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EXISTENCE AND MULTIPLICITY OF NONTRIVIAL SOLUTIONS FOR KLEIN-GORDON-MAXWELL SYSTEM WITH A PARAMETER

Guofeng Che and Haibo Chen

Abstract. This paper is concerned with the following Klein-Gordon-Maxwell system:

$$\begin{cases} -\Delta u + \lambda V(x)u - (2\omega + \phi)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\omega > 0$ is a constant and λ is the parameter. Under some suitable assumptions on $V(x)$ and $f(x,u)$, we establish the existence and multiplicity of nontrivial solutions of the above system via variational methods. Our conditions weaken the Ambrosetti Rabinowitz type condition.

1. Introduction

In this paper, we consider the following Klein-Gordon-Maxwell system:

$$\begin{cases} -\Delta u + \lambda V(x)u - (2\omega + \phi)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\omega > 0$ is a constant, λ is the parameter and $u, \phi : \mathbb{R}^3 \to \mathbb{R}$.

This system appears as a model describing the nonlinear Klein-Gordon field interacting with the electromagnetic field in the electrostatic field. More specifically, it represents a solitary wave $\psi(x) = u(x)e^{i\omega t}$ in equilibrium with a purely electrostatic field $E = -\nabla\phi(x)$ (for more details, see [3, 5, 8, 13] and the references therein). The unknowns of the system are the field u associated with the particle and the electric potential ϕ. The presence of the nonlinear term stimulates the interaction between many particles or external nonlinear perturbations.

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As we know, V. Benci and D. Fortunato [3] are the first to consider the following Klein-Gordon-Maxwell system:

$$\begin{cases} -\Delta u + [m_0^2 - (\omega + \phi)^2]u = f(u), \quad x \in \mathbb{R}^3, \\ \Delta\phi = (\omega + \phi)u^2, \quad x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $f(u) = |u|^{q-2}u$ and $4 < q < 6$, and obtained the existence of infinitely many radially symmetric solutions via the variational methods. Azzollini and Pomponio in [2] established the existence of ground state solutions of system (1.2) under the following conditions:

(i) $4 \le q < 6$ and $m_0 > \omega$;

(ii) $2 < q < 4$ and $m_0\sqrt{q-2} > \omega\sqrt{6-q}$.

In [1], Azzollini etal. improved the existence range of (m_0, ω) for $p \in (2, 4)$ as follows:

$$0 < \omega < m_0 g(p),$$

and

$$g(p) = \begin{cases} \sqrt{(p-2)(4-p)}, & \text{if } \ 2 < p < 3, \\ 1, & \text{if } \ 3 \le p < 4. \end{cases}$$

They also considered the limit case that $m_0 = \omega$. Cassani in [4] considered system (1.2) when $f(u) = \mu|u|^{p-2}u + |u|^{2^*-2}u$, where $\mu \ge 0$ and $p \in (4, 6)$. Moreover, he obtained the existence of trivial solution via a *Pohožaev*-type argument when $\mu = 0$ and proved the existence of nontrivial solutions when one of the following conditions is satisfied:

(i) $p \in (4, 6)$, $|m| > |\omega| > 0$ and $\mu > 0$;

(ii) $p = 4$, $|m| > |\omega| > 0$ and $\mu > 0$ sufficiently large.

Later, Wang in [15] followed the ideas that appeared in [8] and generalized the result of [4]. He established the existence of at least a radially symmetric nontrivial weak solution of system (1.2) when $f(u) = \mu|u|^{p-2}u + |u|^{2^*-2}u$, where $\mu > 0$ and one of the following conditions is satisfied:

(i) $p \in (4, 6)$, $m > \omega > 0$ and $\mu > 0$;

(ii) $p \in (3, 4]$, $m > \omega > 0$ and $\mu > 0$ sufficiently large;

(iii) $p \in (2, 3]$, $m\sqrt{(p-2)(4-p)} > \omega > 0$ and $\mu > 0$ sufficiently large.

Applying the Ekeland's variational principle and the Mountain Pass Theorem in critical point theory, Xu and Chen in [18] obtained the existence of at least two nontrivial solutions of problem (1.1) with $\lambda = 1$ when $f(x, u) = |u|^{p-1}u + h(x)$, $p \in (1, 5)$.

In recent paper [10], the authors studied the existence of infinitely many nontrivial solutions of (1.1) with $\lambda = 1$ under the following assumptions on $V(x)$ and $f(x, u)$,

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) \ge a > 0$, where $a > 0$ is a constant. Moreover, for any $M > 0$, $meas\{x \in \mathbb{R}^3 : V(x) \le M\} < \infty$, where meas denotes the Lebesgue measure in $\mathbb{R}^3$.

