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STRONG CONVERGENCE OF GENERAL ITERATIVE ALGORITHMS FOR NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we introduce two general iterative algorithms (one implicit algorithm and other explicit algorithm) for nonexpansive mappings in a reflexive Banach space with a uniformly Gâteaux differentiable norm. Strong convergence theorems for the sequences generated by the proposed algorithms are established.

1. Introduction

Let E be a real Banach space with the norm $\|\cdot\|$, and let E^* be the dual space of E. Let J denote the normalized duality mapping from E into 2^{E^*} defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|,\ \|f\| = \|x\|\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pair between E and E^*. Let C be a nonempty closed convex subset of E. For the mapping $T : C \to C$, we denote the fixed point set of T by $Fix(T)$, that is, $Fix(T) = \{x \in C : Tx = x\}$. Recall that the mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x,\ y \in C.$$

In a Banach space E having a single-valued normalized duality mapping J, we say that an operator A is *strongly positive* on E if there exists a $\overline{\gamma} > 0$ with the property

$$\langle Ax, J(x)\rangle \ge \overline{\gamma}\|x\|^2 \tag{1.1}$$

and

$$\|aI - bA\| = \sup_{\|x\|\le 1} |\langle (aI - bA)x, J(x)\rangle|, \quad a \in [0, 1], \quad b \in [-1, 1],$$

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for all $x \in E$, where I is the identity mapping. If $E := H$ is a real Hilbert space, then the inequality (1.1) reduce to

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H.$$

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping. More precisely, take $t \in (0,1)$ and define a contraction $T_t : E \to E$ by

$$T_t x = tu + (1-t)Tx, \quad \forall x \in E,$$

where $u \in E$ is an arbitrarily chosen point. Banach's contraction mapping principle guarantees that T_t has unique a fixed point x_t in E, which uniquely solves the following fixed point equation:

$$x_t = tu + (1-t)Tx_t.$$

(Such a path $\{x_t\}$ is said to be an approximating fixed point of T since it possesses the property that if $\{x_t\}$ is bounded, then $\lim_{t\to 0}\|Tx_t - x_t\| = 0$.) It is unclear, in general, what is the behavior of x_t as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [3] proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of T. Reich [10] extended Browder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then $\{x_t\}$ converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from E onto $Fix(T)$. Xu [16] proved Reich's results hold in reflexive Banach space having a weakly continuous duality mapping.

In a real Hilbert space H, in 2000, Moudafi [9] introduced the following viscosity approximation methods for nonexpansive mapping T on C in an implicit way and an explicit way, respectively:

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 0,$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$; and $f : C \to C$ is a contractive mapping (i.e., there exists a constant $k \in (0,1)$ such that $\|f(x)-f(y)\| \leq k\|x-y\|$, $\forall x, y \in H$).

In 2006, Marino and Xu [8] considered the following general iterative algorithm for nonexpansive mapping T on H in an implicit way:

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t, \quad \forall t \in (0, \min\{1, \|A\|^{-1}\}), \tag{1.3}$$

where $A : H \to H$ is a strongly positive linear bounded operator with a coefficient $\overline{\gamma} > 0$; $f : H \to H$ is a contractive mapping; and $\gamma > 0$. In 2011, Wangkeeree *et al.* [13] extended the result of Marino and Xu [8] to a reflexive Banach space having a weakly continuous duality mapping. The results of Marino and Xu [8] and Wangkeeree *et al.* [13] improved upon the corresponding results of Browder [3], Moudafi [9], Reich [10] and Xu [16] to a general approximating fixed point $\{x_t\}$ defined by (1.3). Combining the Moudafi's method

(1.2) with Xu's method [15], Marino and Xu [8] also considered the following general iterative algorithm for a nonexpansive mapping T in an explicit way:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad \forall n \geq 0, \tag{1.4}$$

where f is a contractive mapping on H; and $\gamma > 0$. They proved that if the sequence $\{\alpha_n\}$ in $(0, 1)$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution of a certain variational inequality related to A.

In this paper, as a continuation of study in this direction, we present new general iterative algorithms for the nonexpansive mapping in a reflexive Banach space with a uniformly Gâteaux differentiable norm. First, we introduce a general implicit iterative algorithm. Consequently, by discretizing the continuous implicit method, we provide a general explicit iterative algorithm for finding a fixed point of the nonexpansive mapping. Under some control conditions, we establish the strong convergence of the proposed explicit algorithm to a fixed point of the mapping, which solves a ceratin variational inequality.

2. Preliminaries and lemmas

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual.

A Banach space E is called *strictly convex* if its unit sphere $U = \{x \in E : \|x\| = 1\}$ does not contain any linear segment. For every ε with $0 \leq \varepsilon \leq 2$, the modulus $\delta(\varepsilon)$ of convexity of E is defined by

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : \|x\| \leq 1, \quad \|y\| \leq 1, \quad \|x - y\| \geq \varepsilon\}.$$

E is said to be *uniformly convex* if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then E is reflexive and strictly convex.

The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be *uniformly Gâteaux differentiable* if for each $y \in U$, this limit is attained uniformly for $x \in U$. Finally, the norm is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit in (2.1) is attained uniformly for $(x, y) \in U \times U$. Since the dual E^* of E is uniformly convex if and only if the norm of E is uniformly Fréchet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gâteaux differentiable norm. The converse implication is false. A discussion of these and related concepts may be found in [5].

