf{B}) = v((n/k)^{1/\alpha}, \mathbf{B})$$

for $k, n = 1, 2, \dots$, and

$$v(n^{1/\alpha}, \mathbf{B}) = (1/n)v(1, \mathbf{B}).$$

Therefore,

$$v((n/k)^{1/\alpha}, \mathbf{B}) = (k/n)v(1, \mathbf{B}),$$

that means

$$(2.8) \quad v(q^{1/\alpha}, \mathbf{B}) = q^{-1}v(1, \mathbf{B})$$

for every positive rational number q .

From the continuity of measures and (2.7) we see that, for every fixed \mathbf{B} , the function $v(t, \mathbf{B})$ is continuous from left. For each positive number t , let $\{q_i\}$ be a sequence of rational numbers such that $\{q_i^{1/\alpha}\}$ is an increasing sequence convergent to t . Then (2.8) yields

$$(2.9) \quad v(t, \mathbf{B}) = t^{-\alpha}v(1, \mathbf{B}).$$

It is easy to verify that the set function $\nu(\mathbf{B}) = v(1, \mathbf{B})$, defined for Borel subset \mathbf{B} of \mathbf{M} such that $\mathbf{B} \subset \bigcup_{n=1}^J \mathbf{M}_n$ with some natural number J , can be extended to a Radon measure on \mathbf{M} . Simultaneously, since $\alpha > 0$ by virtue of Lemma 2.1, the set function $\rho_\alpha([t_1; t_2]) = t_1^{-\alpha} - t_2^{-\alpha}$, defined for all pairs of positive numbers t_1 and t_2 such that $t_1 < t_2$, can be extended to a Radon measure on $(0; \infty)$. Therefore, it follows from (2.7) and (2.9) that

$$(2.10) \quad \Gamma = \rho_\alpha \otimes \nu,$$

where \otimes denotes the Radon product of two measures defined on $(0; +\infty)$ and \mathbf{M} , giving a measure on $m(\mathbf{S}) \setminus \{\mathbf{0}\} = (0; +\infty) \times \mathbf{M}$ (see (2.2)).

Combining (1.1) and (2.10) we get

$$\begin{aligned} -\log L(\boldsymbol{\mu}, g) &= \mathbf{a} \cdot g + \int_{m(\mathbf{S}) \setminus \{\mathbf{0}\}} (1 - e^{-x \cdot g}) \Gamma(d\mathbf{x}) \\ &= \mathbf{a} \cdot g + \int_{(0; +\infty) \times \mathbf{M}} (1 - e^{-(t, \mathbf{u}) \cdot g}) \rho_\alpha \otimes \nu(d(t, \mathbf{u})) \\ &= \mathbf{a} \cdot g + \int_{\mathbf{M}} \int_0^\infty (1 - e^{-t\mathbf{u} \cdot g}) \rho_\alpha(dt) \nu(d\mathbf{u}) \\ &= \mathbf{a} \cdot g + \int_{\mathbf{M}} \int_0^\infty (1 - e^{-t\mathbf{u} \cdot g}) d(-t^{-\alpha}) \nu(d\mathbf{u}), \end{aligned}$$

that implies

$$(2.11) \quad -\log L(\boldsymbol{\mu}, g) = \mathbf{a} \cdot g + \int_{\mathbf{M}} (\mathbf{u} \cdot g)^\alpha \int_0^\infty \frac{\alpha \cdot (1 - e^{-t})}{t^{1+\alpha}} dt \nu(d\mathbf{u}).$$

It is clear that the function $(1 - e^{-t})/t^{1+\alpha}$ is integrable if and only if $0 < \alpha < 1$. From the assumption of $\boldsymbol{\mu} \neq \delta(\mathbf{0})$ we see that $L(\boldsymbol{\mu}, g) > 0$ and the right hand side of (2.11) must be a finite quantity. Therefore, $\nu = \mathbf{0}$ and $-\log L(\boldsymbol{\mu}, g) = \mathbf{a} \cdot g$ if $\alpha = 1$.