(f_1) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, $f(x,t) \geq 0$ for $t \geq 0$ and there exist $C_1,\ C_2 > 0$ and $p \in [4,6)$ such that

$$|f(x,u)| \leq C_1|u| + C_2|u|^{p-1},\ \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

(f_2) $\frac{F(x,u)}{|u|^4} \to +\infty$ as $|u| \to +\infty$ uniformly in $x \in \mathbb{R}^3$.

(f_3) Let $\widetilde{F} = \frac{1}{4}f(x,t)t - F(x,t)$, there exists $r_0 > 0$ such that if $|t| \geq r_0$, then $\widetilde{F} \geq 0$ uniformly for $x \in \mathbb{R}^3$.

(f_4) $f(x,-u) = -f(x,u)$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$.

Specifically, the authors established the following theorem in [10].

Theorem 1.1 ([10]). *Under the assumptions* (V_1) *and* (f_1)-(f_4). *Then problem* (1.1) *with* $\lambda = 1$ *has infinitely many nontrivial solutions.*

Motivated by the above facts, in the present paper, we will study the existence and multiplicity of nontrivial solutions of problem (1.1) under the assumptions (V_1), (f_2) and (f_4). Instead of (f_1) and (f_3), we give the following more general assumptions on $f(x,u)$.

(f_1') $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and there exist $c_1,\ c_2 > 0$ and $p \in (2,6)$ such that

$$|f(x,u)| \leq c_1|u| + c_2|u|^{p-1},\ \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

(f_5) $f(x,u) = o(|u|)$ as $|u| \to 0$ uniformly in $x \in \mathbb{R}^3$.

(f_6) There exist $\mu \in (4,6)$ and $r_0 > 0$ such that

$$\inf_{x \in \mathbb{R}^3, |u| = r_0} F(x,u) := \beta > 0,$$

and

$$\mu F(x,u) - f(x,u)u \leq C_0|u|^2,\ \forall\ x \in \mathbb{R}^3 \text{ and } |u| \geq r_0,$$

where $F(x,u) = \int_0^u f(x,s)ds$ and $0 < C_0 < \frac{\beta(\mu-2)}{r_0^2}$.

(f_7) There exist $r > 0$ and $C > 0$ such that

$$4F(x,u) - f(x,u)u \leq C|u|^2,\ \forall\ x \in \mathbb{R}^3 \text{ and } |u| \geq r.$$

Evidently, (f_7) is weaker than the condition (f_3).

Now, we are ready to state the main results of this paper,

Theorem 1.2. *Assume conditions* (V_1), (f_1'), (f_5) *and* (f_6) *hold. Then there exists* $\Lambda_1 > 0$ *such that problem* (1.1) *has at least one nontrivial solution whenever* $\lambda \geq \Lambda_1$.

Theorem 1.3. *Assume conditions* (V_1), (f_1'), (f_2), (f_5) *and* (f_7) *hold. Then there exists* $\Lambda_2 > 0$ *such that problem* (1.1) *has at least one nontrivial solution whenever* $\lambda \geq \Lambda_2$.

To get the existence of infinitely many solutions for the problem (1.1), the assumption (f_5) is not needed. Instead, we need another assumption (V_2), but it is not very restrictive.

(V_2) There exist $d > 0$ and $R_0 > 0$ such that the set $\{x \in \mathbb{R}^3 : V(x) \leq d\}$ is nonempty and $meas\{\{x \in \mathbb{R}^3 : V(x) \leq d\} \setminus (B_{R_0} \cap \{x \in \mathbb{R}^3 : V(x) \leq d\})\} = 0$, where $B_{R_0} = \{x \in \mathbb{R}^3 : |x| < R_0\}$.

Theorem 1.4. *Assume conditions* (V_1), (V_2), (f_1'), (f_4) *and* (f_6) *hold. Then there exists* $\Lambda_3 > 0$ *such that problem* (1.1) *has infinitely many nontrivial weak solutions whenever* $\lambda \geq \Lambda_3$.

Theorem 1.5. *Assume conditions* (V_1), (V_2), (f_1'), (f_2), (f_4) *and* (f_7) *hold. Then there exists* $\Lambda_4 > 0$ *such that problem* (1.1) *has infinitely many nontrivial weak solutions whenever* $\lambda \geq \Lambda_4$.

Notation 1.1. Throughout this paper, we shall denote by $|\cdot|_r$ the L^r-norm and C various positive generic constants, which may vary from line to line. Also if we take a subsequence of a sequence $\{u_n\}$ we shall denote it $\{u_n\}$ again.

The remainder of this paper is as follows. In Section 2, we mainly consider the existence of at least one nontrivial solution. In Section 3, the existence of infinitely many nontrivial solutions is discussed.

2. Existence of nontrivial solutions

In this section, we consider the existence of nontrivial solutions for problem (1.1).