Let J be the normalized duality mapping from E into 2^{E^*}. It is well-known that J is single valued if and only if E is smooth, and that if E has a uniformly Gâteaux differentiable norm, J is uniformly continuous on bounded subsets of

E from the strong topology of E to the weak* topology of E^*. For these facts, see [5, 12].

Let LIM be a linear continuous functional on ℓ^∞. According to time and circumstances, we use $LIM_n(a_n)$ instead of $LIM(a)$ for every $a = \{a_n\} \in \ell^\infty$. LIM is called a *Banach limit* if $\|LIM\| = LIM(1) = 1$ and $LIM_n(a_{n+1}) = LIM_n(a_n)$ for every $a = \{a_n\} \in \ell^\infty$.

Recall that a closed convex subset C of E is said to have the *fixed point property* for nonexpansive self-mappings (FPP for short) if every nonexpansive mapping $T : C \to C$ has a fixed point, that is, there is a point $p \in C$ such that $Tp = p$. It is well-known that every bounded closed convex subset of a uniformly smooth Banach space has the FPP ([7, p. 45]).

The mapping $T : C \to C$ is said to be *pseudocontractive* if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2, \quad \forall x,\ y \in C,$$

and T is said to be *strongly pseudocontractive* it there exists a constant $k \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le k\|x-y\|^2, \quad \forall x,\ y \in C.$$

We need the following lemmas for the proof of our main results.

Lemma 2.1 ([5]). *Let E be a Banach space, let C be a nonempty closed convex subset of E, and let $T : C \to C$ be a continuous strongly pseudocontractive mapping. Then T has a fixed point in C.*

Lemma 2.2 ([4]). *Assume that A is a strongly positive linear bounded operator on a smooth Banach space E with coefficient $\overline{\gamma} > 0$ and $0 < \rho < \|A\|^{-1}$. Then $\|I - \rho A\| \le 1 - \rho\overline{\gamma}$.*

Lemma 2.3 ([14]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \le (1-\lambda_n)s_n + \lambda_n\delta_n + \omega_n, \quad \forall n \ge 1,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and ω_m satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=1}^\infty \lambda_n = \infty$ *or, equivalently,* $\prod_{n=1}^\infty (1-\lambda_n) = 0$;
(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^\infty \lambda_n|\delta_n| < \infty$;
(iii) $\omega_n \ge 0$ *and* $\sum_{n=1}^\infty \omega_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 ([11]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E such that*

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)y_n, \quad \forall n \ge 0,$$

where $\{\lambda_n\}$ is a sequence in $[0,1]$ such that

$$0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 1.$$

Assume that

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Then $\lim_{n\to\infty}\|y_n-x_n\|=0$.

Lemma 2.5 ([1, 2])**.** *Let C be a closed convex of a reflexive and strictly convex Banach space E. Then $C^o=\{x\in C:\|x\|=\inf\{\|y\|:y\in C\}\}$ is a singleton.*

Lemma 2.6. *Let E be a smooth Banach space. Then there holds*

$$\|x+y\|^2\le\|x\|^2+2\langle y,J(x+y)\rangle,\quad \forall x,\ y\in E.$$

3. Main results

Throughout the rest of this paper, we always assume the following:

- E is a real smooth Banach space;
- C is a nonempty closed subspace of E;
- $A:C\to C$ is a strongly positive linear bounded operator with a constant $\overline{\gamma}>0$;
- $h:C\to C$ is a continuous bounded strongly pseudocontractive mapping with a pseudocontractive coefficient $k\in(0,1)$;
- The constant $\gamma>0$ satisfies $0<\gamma<\frac{\overline{\gamma}}{k}$;
- $T:C\to C$ is a nonexpansive mapping with $Fix(T)\neq\emptyset$.

In this section, first, we introduce the following general iterative algorithm that generates a net $\{x_t\}$, $t\in(0,\min\{1,\|A\|^{-1}\})$ in an implicit way:

$$x_t=t\gamma h(x_t)+(I-tA)Tx_t. \tag{3.1}$$

Now, for $t\in(0,\min\{1,\|A\|^{-1}\})$, consider the mapping $G_t:C\to C$ defined by

$$G_t(x):=t\gamma h(x)+(I-tA)Tx,\quad x\in C.$$

Then G_t is a continuous strongly pseudocontractive mapping with a pseudocontractive coefficient $1-t(\overline{\gamma}-\gamma k)\in(0,1)$. Indeed, from Lemma 2.2 we have for each $x,\ y\in C$,

$$\begin{aligned}
&\langle G_tx-G_ty,J(x-y)\rangle\\
&=t\gamma\langle h(x)-h(y),J(x-y)\rangle+\langle (I-tA)(Tx-Ty),J(x-y)\rangle\\
&\le t\gamma k\|x-y\|^2+\|I-tA\|\|Tx-Ty\|\|x-y\|\\
&\le t\gamma k\|x-y\|^2+(1-t\overline{\gamma})\|x-y\|^2\\
&=(1-t(\overline{\gamma}-\gamma k))\|x-y\|^2.
\end{aligned}$$

Thus, by Lemma 2.1, G_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation (3.1).

We summarize the basic properties of $\{x_t\}$.