In the case when $\alpha \neq 1$, the equality (2.5) yields $\mathbf{a} = \mathbf{0}$. Besides, due to the fact that the integral in the right hand side of (2.11) is finite, by virtue of Fubini's Theorem, we can conclude that

$$(2.12) \quad w(\alpha) := \int_0^\infty \frac{\alpha \cdot (1 - e^{-t})}{t^{1+\alpha}} dt < \infty,$$

that ensures $0 < \alpha < 1$. Then, due to (2.11) and (2.12), the condition (2.4) holds with $\mathbf{A}(\alpha) := w(\alpha) \cdot \nu$.

Conversely, let (2.4) be true. Then for every natural number $n = 2, 3, \dots$, and for all $g \in \Phi(\mathbf{S})$, we get

$$\begin{aligned} -\log L(\boldsymbol{\mu}^{*n}, g) &= -\log(L(\boldsymbol{\mu}, g))^n = -n \cdot \log L(\boldsymbol{\mu}, g) \\ &= n \cdot \int_{\mathbf{M}} (\mathbf{u} \cdot g)^\alpha \mathbf{A}(d\mathbf{u}) = \int_{\mathbf{M}} n \cdot (\mathbf{u} \cdot g)^\alpha \mathbf{A}(d\mathbf{u}) \\ &= \int_{\mathbf{M}} (n^{1/\alpha} \cdot \mathbf{u} \cdot g)^\alpha \mathbf{A}(d\mathbf{u}) = \int_{\mathbf{M}} (\mathbf{u} \cdot g)^\alpha \mathbf{A}(d(n^{-1/\alpha} \cdot \mathbf{u})) \\ &= -\log L(T_{n^{1/\alpha}} \boldsymbol{\mu}, g), \end{aligned}$$

that means $\mu^{*n} = T_{n^{1/\alpha}}\mu$, (2.1) is satisfied and μ is α -stable.

By the same argument as the above, we can show that if (2.3) is true then μ is 1-stable. The proof completes. \square

The following corollary is resulted straight from the above theorem.

Corollary 2.3. *The stability exponent α of a r.m. μ is a positive number such that $0 < \alpha \leq 1$.*

Notice 1. The characterization of stable random measures in Theorem 2.2 is similar to those of the works by Nguyen Van Thu [12] and Davydov et al. [3]. However, the one of Theorem 2.2 is more useful to investigate problems of domain of attraction. In particular, it has been used to prove Theorem 3.5 and Theorem 3.6.

From (2.3), it is clear that if μ is 1-stable then it is a trivial probability measure concentrated at a point $\mathbf{a} \in m(\mathbf{S})$, that is $\mu = \delta(\mathbf{a})$. In the follows we will concern with α -stable r.m.'s for $0 < \alpha < 1$.

3. Domain of attraction of stable random measures

Definition. For $\lambda, \mu \in \Pi(\mathbf{S})$, we say that λ belongs to the *domain of attraction* of μ (in symbols $\lambda \in DA(\mu)$) if

$$(3.1) \quad T_{a_n}\lambda^{*n} \rightarrow_w \mu$$

for some sequence of positive numbers $\{a_n\}$.

The next two lemmas are an immediate results of Theorem 4.2 [8].

Lemma 3.1. *Let the condition (3.1) be true with $\mu \neq \delta(\mathbf{0})$. Then $a_n \rightarrow 0$ and $a_n/a_{n+1} \rightarrow 1$ as $n \rightarrow \infty$.*

Lemma 3.2. *Let $\{\lambda_n\}$ be a sequence of r.m.'s, $\{a_n\}$ be a sequence of positive numbers such that $\lambda_n \rightarrow_w \mu$ and $T_{a_n}\lambda_n \rightarrow_w \lambda$ as $n \rightarrow \infty$, where $\mu \neq \delta(\mathbf{0})$ and $\lambda \neq \delta(\mathbf{0})$. Then there exists a positive number c such that $a_n \rightarrow c$ and $\lambda = T_c\mu$.*

The following proposition gives alternative definitions of stability of r.m., which are useful to investigate domains of attraction.