Define the space

$$E_\lambda := \{u \in H^1(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} \lambda V(x) u^2 < +\infty\}.$$

with the inner product and norm

$$\langle u, v\rangle_{E_\lambda} = \int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda V(x) uv) dx, \ \ \|u\|_{E_\lambda} = \langle u, u\rangle_X^{\frac{1}{2}}.$$

Moreover, by Lemma 3.4 in [19], we know that under the assumption (V_1), the embedding $E_\lambda \hookrightarrow L^r(\mathbb{R}^3)$ is continuous for $2 \leq r \leq 6$ and $E_\lambda \hookrightarrow L^r(\mathbb{R}^3)$ is compact for $2 \leq r < 6$, i.e., there exists constants τ_r such that

$$||u||_r \leq \tau_r ||u||_{E_\lambda}. \tag{2.1}$$

Note that problem (1.1) has a variational structure and its solution can be regarded as critical point of the energy functional defined on the space E_λ by

$$J(u, \phi) = \frac{1}{2}||u||^2_{E_\lambda} - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \phi|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} (2\omega + \phi)\phi u^2 dx - \int_{\mathbb{R}^3} F(x, u) dx.$$

Under the assumptions (V_1) and (f_1'), the functional J belongs to $C^1(E_\lambda, \mathbb{R})$ and also exists a strong indefiniteness. To avoid the indefiniteness, we can apply a reduction method described in [6, 18], by which we are led to study a one variable functional that does not present such a strong indefinite nature.

Lemma 2.1 ([8, 18])**.** *For every $u \in E_\lambda$ there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ which solves $\Delta\phi = (\omega+\phi)u^2$. Furthermore*

(i) *In the set $\{x : u(x) \neq 0\}$, we have $-\omega \leq \phi_u \leq 0$ for $\omega > 0$.*

(ii) *If u is radially symmetric, ϕ_u is radial too.*

According to Lemma 2.1, we can consider the functional $I : E_\lambda \to \mathbb{R}$ defined by $I(u) = J(u, \phi_u)$. After multiplying $\Delta\phi_u = (\omega+\phi_u)u^2$ by ϕ_u and integration by parts, we obtain

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx = -\int_{\mathbb{R}^3} (\phi_u)^2 u^2 dx - \int_{\mathbb{R}^3} \omega\phi_u u^2 dx. \tag{2.2}$$

Therefore, the reduced functional takes the form

$$I(u) = \frac{1}{2}||u||^2_{E_\lambda} - \frac{1}{2}\int_{\mathbb{R}^3} \omega\phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx. \tag{2.3}$$

Moreover, I is C^1 and we have for any $u,\ v \in E_\lambda$,

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u\nabla v + \lambda uv - (2\omega+\phi_u)\phi_u uv)dx - \int_{\mathbb{R}^3} f(x,u)vdx. \tag{2.4}$$

Remark 2.1. By (2.2) and Hölder inequality, we have

$$||\phi_u||^2_{D^{1,2}(\mathbb{R}^3)} \leq \int_{\mathbb{R}^3} \omega|\phi_u|u^2 dx \leq \omega||\phi_u||_6||u||^2_{\frac{12}{5}},$$

then

$$\begin{aligned} ||\phi_u||_{D^{1,2}(\mathbb{R}^3)} \leq C||u||^2_{\frac{12}{5}}, \ &\int_{\mathbb{R}^3} \omega|\phi_u|u^2 dx \\ &\leq C||\phi_u||_{D^{1,2}}||u||^2_{\frac{12}{5}} \leq C||u||^4_{\frac{12}{5}} \leq C||u||^4_{E_\lambda}. \end{aligned} \tag{2.5}$$

Now we can apply Lemma 2.2 of [7] or Lemma 2.3 of [18] to our functional I and obtain:

Lemma 2.2 ([7, 18])**.** *The following statements are equivalent*:

(i) *$(u,\phi) \in E_\lambda \times D^{1,2}(\mathbb{R}^3)$ is a critical point of I (i.e., (u,ϕ) is a solution of problem* (1.1)).

(ii) *u is a critical point of I and $\phi = \phi_u$.*

Lemma 2.3 ([11], Mountain Pass Theorem)**.** *Let E be a real Banach space with its dual space E^*, and suppose that $I \in C^1(E,\mathbb{R})$ satisfies*

$$\max\{I(0), I(e)\} \leq \mu < \eta \leq \inf_{||u||=\rho} I(u)$$

for some $\mu < \eta$, $\rho > 0$ and $e \in E$ with $||e|| > \rho$. Let $c \geq \eta$ be characterized by

$$c = \inf_{\gamma\in\Gamma}\max_{0\leq\tau\leq 1} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e, then there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c \text{ and } (1+||u_n||)||I'(u_n)|| \to 0,\ n \to \infty.$$