Proposition 3.1. *Let $\{x_t\}$ be defined via* (3.1)*. Then the following hold*:

(a) *x_t is a unique path $t\mapsto x_t\in C$, $t\in(0,\min\{1,\|A\|^{-1}\})$.*

(b) *If v is a fixed point of T, then for each $t \in (0, \min\{1, \|A\|^{-1}\})$*

$$\langle (A - \gamma h)x_t, J(x_t - v)\rangle \le \langle A(I - T)x_t, J(x_t - v)\rangle.$$

(c) *If T has a fixed point in C, then the path $\{x_t\}$ is bounded and $\|x_t - Tx_t\| \to 0$ as $t \to 0$.*

Proof. (a) To see the continuity of x_t, let $t,\ t_0 \in (0, \min\{1, \|A\|^{-1}\})$. Then we get

$$\begin{aligned}
&\|x_t - x_{t_0}\|^2 \\
&= \langle t\gamma h(x_t) + (I - tA)Tx_t - (t_0\gamma h(x_{t_0}) + (I - t_0A)Tx_{t_0}), J(x_t - x_{t_0})\rangle \\
&= \langle (t - t_0)\gamma h(x_t) + t_0\gamma(h(x_t) - h(x_{t_0})) - (t - t_0)ATx_t, J(x_t - x_{t_0})\rangle \\
&\quad + \langle (I - t_0A)(Tx_t - Tx_{t_0}), J(x_t - x_{t_0})\rangle \\
&\le (\gamma\|h(x_t)\| + \|ATx_t\|)(t - t_0)\|x_t - x_{t_0}\| + t_0\gamma k\|x_t - x_{t_0}\|^2 \\
&\quad + (1 - t_0\overline{\gamma})\|x_t - x_{t_0}\|^2.
\end{aligned}$$

It follows that

$$\|x_t - x_{t_0}\| \le \frac{\gamma\|h(x_t)\| + \|ATx_t\|}{t_0(\overline{\gamma} - \gamma k)}|t - t_0|.$$

This shows that x_t is locally Lipschitzian and hence continuous.

(b) Suppose that v is a fixed point of T. Since T is nonexpansive, we have for all $x,\ y \in C$

$$\begin{aligned}
\langle (I - T)x - (I - T)y, J(x - y)\rangle &= \|x - y\|^2 - \langle Tx - Ty, J(x - y)\rangle \\
&\ge \|x - y\|^2 - \|x - y\|^2 = 0.
\end{aligned}$$

Thus, from (3.1) we obtain

$$\begin{aligned}
\langle (A - \gamma h)x_t, J(x_t - v)\rangle &= -\frac{1}{t}\langle (I - tA)(I - T)x_t, J(x_t - v)\rangle \\
&= -\frac{1}{t}\langle (I - T)x_t - (I - T)v, J(x_t - v)\rangle \\
&\quad + \langle A(I - T)x_t, J(x_t - v)\rangle \\
&\le \langle A(I - T)x_t, J(x_t - v)\rangle.
\end{aligned}$$

(c) Let $v \in Fix(T)$. From strong pseudocontractivity of h, it follows that

$$\langle h(x_t) - h(v), J(x_t - v)\rangle \le k\|x_t - v\|^2.$$

Thus we have

$$\begin{aligned}
\|x_t - v\|^2 &= \langle (I - tA)(Tx_t - v) + t(\gamma h(x_t) - Av), J(x_t - v)\rangle \\
&\le (1 - t\overline{\gamma})\|x_t - v\|^2 + t\langle \gamma h(x_t) - Av, J(x_t - v)\rangle \\
&= (1 - t\overline{\gamma})\|x_t - v\|^2 + t\gamma\langle h(x_t) - h(v), J(x_t - v)\rangle \\
&\quad + t\langle \gamma h(v) - Av, J(x_t - v)\rangle \\
&\le (1 - t\overline{\gamma})\|x_t - v\|^2 + t\gamma k\|x_t - v\|^2 + t\|\gamma h(v) - Av\|\|x_t - v\|.
\end{aligned}$$

It follows that

$$\|x_t - v\| \le \frac{\|\gamma h(v) - Av\|}{\overline{\gamma} - \gamma k}.$$

Hence $\{x_t\}$ is bounded for $t \in (0, \min\{1, \|A\|^{-1}\})$. Since $\|Tx_t - v\| \le \|x_t - v\|$, $\{Tx_t\}$ is bounded and so are $\{ATx_t\}$ and $\{Ax_t\}$. Moreover, since h is a bounded mapping, $\{h(x_t)\}$ is bounded. This implies that

$$\|x_t - Tx_t\| = t\|\gamma h(x_t) - ATx_t\| \to 0 \quad \text{as } t \to 0. \qquad \square$$

Using Proposition 3.1, we establish strong convergence of $\{x_t\}$.

Theorem 3.2. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Assume that every weakly compact convex subset of E has the FPP for nonexpansive mappings. Let $\{x_t\}$ be defined via* (3.1). *Then, as $t \to 0$, $\{x_t\}$ converges strongly to a fixed point p of T, which is the unique solution in $Fix(T)$ of the variational inequality*

$$\langle (A - \gamma h)p, J(p - q)\rangle \le 0, \quad \forall q \in Fix(T). \tag{3.2}$$

Proof. First, we show the uniqueness of the solution of the variational inequality (3.2). Suppose both $p_1 \in Fix(T)$ and $p_2 \in Fix(T)$ are solutions of the variational inequality (3.2). We have

$$\langle (A - \gamma h)p_1, J(p_1 - p_2)\rangle \le 0$$

and

$$\langle (A - \gamma h)p_2, J(p_2 - p_1)\rangle \le 0.$$

Adding up the above two inequalities, we obtain

$$\langle (A - \gamma h)p_1 - (A - \gamma h)p_2, J(p_1 - p_2)\rangle \le 0.$$

Note that

$$\begin{aligned}\langle (A - \gamma h)p_1 - (A - \gamma h)p_2, J(p_1 - p_2)\rangle &= \langle A(p_1 - p_2)J(p_1 - p_2)\rangle \\ &\quad - \gamma\langle h(p_1) - h(p_2), J(p_1 - p_2)\rangle \\ &\ge \overline{\gamma}\|p_1 - p_2\|^2 - \gamma k\|p_1 - p_2\|^2 \\ &= (\overline{\gamma} - \gamma k)\|p_1 - p_2\|^2 \ge 0.\end{aligned}$$

Consequently, we have $p_1 = p_2$ and the uniqueness is proved. We use $\widetilde{p}$ to the unique solution of the variational inequality (3.2).