Proposition 3.3. *Let μ be an arbitrary inf.div. r.m. Then the following conditions are equivalent:*

- i) *The r.m. μ is α -stable for some real number $\alpha \in (0; 1)$;*
- ii) *For every natural number n there is a positive number a_n such that*

$$(3.2) \quad \mu^{*n} = T_{a_n}\mu;$$

- iii) *For every positive number r there is a positive number b_r such that*

$$(3.3) \quad \mu^{*r} = T_{b_r}\mu.$$

Proof. The implications i)⇒ii) and iii)⇒ii) are trivial, we will prove the implications ii)⇒iii) and iii)⇒i). Suppose that (3.2) is true and a positive number r is given. Then there exist two sequences n_i and n_k of natural numbers such that $n_i/n_k \rightarrow r$. From (3.2) we get

$$\mu = T_{1/a_{n_k}} \mu^{*n_k} = T_{a_{n_i}/a_{n_k}} (T_{1/a_{n_i}} \mu^{*n_i})^{*n_k/n_i} = T_{a_{n_i}/a_{n_k}} \mu^{*n_k/n_i}$$

and Lemma 3.2 implies $\mu = T_c \mu^{*1/r}$ for some positive c . That means the condition (3.3) holds with $b_r = c$, the implication ii)⇒iii) is proved.

Suppose that (3.3) is correct. Let r and s be two arbitrary positive number. Then

$$\mu = T_{b_r^{-1}} \mu^{*r} = T_{b_s^{-1}} \mu^{*s}.$$

Taking $\alpha = 1/\log_r(b_r)$ and $\beta = 1/\log_s(b_s)$, we get

$$\mu = T_{r^{-1/\alpha}} \mu^{*r} = T_{s^{-1/\beta}} \mu^{*s}.$$

We will show $\beta = \alpha$, that ensures

$$\mu = T_{n^{-1/\alpha}} \mu^{*n} = T_{m^{-1/\alpha}} \mu^{*m}$$

saying that (2.1) is valid with an unique α for all natural numbers n , the implication iii)⇒i) holds.

At first, for the case when $s = r^k$, $k = 1, 2, \dots$, we see that

$$\begin{aligned} \mu &= T_{s^{-1/\beta}} \mu^{*r^k} = T_{s^{-1/\beta}} T_{r^{1/\alpha}} T_{r^{-1/\alpha}} \mu^{*r^k} = T_{s^{-1/\beta}} (T_{r^{1/\alpha}} \mu)^{*r^{k-1}} \\ &= \dots = T_{s^{-1/\beta}} T_{r^{k/\alpha}} \mu = T_{r^{-k/\beta, r^{k/\alpha}}} \mu = T_{r^{(-\frac{k}{\beta} + \frac{k}{\alpha})}} \mu. \end{aligned}$$

This yields

$$r^{(-\frac{k}{\beta} + \frac{k}{\alpha})} = 1 \quad \text{and} \quad -\frac{k}{\beta} + \frac{k}{\alpha} = 0,$$

that means $\alpha = \beta$.

Secondly, for $s = r^{1/m}$, $m = 1, 2, \dots$, by symmetry we also have $\alpha = \beta$. Therefore, the equality $\alpha = \beta$ is correct for the case when $s = r^{k/m}$, i.e., when $s = r^q$ with an arbitrary positive rational number q .

Finally, let $s = r^c$ with $c > 0$. Then $c = \lim_n q_n$ for some increasing sequence $\{q_n\}$ of rational numbers with $0 < q_n < c$ for all n . We have $c = q_n + d_n$ with $d_n > 0$ and $d_n \rightarrow 0+$ as $n \rightarrow \infty$. In that case,

$$\mu = T_{s^{-1/\beta}} \mu^{*s} = T_{s^{-1/\beta}} \mu^{*r^c} = T_{s^{-1/\beta}} \mu^{*r^{q_n+d_n}} = T_{r^{-c/\beta}} T_{r^{q_n/\alpha}} \mu^{*r^{d_n}}.$$

When $n \rightarrow \infty$, $d_n \rightarrow 0$, $q_n \rightarrow c$ and $r^{d_n} \rightarrow 1$. Then $\mu = T_{r^{(-c/\beta+c/\alpha)}} \mu$, which yields $r^{(-c/\beta+c/\alpha)} = 1$, that means $\alpha = \beta$. The proof completes. \square

Proposition 3.4. *Let $\mu \in \Pi(\mathbf{S})$ be a r.m. such that $\mu \neq \delta(\mathbf{0})$. Then μ is α -stable, for some $\alpha \in (0; 1)$, if and only if its domain of attraction is not empty, $DA(\mu) \neq \emptyset$.*

Proof. From (2.1) it is clear that if μ is α -stable then μ belongs to its domain of attraction, $\mu \in AD(\mu)$, and $DA(\mu) \neq \emptyset$.