Lemma 2.4 ([9, Theorem A.2])**.** *Let Ω be an open set in $\mathbb{R}^N$ and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ be a function such that $|f(x,u)| \leq c(|u|^r + |u|^s)$ for some $c > 0$ and $1 \leq r < s < \infty$. Suppose that $s \leq p < \infty$, $r \leq t < \infty$, $t > 1$, $\{u_n\}$ is a bounded sequence in $L^p(\Omega) \cap L^t(\Omega)$, $u_n \to u$ a.e. in Ω and in $L^p(\Omega \cap B_R) \cap L^t(\Omega \cap B_R)$ for all $R > 0$. Then, passing to a subsequence, there exists a sequence $\{v_n\}$ such that*

$$v_n \to u \quad in \quad L^p(\Omega) \cap L^t(\Omega)$$

and

$$f(x, u_n) - f(x, u_n - v_n) - f(x, u) \to 0, \quad in \quad L^{t/r}(\Omega) + L^{p/s}(\Omega),$$

where $v_n(x) = \chi(2|x|/R_n)u(x)$, $\chi \in C^\infty(\mathbb{R}, [0,1])$ be such that $\chi(t) = 1$ for $t \leq 1$, $\chi(t) = 0$ for $t \geq 2$, $R_n > 0$ is a sequence of constants with $R_n \to \infty$ as $n \to \infty$, the space $L^p(\Omega) \cap L^t(\Omega)$ has the norm $||u||_{p\wedge t} := ||u||_p + ||u||_t$ and the space $L^p(\Omega) + L^t(\Omega)$ with the norm

$$||u||_{p\vee t} := \inf\{||v||_p + ||w||_t : \; v \in L^p(\Omega), \; w \in L^t(\Omega), \; u = v + w\}.$$

Lemma 2.5. *Assume that (V_1), (f_1') and (f_6) hold. Then there exists $\Lambda > 0$ such that I satisfies the $(PS)_c$ condition for all $\lambda \geq \Lambda$.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence. Firstly, we prove that $\{u_n\}$ is bounded in E_λ for $\lambda > 0$ large enough. Arguing by contradiction, we can assume that $||u_n||_{E_\lambda} \to +\infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{||u_n||}$. Then $||v_n|| = 1$ and $||v_n||_r \leq \tau_r ||v_n||_{E_\lambda} = \tau_r$ for $2 \leq r \leq 6$. Set

$$h(t) := F(x, t^{-1}z)t^\mu, \; \forall \; t \in [1, +\infty) \text{ and } (x,z) \in \mathbb{R}^3 \times \mathbb{R}.$$

Then by (f_6), we have

$$\begin{aligned} h'(t) &= f(x, t^{-1}z)(-\frac{z}{t^2})t^\mu + F(x, t^{-1}z)\mu t^{\mu-1} \\ &= t^{\mu-1}\left[\mu F(x, t^{-1}z) - t^{-1}zf(x, t^{-1}z)\right] \\ &\leq C_0 t^{\mu-3}|z|^2, \end{aligned}$$

where $|z| \geq r_0$ and $t \in [1, \frac{|z|}{r_0}]$. Then

$$h(\frac{|z|}{r_0}) - h(1) = \int_1^{\frac{|z|}{r_0}} h'(t)dt \leq \int_1^{\frac{|z|}{r_0}} C_0 t^{\mu-3}|z|^2 dt = \frac{C_0|z|^\mu}{(\mu-2)r_0^{\mu-2}} - \frac{C_0|z|^2}{\mu-2}.$$

Therefore, we have

$$F(x,z) = h(1) \geq h(\frac{|z|}{r_0}) - \frac{C_0|z|^\mu}{(\mu-2)r_0^{\mu-2}} \geq (\frac{\beta}{r_0^\mu} - \frac{C_0}{(\mu-2)r_0^{\mu-2}})|z|^\mu.$$

Thus $\frac{\beta}{r_0^\mu} - \frac{C_0}{(\mu-2)r_0^{\mu-2}} > 0$ for $C_0 < \frac{\beta(\mu-2)}{r_0^2}$. Since $\mu > 4$, then there exists a constant $4 < \theta < 6$ such that $\theta < \mu$, and hence

$$\lim_{|u|\to\infty} \frac{F(x,u)}{|u|^\theta} = +\infty. \tag{2.6}$$

In particularly, we have

$$\lim_{|u|\to\infty}\frac{F(x,u)}{|u|^4}=+\infty. \tag{2.7}$$

From (f_1'), we have

$$F(x,u)\le\frac{c_1}{2}|u|^2+\frac{c_2}{p}|u|^p. \tag{2.8}$$

It follows from (2.6) and (2.8) that for any $M>0$, there exists a constant $C(M)>0$ such that

$$F(x,u)\ge M|u|^\theta-C(M)|u|^2. \tag{2.9}$$

Furthermore, we have

$$\frac{I(u_n)}{||u_n||^\theta_{E_\lambda}}=\frac{1}{2||u_n||^{\theta-2}_{E_\lambda}}-\frac{1}{2||u_n||^\theta_{E_\lambda}}\int_{\mathbb{R}^3}\omega\phi_u u^2dx-\int_{\mathbb{R}^3}\frac{F(x,u_n)}{||u_n||^\theta_{E_\lambda}}dx.$$