Now, we may assume, without loss of generality, that $t \le \|A\|^{-1}$. From Proposition 3.1(c), we have that $\{x_t\}$ is bounded.

Assume that $t_n \to 0$ as $n \to \infty$. Set $x_n := x_{t_n}$. We use the so-called optimization method. Define $\phi : C \to \mathbb{R}$ by $\phi(z) = LIM_n(\|x_n - z\|^2)$, $z \in C$, where LIM is a Banach limit on l^∞. Then ϕ is continuous and convex, $\phi(z) \to \infty$ as $\|z\| \to \infty$. Since E is reflexive, ϕ attains its infimum over C ([2, p. 79]). Let

$$K = \{u \in C : \phi(u) = \min_{z \in C} \phi(z)\}.$$

We see easily that K is a nonempty closed bounded convex subset of E. Note that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ by Proposition 3.1(c). Thus, it follows that for each $u \in K$,

$$\begin{aligned}\phi(Tu) &= LIM_n(\|x_n - Tu\|^2)\\ &= LIM_n(\|Tx_n - Tu\|^2)\\ &\le LIM_n(\|x_n - u\|^2) = \phi(u),\end{aligned}$$

which implies that $T(K) \subset K$, that is, K is invariant under T. So, by the hypothesis, T has a fixed point $p \in K$. For $x - Ap \in C$ and t with $0 < t < \min\{1, \|A\|^{-1}\}$, by Lemma 2.6, we get

$$\|x_n - p - t(x - Ap)\|^2 \le \|x_n - p\|^2 - 2t\langle x - Ap, J(x_n - p - t(x - Ap)\rangle.$$

Let $\varepsilon > 0$ be given. Since the norm of E is uniformly Gâteaux differentiable, the duality mapping J is norm-to-weak* uniformly continuous on bounded subsets of E. Therefore

$$|\langle x - Ap, J(x_n - p - t(x - Ap) - J(x_n - p))\rangle| < \varepsilon$$

for t is close enough to 0. Consequently, we have

$$\begin{aligned}\langle x - Ap, J(x_n - p)\rangle &< \varepsilon + \langle x - Ap, J(x_n - p - t(x - Ap))\rangle\\ &\le \varepsilon + \frac{1}{2t}(\|x_n - p\|^2 - \|x_n - p - t(x - Ap)\|^2).\end{aligned}$$

Since p is a minimizer of ϕ over C, we have

$$\begin{aligned}&LIM_n(\langle x - Ap, J(x_n - p)\rangle)\\ &\le \varepsilon + \frac{1}{2t}(LIM_n(\|x_n - p\|^2) - LIM_n(\|x_n - p - t(x - Ap)\|^2))\\ &\le \varepsilon.\end{aligned}$$

Thus, we obtain

$$LIM_n(\langle x - Ap, J(x_n - p)\rangle) \le 0, \quad \forall x \in C. \tag{3.3}$$

On the other hand, since $x_n - p = t_n(\gamma h(x_n) - Ap) + (I - t_nA)(Tx_n - p)$, it follows that

$$\begin{aligned}\|x_n - p\|^2 &= t_n\langle \gamma h(x_n) - Ap, J(x_n - p)\rangle + \langle (I - t_nA)(Tx_n - p), J(x_n - p)\rangle\\ &\le t_n\langle \gamma h(x_n) - Ap, J(x_n - p)\rangle + (1 - t_n\overline{\gamma})\|x_n - p\|^2,\end{aligned}$$

which implies that for $x \in C$,

$$\begin{aligned}\|x_n - p\|^2 &\le \frac{1}{\overline{\gamma}}\langle \gamma h(x_n) - Ap, J(x_n - p)\rangle\\ &= \frac{1}{\overline{\gamma}}\langle \gamma h(x_n) - x, J(x_n - p)\rangle + \frac{1}{\overline{\gamma}}\langle x - Ap, J(x_n - p)\rangle.\end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$\begin{aligned} &LIM_n(\|x_n - p\|^2) \\ &\leq \frac{1}{\overline{\gamma}} LIM_n(\langle \gamma h(x_n) - x, J(x_n - p)\rangle) + \frac{1}{\overline{\gamma}} LIM_n(\langle x - Ap, J(x_n - p)\rangle) \\ &\leq \frac{1}{\overline{\gamma}} LIM_n(\langle \gamma h(x_n) - x, J(x_n - p)\rangle). \end{aligned}$$

In particular,

$$\overline{\gamma} LIM_n(\|x_n - p\|^2) \leq LIM_n(\langle \gamma h(x_n) - \gamma h(p), J(x_n - p)\rangle) \leq \gamma k LIM_n(\|x_n - p\|^2).$$

Hence, $(\overline{\gamma} - \gamma k) LIM_n(\|x_n - p\|^2) \leq 0$. Since $\overline{\gamma} > \gamma k$, we have

$$LIM_n(\|x_n - p\|^2) = 0,$$

and hence there exists a subsequence which is still denoted $\{x_n\}$ such that $x_n \to p$