Conversely, if (3.1) fulfils with some $\lambda \in \Pi(\mathbf{S})$ then for every fixed natural number m we have

$$T_{a_{m,n}} \lambda^{*m,n} \rightarrow_w \mu$$

as $n \rightarrow \infty$, that yields

$$T_{a_{m,n}/a_n} (T_{a_n} \lambda^{*n})^{*m} \rightarrow_w \mu.$$

Therefore, by virtue of Lemma 3.2, there exists a positive number c such that $a_{m,n}/a_n \rightarrow c$ and

$$T_{1/a_m} \mu^{*m} = \mu$$

with $a_m = 1/c$, this implies (3.2). Then Proposition 3.3 ensures that μ is α -stable for some positive α , the proof is finished. \square

In the what follows we will characterize domains of attraction by using the concept of regular variation of functions, which is given in the next definition.

Definition. A positive measurable function f is said to be *regularly varying at ∞* if there exists a real constant γ such that, for every $s > 0$,

$$\lim_{t \rightarrow \infty} \frac{f(t \cdot s)}{f(t)} = s^\gamma.$$

The real number γ is called *variation order*.

Theorem 3.5. *Let $\lambda, \mu \in \Pi(\mathbf{S})$, $\alpha \in (0; 1)$ be given. Suppose that μ is α -stable and $\lambda \in DA(\mu)$. Then, for every function $g \in \Phi(\mathbf{S})$ and for every positive number s , we have*

$$\lim_{t \rightarrow \infty} \frac{-\log L(\lambda, t \cdot s \cdot g)}{-\log L(\lambda, t \cdot g)} = s^\alpha,$$

that means the function $f_g(t) = -\log L(\lambda, t \cdot g)$ is regularly varying at ∞ with order α .

Proof. From the definition of Laplace transform, it is clear that $f_g(t)$ is an increasing continuous function. Besides, by the assumption of $\lambda \in DA(\mu)$, (3.1) is true for some sequence $\{a_n\}$ of positive numbers. Then from [7, Theorem 4.2] and Theorem 2.2, we get

$$\lim_{n \rightarrow \infty} \frac{-\log L(T_{a_n} \lambda^{*n}, s \cdot g)}{-\log L(T_{a_n} \lambda^{*n}, g)} = \frac{-\log L(\mu, s \cdot g)}{-\log L(\mu, g)} = s^\alpha$$

for all positive numbers s . Therefore,

$$(3.4) \quad \lim_{n \rightarrow \infty} \frac{f_g(a_n^{-1} \cdot s)}{f_g(a_n^{-1})} = \lim_{n \rightarrow \infty} \frac{-\log L(\lambda, a_n^{-1} \cdot s \cdot g)}{-\log L(\lambda, a_n^{-1} \cdot g)} = s^\alpha.$$

We affirm that

$$(3.5) \quad \lim_{t \rightarrow \infty} \frac{f_g(t \cdot s)}{f_g(t)} = s^\alpha.$$

Indeed, from Lemma 3.1 we see that $a_n \rightarrow 0$ and $a_n^{-1} \rightarrow \infty$ when $n \rightarrow \infty$. For every positive number t such that $t > \min\{a_1^{-1}, a_2^{-1}, \dots\}$ let

$$n(t) = \max\{m : a_m^{-1} < t\}.$$

Then $a_{n(t)}^{-1} < t \leq a_{n(t)+1}^{-1}$ and, since $f_g(t)$ is increasing,

$$(3.6) \quad \frac{f_g(a_{n(t)}^{-1} \cdot s)}{f_g(a_{n(t)+1}^{-1})} \leq \frac{f_g(t \cdot s)}{f_g(t)} \leq \frac{f_g(a_{n(t)+1}^{-1} \cdot s)}{f_g(a_{n(t)}^{-1})}.$$

Besides, because the function $x \mapsto x^{1/c}$ is convex for every positive number $c < 1$, from Jensen inequality we have

$$\begin{aligned} f_g(c \cdot t) &= -\log L(\boldsymbol{\lambda}, c \cdot t \cdot g) = -\log \int_{m(\mathbf{S})} e^{-\mathbf{x} \cdot c \cdot t \cdot g} \boldsymbol{\lambda}(d\mathbf{x}) \\ &\geq -\log \left(\int_{m(\mathbf{S})} e^{-\mathbf{x} \cdot t \cdot g} \boldsymbol{\lambda}(d\mathbf{x}) \right)^c = -c \cdot \log L(\boldsymbol{\lambda}, t \cdot g) = c \cdot f_g(t). \end{aligned}$$