Then by (2.5) and $\theta>4$, we deduce that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}\frac{F(x,u_n)}{||u_n||^\theta_{E_\lambda}}dx=0.$$

Since $||v_n||_{E_\lambda}=1$, going if necessary to a subsequence, we can assume that $v_n\rightharpoonup v$ in E_λ, $v_n\to v$ in $L^r(\mathbb{R}^3)$ for $2\le r<6$ and $v_n\to v$ a.e. in $\mathbb{R}^3$. Set $\Omega=\{x\in\mathbb{R}^3:v(x)\ne0\}$. If $meas(\Omega)>0$, then $\int_\Omega|v|^\theta dx>0$. By (2.9), we have

$$\int_{\mathbb{R}^3}\frac{F(x,u_n)}{||u_n||^\theta_{E_\lambda}}dx\ge M||v_n||^\theta_\theta-C(M)\frac{||v_n||_2^2}{||u_n||^{\theta-2}_{E_\lambda}}.$$

Therefore

$$\begin{aligned}0&=\liminf_{n\to\infty}\Big(\int_{\mathbb{R}^3}\frac{F(x,u_n)}{||u_n||^\theta_{E_\lambda}}dx+C(M)\frac{||v_n||_2^2}{||u_n||^{\theta-2}_{E_\lambda}}\Big)\\&\ge\liminf_{n\to\infty}M||v_n||^\theta_\theta\ge M\int_\Omega|v|^\theta dx>0,\end{aligned}$$

which is a contradiction, then $meas(\Omega)=0$, and as a result $v=0$ a.e. in $\mathbb{R}^3$. Therefore, from (V_1), we have

$$||v_n||_2^2=\int_{V(x)\ge1}|v_n|^2dx+\int_{V(x)<1}|v_n|^2dx\le\frac{1}{\lambda}||v_n||^2_{E_\lambda}+o(1)\le\frac{2}{\lambda}$$

for n large enough. It follows from (f_1') and (f_6) that there exists a constant $c>0$ such that

$$\mu F(x,u)-uf(x,u)\le c|u|^2$$

for all $(x,u)\in\mathbb{R}^3\times\mathbb{R}$. Therefore, by Lemma 2.1 and $\mu\in(4,6)$, we have

$$\begin{aligned}0 &\leftarrow \frac{1}{||u_n||^2_{E_\lambda}}\big[\mu I(u_n)-\langle I'(u_n),u_n\rangle\big]\\ &= \frac{1}{||u_n||^2_{E_\lambda}}\big[\frac{\mu-2}{2}||u_n||^2_{E_\lambda}+\frac{4-\mu}{2}\int_{\mathbb{R}^3}\omega\phi_{u_n}u_n^2dx\\ &\quad+\int_{\mathbb{R}^3}\big(f(x,u_n)u_n-\mu F(x,u_n)\big)dx+\int_{\mathbb{R}^3}\phi^2_{u_n}u_n^2dx\big]\\ &\geq \frac{\mu-2}{2}-c\int_{\mathbb{R}^3}|v_n|^2dx\\ &\geq \frac{\mu-2}{2}-\frac{2c}{\lambda}.\end{aligned}$$

Let $\lambda>0$ so large that the term $\frac{\mu-2}{2}-\frac{2c}{\lambda}>0$, then we get a contradiction. Hence, $\{u_n\}$ is bounded in E_λ for large λ. Therefore, going if necessary to a subsequence, there exists $u\in E_\lambda$ such that

$$u_n\rightharpoonup u,\quad \text{in}\quad E_\lambda. \tag{2.10}$$

$$u_n\to u,\quad \text{in}\quad L^r(\mathbb{R}^3),\ 2\leq r<6. \tag{2.11}$$

$$u_n\to u,\ \text{a.e. in}\ \ \mathbb{R}^3. \tag{2.12}$$

Take $v_n(x)=\chi(\frac{2|x|}{R_n})u(x)$, where $R_n>0$ is a sequence of constants with $R_n\to+\infty$ as $n\to+\infty$. We claim that $v_n\to u$ in E_λ. Indeed, $u\in E_\lambda$ implies that for any $\varepsilon>0$, there exists a $\rho=\rho(\varepsilon)$ such that

$$\int_{\mathbb{R}^3\setminus B_\rho(0)}|\nabla u_n|^2dx\leq\varepsilon\ \text{ and }\ \int_{\mathbb{R}^3\setminus B_\rho(0)}\lambda V(x)|u_n|^2dx\leq\varepsilon. \tag{2.13}$$