Next, we prove that p solves the variational inequality (3.2). Indeed, from Proposition 3.1(b), we have for $q \in Fix(T)$,

$$\langle (A - \gamma h)x_t, J(x_t - q)\rangle \leq \langle A(I - T)x_t, J(x_t - q)\rangle.$$

Replacing t with t_n, letting $n \to \infty$ and noting that $(I-T)x_{t_n} \to (I-T)p = 0$, we obtain

$$\langle (A - \gamma h)p, J(p - q)\rangle \leq 0.$$

That is, $p \in Fix(T)$ is a solution of the variational inequality (3.2). Then $p = \widetilde{p}$. In summary, we have that each cluster point of $\{x_n\}$ converges strongly to p as $t_n \to 0$. This complete the proof. □

Next, we substitute the fixed point property assumption, mentioned in Theorem 3.2, by assuming that the space E is strict convex.

Theorem 3.3. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Let $\{x_t\}$ be defined via* (3.1)*. Then, as $t \to 0$, $\{x_t\}$ converges strongly to a fixed point p of T, which is the unique solution in $Fix(T)$ of the variational inequality* (3.2)*.*

Proof. Let $w \in Fix(T)$. As in the proof of Theorem 3.2, we define $\phi : C \to \mathbb{R}$ by $\phi(z) = LIM_n(\|x_n - z\|^2)$, $z \in C$, where LIM is a Banach limit on l^∞. Let

$$K = \{u \in C : \phi(u) = \min_{z \in C} \phi(z)\}.$$

Then, by the proof of Theorem 3.2, K is invariant under T, Moreover K contains a fixed point of T. To this end, define the function $g : K \to \mathbb{R}$ by $g(u) = \|u - w\|$. Then, by Theorem 1.2 of [2] (or Theorem 2.5.7 of [1]) we conclude that the set

$$K^o = \{v \in K : g(v) = \min\{g(u) : u \in K\}\}$$

is nonempty, and by Lemma 2.5, K^o is singleton. Denote such a singleton by $p \in K$. Then we also know that $Tw = w$ and

$$\|Tp - w\| = \|Tp - Tw\| \leq \|p - w\|.$$

Therefore $Tp = p$. We now follows the proof of Theorem 3.2. □

Now, we propose the following general iterative algorithm which generates a sequence in an explicit way:

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = \alpha_n \gamma h(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 1, \end{cases} \tag{3.5}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

Using Theorem 3.2 and Theorem 3.3, we obtain strong convergence of the sequence $\{x_n\}$ generated by (3.5).

Theorem 3.4. *Let $\{x_n\}$ be a sequence generated by the explicit algorithm* (3.5)*. Let $\{\alpha_n\}$ satisfy the following conditions*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $|\alpha_{n+1} - \alpha_n| \leq o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=1}^{\infty} \sigma_n < \infty$.

If one of the following assumptions holds:

(H1) *E is a reflexive Banach space with a uniformly Gâteaux differentiable norm, and every weakly compact convex subset of E has the FPP for nonexpansive mappings*;

(H2) *E is a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm,*

then $\{x_n\}$ converges strongly to a fixed point p of T, which is the unique solution in $Fix(T)$ of the variational inequality (3.2)*.*

Proof. By condition (C1), we may assume, without loss of generality, that $\alpha_n < \|A\|^{-1}$ for all $n \geq 1$. By Lemma 2.2, we have $\|I - \alpha_n A\| \leq (1 - \alpha_n \overline{\gamma})$.

Now we divide the proof into five steps.

Step 1. We show that $\{x_n\}$ is bounded. Indeed, pick any $p \in Fix(T)$ to obtain

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \|\alpha_n \gamma h(x_n) + (I - \alpha_n A)Tx_n - p\| \\
&= \|\alpha_n(\gamma h(x_n) - \gamma h(p)) + \alpha_n(\gamma h(p) - Ap) + (I - \alpha_n A)(Tx_n - p)\| \\
&\leq \alpha_n \gamma k \|x_n - p\| + \alpha_n \|\gamma h(p) - Ap\| + (1 - \alpha_n \overline{\gamma})\|x_n - p\| \\
&\leq (1 - \alpha_n(\overline{\gamma} - \gamma k))\|x_n - p\| + \alpha_n(\overline{\gamma} - \gamma k)\frac{\|\gamma h(p) - Ap\|}{\overline{\gamma} - \gamma k}.
\end{aligned}$$

It follows from induction that

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\gamma h(p) - Ap\|}{\overline{\gamma} - \gamma k}\right\}, \quad \forall n \geq 1.$$

Hence $\{x_n\}$ is bounded. Moreover, since h is a bounded mapping, $\{h(x_n)\}$ is bounded. Also, $\{Tx_n\}$ and $\{ATx_n\}$ are bounded.