Therefore, (3.6) implies

$$\frac{a_{n(t)+1}}{a_{n(t)}} \cdot \frac{f_g(a_{n(t)}^{-1} \cdot s)}{f_g(a_{n(t)}^{-1})} \leq \frac{f_g(t \cdot s)}{f_g(t)} \leq \frac{a_{n(t)}}{a_{n(t)+1}} \cdot \frac{f_g(a_{n(t)}^{-1} \cdot s)}{f_g(a_{n(t)}^{-1})},$$

that together with (3.4) and Lemma 3.1 yields (3.5), because $n(t) \rightarrow \infty$ when $t \rightarrow \infty$. The proposition is proved. \square

For every natural number k ; every Borel subset \mathbf{B} of \mathbf{M} ; and every pair of positive numbers u, v such that $u < v$, let denote

$$\begin{aligned} \mathbf{M}_{[u; \infty)}^k &= \{t \cdot \mathbf{x} : \mathbf{x} \in \mathbf{M}_1 \cup \dots \cup \mathbf{M}_k, t \in [u; \infty)\}, \\ \mathbf{B}_{[u; v)} &= \{t \cdot \mathbf{x} : \mathbf{x} \in \mathbf{B}, t \in [u; v)\}, \\ \mathbf{B}_{[u; \infty)} &= \{t \cdot \mathbf{x} : \mathbf{x} \in \mathbf{B}, t \in [u; \infty)\}. \end{aligned}$$

To characterize domains of attraction in more details, we introduce the concept of regular variation of measures in the next definition.

Definition. A Radon measure $\boldsymbol{\lambda}$ on $m(\mathbf{S})$ is said to be *regularly varying at ∞* if there exists a non-zero Radon measure \mathbf{G} on $m(\mathbf{S}) \setminus \{\mathbf{0}\}$; a Borel subset \mathbf{B}^1 of \mathbf{M} ; and a positive number c such that

$$0 < \mathbf{G}(\mathbf{B}_{[c; \infty)}^1) < \infty ; \mathbf{G}(\partial \mathbf{B}_{[c; \infty)}^1) = 0,$$

where $\partial \mathbf{A}$ denotes the boundary of the set \mathbf{A} , and satisfying

$$(3.7) \quad \lim_{t \rightarrow \infty} \frac{\boldsymbol{\lambda}(\mathbf{B}_{[t \cdot u; t \cdot v)}^1)}{\boldsymbol{\lambda}(\mathbf{B}_{[t \cdot c; \infty)}^1)} = \mathbf{G}(\mathbf{B}_{[u; v)})$$

for every pair of positive numbers u, v such that $u < v$, and every Borel subset \mathbf{B} of \mathbf{M} such that $\mathbf{G}(\partial \mathbf{B}_{[u; v)}) = 0$.

Because the sets of the form $\mathbf{B}_{[u;v]}$ generate all Borel subsets of $m(\mathbf{S}) \setminus \{\mathbf{0}\}$, it is clear that (3.7) is equivalent to the next condition,

$$(3.8) \quad \lim_{t \rightarrow \infty} \frac{\lambda(t \cdot \mathbf{C})}{\lambda(\mathbf{B}_{[t;c;\infty)}^1)} = G(\mathbf{C})$$

for every Borel subset \mathbf{C} of $m(\mathbf{S}) \setminus \{\mathbf{0}\}$ with $G(\partial\mathbf{C}) = 0$. Moreover, by virtue of Proposition 2.3 [13], we can prove that there exists a real number ρ such that, for every positive number s ,

$$(3.9) \quad G(s \cdot \mathbf{C}) = s^\rho \cdot G(\mathbf{C}).$$

Besides, (3.8) and (3.9) yield

$$(3.10) \quad G(\mathbf{B}_{[r;\infty)}^1) < \infty$$

for each $r > 0$.