Hence, by (2.13), we have

$$\begin{aligned}||v_n-u||^2_{E_\lambda} &= \int_{\mathbb{R}^3}|\nabla(v_n-u)|^2dx+\int_{\mathbb{R}^3}\lambda V(x)|v_n-u|^2dx\\ &= \int_{\mathbb{R}^3}|\nabla(\chi(\frac{2|x|}{R_n})u-u)|^2dx+\int_{\mathbb{R}^3}\lambda V(x)|\chi(\frac{2|x|}{R_n})u-u|^2dx\\ &\leq \int_{\mathbb{R}^3}|\chi(\frac{2|x|}{R_n})-1|^2|\nabla u|^2dx+(\frac{2}{R_n})^2\int_{\mathbb{R}^3}|\chi'(\frac{2|x|}{R_n})|^2|u|^2dx\\ &\quad+\int_{\mathbb{R}^3}\lambda V(x)|\chi(\frac{2|x|}{R_n})-1|^2|u|^2dx\\ &\leq \int_{B_\rho(0)}|\chi(\frac{2|x|}{R_n})-1|^2|\nabla u|^2dx+(\frac{2}{R_n})^2\int_{\mathbb{R}^3}|\chi'(\frac{2|x|}{R_n})|^2|u|^2dx\\ &\quad+\int_{B_\rho(0)}\lambda V(x)|\chi(\frac{2|x|}{R_n})-1|^2|u|^2dx+c\varepsilon.\end{aligned}$$

Therefore, by the Lebesgue dominated convergence theorem, we have

$$||v_n - u||_{E_\lambda} \to 0 \quad \text{as } n \to +\infty. \tag{2.14}$$

Furthermore, by the Hölder inequality, we have

$$\begin{aligned}
&\langle I'(u_n) - I'(v_n), u_n - v_n\rangle \\
&= ||u_n - v_n||^2_{E_\lambda} - \int_{\mathbb{R}^3} (f(x, u_n) - f(x, v_n))(u_n - v_n)dx \\
&\quad + 2\omega \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_{v_n} v_n)(u_n - v_n)dx \\
&\quad + \int_{\mathbb{R}^3} (\phi^2_{u_n} u_n - \phi^2_{v_n} v_n)(u_n - v_n)dx.
\end{aligned} \tag{2.15}$$

Since $u_n \rightharpoonup u$ in E_λ and $I'(u_n) \to 0$, we have $\langle I'(u_n) - I'(u), u_n - u\rangle \to 0$ as $n \to +\infty$. By $||v_n - u||_{E_\lambda} \to 0$, $I \in C^1(E_\lambda, \mathbb{R})$ and the boundedness of $\{u_n\}$ in E_λ, we have

$$\begin{aligned}
&\big|\langle I'(u_n) - I'(v_n), u_n - v_n\rangle\big| \\
&\le \big|\langle I'(u_n) - I'(u), u_n - v_n\rangle\big| + \big|\langle I'(u) - I'(v_n), u_n - v_n\rangle\big| \\
&\le \big|\langle I'(u_n) - I'(u), u_n - u\rangle\big| + \big|\langle I'(u_n) - I'(u), u - v_n\rangle\big| \\
&\quad + \big|\langle I'(u) - I'(v_n), u_n - v_n\rangle\big| \\
&\to 0 \quad \text{as} \quad n \to +\infty.
\end{aligned} \tag{2.16}$$

Meanwhile, by (2.5), (2.11), (2.14) and Lemma 2.1, we have

$$\begin{aligned}
&|2\omega \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_{v_n} v_n)(u_n - v_n)dx| \\
&= |2\omega \int_{\mathbb{R}^3} \phi_{u_n} u_n(u_n - v_n)dx - 2\omega \int_{\mathbb{R}^3} \phi_{v_n} v_n(u_n - v_n)dx| \\
&\le 2\omega||\phi_{u_n} u_n||_2||u_n - u||_2 + 2\omega||\phi_{u_n} u_n||_2||u - v_n||_2 \\
&\quad + 2\omega||\phi_{v_n} v_n||_2||u_n - u||_2 + 2\omega||\phi_{v_n} v_n||_2||u - v_n||_2 \\
&\le C||\phi_{u_n}||_6||u_n||_3(||u_n - u||_2 + ||v_n - u||_2) \\
&\quad + C||\phi_{v_n}||_6||v_n||_3(||u_n - u||_2 + ||v_n - u||_2) \\
&\to 0 \quad \text{as} \quad n \to \infty,
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
&| \int_{\mathbb{R}^3} (\phi^2_{u_n} u_n - \phi^2_{v_n} v_n)(u_n - v_n)dx| \\
&= | \int_{\mathbb{R}^3} \phi^2_{u_n} u_n(u_n - v_n)dx - \int_{\mathbb{R}^3} \phi^2_{v_n} v_n(u_n - v_n)dx| \\
&\le \big(\int_{\mathbb{R}^3} \phi^6_{u_n} dx\big)^{\frac{1}{3}} \big(\int_{\mathbb{R}^3} |u_n - u|^{\frac{3}{2}} |u_n|^{\frac{3}{2}} dx\big)^{\frac{2}{3}}
\end{aligned} \tag{2.18}$$