As a direct consequence, from condition (C1) we get

$$\|x_{n+1} - Tx_n\| = \alpha_n\|\gamma h(x_n) - ATx_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.6}$$

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Indeed, from (3.5), it is easily seen that

$$\begin{aligned}
&\|x_{n+2} - x_{n+1}\| \\
&= \|\alpha_{n+1}\gamma h(x_{n+1}) + (I - \alpha_{n+1}A)Tx_{n+1} - \alpha_n\gamma h(x_n) - (I - \alpha_n A)Tx_n\| \\
&= \|(I - \alpha_{n+1}A)(Tx_{n+1} - Tx_n) + (\alpha_n - \alpha_{n+1})(ATx_n - \gamma h(x_n)) \\
&\qquad + \alpha_{n+1}\gamma(h(x_{n+1} - h(x_n))\| \\
&\le (1 - \alpha_n\overline{\gamma})\|x_{n+1} - x_n\| + |\alpha_n - \alpha_{n+1}|\|ATx_n - \gamma h(x_n)\| \\
&\quad + \alpha_{n+1}\gamma k\|x_{n+1} - x_n\| \\
&= (1 - \alpha_{n+1}(\overline{\gamma} - \gamma k))\|x_{n+1} - x_n\| + |\alpha_n - \alpha_{n+1}|\|ATx_n - \gamma h(x_n)\|
\end{aligned}$$

for $\forall n \ge 1$. So, from the condition (C2), we obtain

$$\|x_{n+2} - x_{n+1}\| \le (1 - \alpha_{n+1}(\overline{\gamma} - \gamma k))\|x_{n+1} - x_n\| + (o(\alpha_{n+1}) + \sigma_n)M \tag{3.7}$$

for $\forall n \ge 1$, where $M = \sup_{n\ge1}\{\|ATx_n - \gamma h(x_n)\|\}$. Put $s_n = \|x_{n+1} - x_n\|$, $\lambda_n = \alpha_{n+1}(\overline{\gamma} - \gamma k))$, $\lambda_n\delta_n = o(\alpha_{n+1})M$ and $\omega_n = \sigma_n M$. Then, from the conditions (C1) and (C2), it follows that $\lambda_n \to 0$ as $n \to \infty$, $\sum_{n=1}^\infty \lambda_n = \infty$ and $\sum_{n=1}^\infty \omega_n = M\sum_{n=1}^\infty \sigma_n < \infty$. Since (3.7) reduces

$$s_{n+1} = (1 - \lambda_n)s_n + \lambda_n\delta_n + \omega_n,$$

it follows from Lemma 2.3 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Step 3. We show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. In fact, from (3.6) and Step 2 it follows that

$$\|Tx_n - x_n\| \le \|Tx_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \ \text{ as } \ n \to \infty.$$

Step 4. We show that $\limsup_{n\to\infty}\langle\gamma h(p) - Ap, J(x_n - p)\rangle \le 0$, where $p = \lim_{t\to0} x_t$ and x_t is defined by (3.1). In fact, let $x_t = t\gamma h(x_t) + (I - tA)Tx_t$. Then, it follows from Theorem 3.2 or Theorem 3.3 that $\{x_t\}$ converges strongly to $p \in Fix(T)$ which is the unique solution of the variational inequality (3.2). Noting that

$$\begin{aligned}
&x_t - x_n \\
&= t\gamma h(x_t) + Tx_t - tATx_t - x_n \\
&= t(\gamma h(x_t) - Ax_t) + (Tx_t - x_n) - t(ATx_t - Ax_t) \\
&= t(\gamma h(x_t) - Ax_t) + (Tx_t - Tx_n) + (Tx_n - x_n) + t^2A(\gamma h(x_t) - ATx_t),
\end{aligned}$$

we have

$$\begin{aligned}
&\|x_t - x_n\|^2 \\
&= t\langle \gamma h(x_t) - Ax_t, J(x_t - x_n)\rangle + \langle Tx_t - Tx_n, J(x_t - x_n)\rangle \\
&\quad + \langle Tx_n - x_n, J(x_t - x_n)\rangle + t^2\langle A(\gamma h(x_t) - ATx_t), J(x_t - x_n)\rangle \\
&\le t\langle \gamma h(x_t) - Ax_t, J(x_t - x_n)\rangle + \|x_t - x_n\|^2 \\
&\quad + \|Tx_n - x_n\|\|x_t - x_n\| + t^2\|A(\gamma h(x_t) - ATx_t)\|\|x_t - x_n\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\langle \gamma h(x_t) - Ax_t, J(x_n - x_t)\rangle \\
&\le \frac{\|Tx_n - x_n\|}{t} + t\|A(\gamma h(x_t) - ATx_t)\|\|x_t - x_n\| \\
&\le \frac{\|Tx_n - x_n\|}{t} + tL,
\end{aligned} \tag{3.8}$$

where $L > 0$ is a constant such that $L = \sup\{\|A(\gamma h(x_t) - ATx_t)\|\|x_t - x_n\| : n \ge 1 \text{ and } t \in (0, \min\{1, \|A\|^{-1}\})\}$. Since $x_n - Tx_n \to 0$ by Step 3, taking the upper limit as $n \to \infty$ in (3.8), we derive

$$\limsup_{n\to\infty}\langle \gamma h(x_t) - Ax_t, J(x_n - x_t)\rangle \le tM. \tag{3.9}$$

Taking the lim sup as $t \to 0$ in (3.9) and noticing that the fact that the two limits are interchangeable due to the fact that J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak* topology of E^*, we obtain

$$\limsup_{n\to\infty}\langle \gamma h(p) - Ap, J(x_n - p)\rangle \le 0.$$

Step 5. We show that $\lim_{n\to\infty} x_n = p$, where $p = \lim_{t\to 0} x_t \in Fix(T)$, x_t being defined by (3.1), which is the unique solution of the variational inequality (3.2). Indeed, from (3.5), Lemma 2.2 and Lemma 2.6, we derive