Theorem 3.6. *Let $\lambda \in \Pi(\mathbf{S})$ be given. If λ varies regularly at ∞ , then $\lambda \in DA(\mu)$ for some non-trivial stable r.m. μ . Conversely, if $\lambda \in DA(\mu)$ for some non-trivial stable r.m. μ , then λ is regularly varying at ∞ .*

Proof. Let (3.7) hold, then (3.9) implies $G(\partial\mathbf{B}_{[t;c;\infty)}^1) = 0$ for every positive t . Therefore, (3.8) ensures that the function $h(t) = \lambda(\mathbf{B}_{[t;c;\infty)}^1)$ varies regularly at ∞ . Then, defining

$$a_n = \inf\{s > 0 : n \cdot \lambda(\mathbf{B}_{[s^{-1} \cdot c; \infty)}^1) \geq 1\}$$

for every large enough n , we get $n \cdot \lambda(\mathbf{B}_{[a_n^{-1} \cdot c; \infty)}^1) \rightarrow 1$ as $n \rightarrow \infty$, which together with (3.8) yields

$$\lim_{n \rightarrow \infty} n \cdot \lambda(a_n^{-1} \cdot \mathbf{C}) = G(\mathbf{C})$$

for every Borel subset \mathbf{C} of $m(\mathbf{S}) \setminus \{\mathbf{0}\}$ with $G(\partial\mathbf{C}) = 0$. Combining this with (3.10); Portmanteau Theorem (see Theorem 2.4 in [5] for instance); Theorem 4.2 and Theorem 6.1 in [7], we can ensure that

$$\mathbf{I}(\mathbf{0}, n \cdot T_{a_n} \lambda) \rightarrow_w \mathbf{I}(\mathbf{0}, G).$$

However, because of the equality $\mathbf{I}(\mathbf{0}, n \cdot T_{a_n} \lambda) = \mathbf{I}(\mathbf{0}, T_{a_n} \lambda)^{*n}$ and by virtue of Corollary 6.4 [7], the above convergence implies

$$T_{a_n} \lambda^{*n} \rightarrow_w \mathbf{I}(\mathbf{0}, G),$$

that means $\lambda \in DA(\mu)$, where $\mu = \mathbf{I}(\mathbf{0}, G)$ is a non-trivial stable r.m. due to Proposition 3.2. The first part of theorem is proved.

Let now $\lambda \in DA(\mu)$ where μ is a non-trivial stable r.m., then μ is inf.div. and by virtue of (1.1) and Theorem 2.2 we get $\mu = \mathbf{I}(\mathbf{0}, \Gamma)$ for some spectral measure Γ . Then from Proposition 3.2 there exists a sequence $\{a_n\}$ of positive numbers such that

$$(3.11) \quad T_{a_n} \lambda^{*n} \rightarrow_w \mathbf{I}(\mathbf{0}, \Gamma).$$

Besides, (1.2) and (2.10) imply $0 < \Gamma(\mathbf{M}_{[c;\infty)}^k) < \infty$ for some natural number k . Then, because $m(\mathbf{S})$ is a metric space, there exist a Borel subset \mathbf{B}^1 of \mathbf{M} and a positive number c such that $0 < \Gamma(\mathbf{B}_{[c;\infty)}^1) < \infty$ and $\Gamma(\partial\mathbf{B}_{[c;\infty)}^1) = 0$.

Moreover, (3.11) together with Corollary 6.4 [7] yields

$$I(\mathbf{0}, n \cdot T_{a_n} \lambda) \rightarrow_w I(\mathbf{0}, \Gamma).$$

Combining Theorem 2.4 [5] with Theorem 4.2 and Theorem 6.1 in [7], we ensure that the above convergence implies

$$(3.12) \quad n \cdot T_{a_n} \lambda(\mathbf{C}) \rightarrow \Gamma(\mathbf{C})$$

for every Borel subset \mathbf{C} of $m(\mathbf{S}) \setminus \{\mathbf{0}\}$ such that $\Gamma(\partial\mathbf{C}) = 0$.