$$
\begin{aligned}
&+ \Big(\int_{\mathbb{R}^3} \phi_{u_n}^6 dx\Big)^{\frac{1}{3}} \Big(\int_{\mathbb{R}^3} |v_n-u|^{\frac{3}{2}}|u_n|^{\frac{3}{2}}dx\Big)^{\frac{2}{3}} \\
&+ \Big(\int_{\mathbb{R}^3} \phi_{v_n}^6 dx\Big)^{\frac{1}{3}} \Big(\int_{\mathbb{R}^3} |u_n-u|^{\frac{3}{2}}|v_n|^{\frac{3}{2}}dx\Big)^{\frac{2}{3}} \\
&+ \Big(\int_{\mathbb{R}^3} \phi_{v_n}^6 dx\Big)^{\frac{1}{3}} \Big(\int_{\mathbb{R}^3} |v_n-u|^{\frac{3}{2}}|v_n|^{\frac{3}{2}}dx\Big)^{\frac{2}{3}} \\
\le\ & C||u_n||_{E_\lambda}^4||u_n||_3(||u_n-u||_3+||v_n-u||_3) \\
&+ C||v_n||_{E_\lambda}^4||v_n||_3(||u_n-u||_3+||v_n-u||_3) \\
\to\ & 0 \ \text{ as } \ n\to+\infty.
\end{aligned}
$$

Now, we prove that $|\int_{\mathbb{R}^3}(f(x,u_n)-f(x,v_n))(u_n-v_n)dx| \to 0$ as $n\to+\infty$. Take $r=1$, $s=p-1$. It follows from Lemma 2.4 that

$$g_n(x)\to 0,\ in\ L^2(\mathbb{R}^3)+L^{\frac{p}{p-1}}(\mathbb{R}^3),$$

where $g_n(x)=f(x,u_n)-f(x,u)-f(x,u_n-v_n)$. Then

$$\int_{\mathbb{R}^3}|f(x,u_n)-f(x,u)-f(x,u_n-v_n)||u_n-v_n|dx \le ||g_n||_{2\vee p'}||u_n-v_n||_{2\vee p}\to 0,$$

as $n\to+\infty$, where $p'=\frac{p}{p-1}$. Take $u_n=v_n$ for all $n>0$ in Lemma 2.4. Then

$$f(x,v_n)-f(x,u)\to 0, \quad \text{in} \ \ L^2(\mathbb{R}^3)+L^{\frac{p}{p-1}}(\mathbb{R}^3).$$

Consequently, we have $\int_{\mathbb{R}^3}|f(x,v_n)-f(x,u)||u_n-v_n|dx\to 0$ as $n\to+\infty$. Then one has

$$
\begin{aligned}
&\int_{\mathbb{R}^3}|f(x,u_n)-f(x,v_n)-f(x,u_n-v_n)||u_n-v_n|dx \\
\le &\int_{\mathbb{R}^3}|f(x,u_n)-f(x,u)-f(x,u_n-v_n)||u_n-v_n|dx \\
&+\int_{\mathbb{R}^3}|f(x,v_n)-f(x,u)||u_n-v_n|dx \\
\to\ & 0 \ \text{ as } n\to+\infty.
\end{aligned} \tag{2.19}
$$

Set $\omega_n=u_n-v_n$. Then by (V_1) and $\omega_n\rightharpoonup 0$, we have

$$||\omega_n||_2^2=\int_{V(x)\ge 1}|\omega_n|^2dx+\int_{V(x)<1}|\omega_n|^2dx\le\frac{1}{\lambda}||\omega_n||_{E_\lambda}^2+o(1) \tag{2.20}$$

for n large enough. Take $0<\alpha<\min\{1,\frac{6-p}{2}\}$. Then $2<\frac{2(p-\alpha)}{2-\alpha}<6$. By (2.20) and the Hölder inequality, we have

$$||\omega_n||_p^p\le||\omega_n||_2^\alpha||\omega_n||_{\frac{2(p-\alpha)}{2-\alpha}}^{p-\alpha}\le c(\lambda)^{-\frac{\alpha}{2}}||\omega_n||_{E_\lambda}^p+o(1) \tag{2.21}$$

for n large enough. Consequently, by (2.11), (2.21) and (f_1'), one has

$$|\int_{\mathbb{R}^3}f(x,\omega_n)\omega_n dx|\le c_1||\omega_n||_2^2+c_2||\omega_n||_p^p\le\frac{c_1}{\lambda}||\omega_n||_{E_\lambda}^2+\frac{cc_2}{(\lambda)^{\frac{\alpha}{2}}}||\omega_n||_{E_\lambda}^p+o(1) \tag{2.22}$$

for n large enough.