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\alpha_n(\gamma h(x_n) - Ap) + (I - \alpha_n A)Tx_n - (I - \alpha_n A)p\|^2 \\
&\le \|(I - \alpha_n A)(Tx_n - p)\|^2 + 2\alpha_n\langle \gamma h(x_n) - Ap, J(x_{n+1} - p)\rangle \\
&\le (1 - \alpha_n\overline{\gamma})^2\|x_n - p\|^2 + 2\alpha_n\langle \gamma h(x_n) - \gamma h(p), J(x_{n+1} - p)\rangle \\
&\quad + 2\alpha_n\langle \gamma h(p) - Ap, J(x_{n+1} - p)\rangle \\
&\le (1 - \alpha_n\overline{\gamma})^2\|x_n - p\|^2 + 2\alpha_n\gamma k\|x_n - p\|\|x_{n+1} - p\| \\
&\quad + 2\alpha_n\langle \gamma h(p) - Ap, J(x_{n+1} - p)\rangle \\
&\le (1 - \alpha_n\overline{\gamma})^2\|x_n - p\|^2 + \alpha_n\gamma k(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) \\
&\quad + 2\alpha_n\langle \gamma h(p) - Ap, J(x_{n+1} - p)\rangle.
\end{aligned}$$

This implies that
(3.10)

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\le \frac{(1-\alpha_n\overline{\gamma})^2+\alpha_n\gamma k}{1-\alpha_n\gamma k}\|x_n-p\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma k}\langle \gamma h(p)-Ap, J(x_{n+1}-p)\rangle\\
&=\left(1-\frac{2\alpha_n(\overline{\gamma}-\gamma k)}{1-\alpha_n\gamma k}\right)\|x_n-p\|^2+\frac{\alpha_n^2\overline{\gamma}^2}{1-\alpha_n\gamma k}\|x_n-p\|^2\\
&\quad+\frac{2\alpha_n}{1-\alpha_n\gamma k}\langle \gamma h(p)-Ap, J(x_{n+1}-p)\rangle\\
&\le \left(1-\frac{2\alpha_n(\overline{\gamma}-\gamma k)}{1-\alpha_n\gamma k}\right)\|x_n-p\|^2+\frac{2\alpha_n(\overline{\gamma}-\gamma k)}{1-\alpha_n\gamma k}\cdot\frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma}-\gamma k)}L\\
&\quad+\frac{2\alpha_n(\overline{\gamma}-\gamma k)}{1-\alpha_n\gamma k}\cdot\frac{1}{\overline{\gamma}-\gamma k}\langle \gamma h(p)-Ap, J(x_{n+1}-p)\rangle,
\end{aligned}$$

where $L=\sup\{\|x_n-p\|: n\ge 1\}$. Put $\lambda_n=\frac{2\alpha_n(\overline{\gamma}-\gamma k)}{1-\alpha_n\gamma k}$ and

$$\delta_n=\frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma}-\gamma k)}L+\frac{1}{\overline{\gamma}-\gamma k}\langle \gamma h(p)-Ap, J(x_{n+1}-p)\rangle.$$

Then it follows from the condition (C1) and Step 4 that $\lim_{n\to\infty}\lambda_n=0$, $\sum_{n=1}^{\infty}\lambda_n=\infty$, and $\limsup_{n\to\infty}\delta_n\le 0$. (3.10) reduces to

$$\|x_{n+1}-p\|^2\le(1-\lambda_n)\|x_n-p\|^2+\lambda_n\delta_n. \tag{3.11}$$

Thus, applying Lemma 2.3 together with $\omega_n=0$ to (3.11), we conclude that $\lim_{n\to\infty}x_n=p$. This completes the proof. □

Corollary 3.5. *Let E be a uniformly smooth Banach space. Let $\{x_n\}$ be a sequence generated by the explicit algorithm* (3.5). *Let $\{\alpha_n\}$ satisfy the conditions* (C1) *and* (C2) *in Theorem 3.4. Then $\{x_n\}$ converges strongly to a fixed point p of T, which is the unique solution in $Fix(T)$ of the variational inequality* (3.2).

Removing the condition $|\alpha_{n+1}-\alpha_n|\le o(\alpha_{n+1})+\sigma_n$, $\sum_{n=1}^{\infty}\sigma_n<\infty$ on the sequence $\{\alpha_n\}$ in Theorem 3.4, we have the following result.

Theorem 3.6. *Let $\{x_n\}$ be a sequence generated by the following explicit algorithm*:

$$\begin{cases} x_1=x\in C\\ x_{n+1}=\alpha_n\gamma h(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)Tx_n, \quad n\ge 1,\end{cases} \tag{3.12}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, which satisfy the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;
(C2) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$.

If one of the following assumptions holds:

(H1) *E is a reflexive Banach space with a uniformly Gâteaux differentiable norm, and every weakly compact convex subset of E has the FPP for nonexpansive mappings*;

(H2) *E is a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm,*

then $\{x_n\}$ *converges strongly to a fixed point* p *of* T, *which is the unique solution in* $Fix(T)$ *of the variational inequality* (3.2).

Proof. We only include the difference from the proof of Theorem 3.5. By conditions (C1) and (C2), we may assume, without loss of generality, that $\frac{\alpha_n}{1-\beta_n} < \|A\|^{-1}$ for all $n \geq 1$. By Lemma 2.2, we have $\|(1-\beta_n)I - \alpha_n A\| \leq (1-\beta_n-\alpha_n\overline{\gamma})$.