Let $\{t_k\}$ be any increasing sequence of positive numbers tending to ∞ as $k \rightarrow \infty$. Then Lemma 3.1 ensures $a_n^{-1} \rightarrow \infty$; $a_n^{-1}/a_{n+1}^{-1} \rightarrow 1$ as $n \rightarrow \infty$; and for each k there is a natural number $n(k)$ such that $a_{n(k)}^{-1} \leq t_k < a_{n(k)+1}^{-1}$. It is clear that $n(k) \rightarrow \infty$ when $k \rightarrow \infty$. Moreover, for each Borel subset \mathbf{B} of \mathbf{M} and each positive number r such that $\Gamma(\partial\mathbf{B}_{[r;\infty)}) = 0$, we have

$$\mathbf{B}_{[a_{n(k)}^{-1}; r; \infty)} \supset \mathbf{B}_{[t_k; r; \infty)} \supset \mathbf{B}_{[a_{n(k)+1}^{-1}; r; \infty)},$$

and therefore,

$$n(k)\lambda(\mathbf{B}_{[a_{n(k)}^{-1}; r; \infty)}) \geq n(k)\lambda(\mathbf{B}_{[t_k; r; \infty)}) \geq \frac{n(k)}{n(k)+1} \cdot (n(k)+1)\lambda(\mathbf{B}_{[a_{n(k)+1}^{-1}; r; \infty)}).$$

Hence, it follows from (3.12) that

$$(3.13) \quad n(k) \cdot \lambda(\mathbf{B}_{[t_k; r; \infty)}) \rightarrow \Gamma(\mathbf{B}_{[r; \infty)})$$

when $k \rightarrow \infty$.

On the other hand, if \mathbf{B} is a Borel subset of \mathbf{M} such that $\Gamma(\partial\mathbf{B}_{[u;v)}) = 0$ for a pair of positive numbers u, v such that $u < v$, then from (2.10) we get $\Gamma(\partial\mathbf{B}_{[r; \infty)}) = 0$ for every positive number r . This together with (3.13) and the fact that $\mathbf{B}_{[u;v)} = \mathbf{B}_{[u; \infty)} \setminus \mathbf{B}_{[v; \infty)}$ implies

$$\lim_{k \rightarrow \infty} n(k) \cdot \lambda(t_k \mathbf{B}_{[u;v)}) = \Gamma(\mathbf{B}_{[u;v)})$$

and

$$\lim_{k \rightarrow \infty} n(k) \cdot \lambda(t_k \mathbf{B}_{[c; \infty)}^1) = \Gamma(\mathbf{B}_{[c; \infty)}^1).$$

Consequently,

$$(3.14) \quad \lim_{k \rightarrow \infty} \frac{\lambda(t_k \cdot \mathbf{B}_{[u;v)})}{\lambda(t_k \cdot \mathbf{B}_{[c; \infty)}^1)} = \lim_{k \rightarrow \infty} \frac{n(k) \cdot \lambda(t_k \cdot \mathbf{B}_{[u;v)})}{n(k) \cdot \lambda(t_k \cdot \mathbf{B}_{[c; \infty)}^1)} = \frac{\Gamma(\mathbf{B}_{[u;v)})}{\Gamma(\mathbf{B}_{[c; \infty)}^1)}.$$

However, because (3.14) is valid for every sequence $\{t_k\}$ increasingly convergent to ∞ , we get

$$\lim_{t \rightarrow \infty} \frac{\lambda(\mathbf{B}_{[t \cdot u; t \cdot v)})}{\lambda(\mathbf{B}_{[t \cdot c; \infty)}^1)} = \frac{\Gamma(\mathbf{B}_{[u;v)})}{\Gamma(\mathbf{B}_{[c; \infty)}^1)}.$$

By setting $\mathbf{G} = \Gamma/\Gamma(\mathbf{B}_{[c;\infty)}^1)$, the above equality confirms (3.7), that means λ is regularly varying at ∞ . The proof completes. \square

Notice 2. Davydov et al. [2] gave results similar to Theorem 3.6 for point processes (Theorem 4.3) and for probability measures on convex cone with sub-invariant norm (Theorem 4.7). It is clear that point processes are special cases of random measures, Theorem 3.6 is a generalization of [2, Theorem 4.3]. Meantime, the convex cone $m(\mathbf{S})$ of non negative Radon measures concerned in this study has no sub-invariant norm, [2, Theorem 4.7] can not be an extension of Theorem 3.6. Moreover, Theorem 3.6 represented both necessary and sufficient condition for random measure belonging to stable random measure, whilst [2, Theorem 4.7] gave only a sufficient one.

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