Step 1. We show that $\{x_n\}$, $\{h(x_n)\}$, $\{Tx_n\}$ and $\{ATx_n\}$ are bounded. Indeed, pick any $p \in Fix(T)$ to obtain

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n\gamma h(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)Tx_n - p\| \\ &= \|\alpha_n(\gamma h(x_n) - \gamma h(p)) + \alpha_n(\gamma h(p) - Ap) + \beta_n(x_n - p) \\ &\quad + ((1-\beta_n)I - \alpha_n A)(Tx_n - p)\| \\ &\leq \alpha_n\gamma k\|x_n - p\| + \alpha_n\|\gamma h(p) - Ap\| \\ &\quad + \beta_n\|x_n - p\| + (1-\beta_n-\alpha_n\overline{\gamma})\|x_n - p\| \\ &= (1-\alpha_n(\overline{\gamma}-\gamma k))\|x_n - p\| + \alpha_n(\overline{\gamma}-\gamma k)\frac{\|\gamma h(p) - Ap\|}{\overline{\gamma}-\gamma k}.\end{aligned}$$

The rest follows from Step 1 of the proof of Theorem 3.4.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. To this end, define a sequence $\{z_n\}$ by $z_n = (x_{n+1}-\beta_n x_n)/(1-\beta_n)$ so that

$$x_{n+1} = \beta_n x_n + (1-\beta_n)z_n. \tag{3.13}$$

We now observe that

(3.14)

$$\begin{aligned}&z_{n+1} - z_n \\ &= \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1}-\beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}\gamma h(x_{n+1}) + \beta_{n+1}x_{n+1} + ((1-\beta_{n+1})I - \alpha_{n+1}A)Tx_{n+1} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} \\ &\quad - \frac{\alpha_n\gamma h(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)Tx_n - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma h(x_{n+1}) - ATx_{n+1}) + Tx_{n+1} - Tx_n \\ &\quad + \frac{\alpha_n}{1-\beta_n}(ATx_n) - \gamma h(x_n)).\end{aligned}$$

It follows from (3.14) that
(3.15)

$$\begin{aligned}&\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\\ \leq\ & \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma h(x_{n+1})\|+\|ATx_{n+1}\|)+\frac{\alpha_n}{1-\beta_n}(\|\gamma h(x_n)\|+\|ATx_n\|).\end{aligned}$$

By conditions (C1), (C2) and (3,15), we obtain that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

Hence by Lemma 2.4, we have

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{3.16}$$

It then follows from condition (C2), (3.13) and (3.16) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\|=0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. In fact, from (3.12) it follows that

$$\begin{aligned}\|Tx_n-x_n\|&\leq\|Tx_n-x_{n+1}\|+\|x_{n+1}-x_n\|\\&\leq\|\alpha_n\gamma h(x_n)-\alpha_nATx_n\|+\beta_n\|x_n-Tx_n\|+\|x_{n+1}-x_n\|.\end{aligned}$$

This implies that

$$(1-\beta_n)\|Tx_n-x_n\|\leq\alpha_n(\gamma\|h(x_n)\|+\|ATx_n\|)+\|x_{n+1}-x_n\|.$$

Thus, by conditions (C1) and (C2) and Step 2, we have

$$\lim_{n\to\infty}\|Tx_n-x_n\|=0.$$

Step 4. We show that $\limsup_{n\to\infty}\langle\gamma h(p)-Ap,J(x_n-p)\rangle\leq 0$, where $p=\lim_{t\to 0}x_t$ and x_t is defined by (3.1). The result follows from Step 4 in the proof of Theorem 3.4.

Step 5. We show that $\lim_{n\to\infty}x_n=p$, where $p=\lim_{t\to 0}x_t\in Fix(T)$, x_t being defined by (3.1), which is the unique solution of the variational inequality (3.2). Indeed, from (3.12), observe that

$$x_{n+1}-p=\ \alpha_n(\gamma h(x_n)-Ap)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_nA)(Tx_n-p).$$

By Lemma 2.2 and Lemma 2.6, we derive

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (\beta_n\|x_n-p\| + \|((1-\beta_n)I-\alpha_n A)(Tx_n-p)\|)^2 \\ &\quad + 2\alpha_n\langle \gamma h(x_n)-Ap, J(x_{n+1}-p)\rangle \\ &\le (\beta_n\|x_n-p\| + (1-\beta_n-\alpha_n\overline{\gamma})\|x_n-p\|)^2 \\ &\quad + 2\alpha_n\langle \gamma h(x_n)-Ap, J(x_{n+1}-p)\rangle \\ &= (1-\alpha_n\overline{\gamma})^2\|x_n-p\|^2 + 2\alpha_n\langle \gamma h(x_n)-\gamma h(p), J(x_{n+1}-p)\rangle \\ &\quad + 2\alpha_n\langle \gamma h(p)-Ap, J(x_{n+1}-p)\rangle \\ &\le (1-\alpha_n\overline{\gamma})^2\|x_n-p\|^2 + 2\alpha_n\gamma k\|x_n-p\|\|x_{n+1}-p\| \\ &\quad + 2\alpha_n\langle \gamma h(p)-Ap, J(x_{n+1}-p)\rangle \\ &\le (1-\alpha_n\overline{\gamma})^2\|x_n-p\|^2 + \alpha_n\gamma k(\|x_n-p\|^2+\|x_{n+1}-p\|^2) \\ &\quad + 2\alpha_n\langle \gamma h(p)-Ap, J(x_{n+1}-p)\rangle.\end{aligned}$$

The remainder follows from the proof of Theorem 3.4. □

Remark 3.7. Our results in this paper extend, improve and develop the corresponding results in [8, 9, 10, 13] and the references therein.

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