Therefore, it follows from (2.15), (2.16), (2.17), (2.18), (2.22) and the boundedness of $\{\omega_n\}$ that

$$\begin{aligned} o(1) &\geq ||u_n - v_n||^2_{E_\lambda} - \int_{\mathbb{R}^3} f(x, u_n - v_n)(u_n - v_n)dx \\ &\geq (1 - \frac{c_1}{\lambda} - \frac{cc_2}{(\lambda)^{\frac{\alpha}{2}}}||\omega_n||^{p-1}_{E_\lambda})||\omega_n||^2_{E_\lambda}. \end{aligned}$$

Let $\Lambda > 0$ be so large that the term in the brackets above is positive when $\lambda \geq \Lambda$, thus we get $\omega_n \to 0$ as $n \to +\infty$ in E_λ. Since $\omega_n = u_n - v_n$ and $v_n \to u$ in E_λ, then we have $u_n \to u$ in E_λ. The proof is complete. □

Proof of Theorem 1.2. For any $0 < \varepsilon < \frac{1}{\tau_2^2}$, it follows from (f_1') and (f_5) that there exists $c(\varepsilon) > 0$ such that

$$|F(x, u)| \leq \frac{\varepsilon}{2}|u|^2 + \frac{\varepsilon}{p}|u|^p.$$

Therefore, for small $\rho > 0$,

$$\begin{aligned} I(u) &= \frac{1}{2}||u||^2_{E_\lambda} - \frac{1}{2}\int_{\mathbb{R}^3} \omega\phi_u u^2 dx - \int_{\mathbb{R}^3} F(x, u)dx \\ &\geq \frac{1}{2}(||u||^2_{E_\lambda} - \varepsilon\tau_2^2||u||^2_{E_\lambda}) - \frac{\varepsilon}{p}\tau_p^p||u||^p_{E_\lambda} \\ &\geq \frac{1}{4}(||u||^2_{E_\lambda} - \varepsilon\tau_2^2||u||^2_{E_\lambda}) \end{aligned}$$

for all $u \in \overline{B_\rho}$, where $B_\rho = \{u \in E_\lambda : ||u||_{E_\lambda} < \rho\}$. Hence,

$$I|_{\partial B_\rho} \geq \frac{1}{4}(1 - \varepsilon\tau_2^2)\rho^2 := \eta > 0.$$

Take $0 \neq u \in E_\lambda$. It follows from (f_1') and (2.6) that for any $M > 0$, there exists $C(M) > 0$ such that

$$F(x, u) \geq M|u|^4 - C(M)|u|^2.$$

Then by Lemma 2.1, one has

$$\begin{aligned} I(tu) &= \frac{t^2}{2}||u||^2_{E_\lambda} - \frac{t^2}{2}\int_{\mathbb{R}^3} \omega\phi_{tu} u^2 dx - \int_{\mathbb{R}^3} F(x, tu)dx \\ &\leq \frac{t^2}{2}||u||^2_{E_\lambda} + \frac{t^2}{2}\int_{\mathbb{R}^3} \omega^2 u^2 dx + C(M)t^2\int_{\mathbb{R}^3} u^2 dx - Mt^4\int_{\mathbb{R}^3} u^4 dx \\ &\to -\infty \end{aligned}$$

as $t \to +\infty$. Therefore, there exists a point $e \in E_\lambda \setminus \overline{B_\rho}$ such that $I(e) \leq 0$. By Lemma (2.3), I satisfies the $(PS)_c$ condition for large $\lambda > 0$. Furthermore, it is obvious that $I(0) = 0$. Hence I possesses a critical value $c \geq \eta$ by Lemma 2.3, i.e., problem (1.1) has a nontrivial weak solution in E_λ. The proof is complete. □

Proof of Theorem 1.3. From the proof of Theorem 1.2, we know that there exist constants $\rho > 0$ and $\eta > 0$ such that $I|_{\partial B_\rho} \geq \eta > 0$ and there is a point $e \in E_\lambda \setminus \overline{B}$ such that $I(e) \leq 0$. Now we prove that I satisfies the $(PS)_c$ condition for large n. We need to prove that $\{u_n\}$ is bounded in E_λ. If $\{u_n\}$ is unbounded in E_λ, we can assume that $||u_n||_{E_\lambda} \to +\infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{||u_n||}$. Then $||v_n|| = 1$ and $||v_n||_r \leq \tau_r ||v_n||_{E_\lambda} = \tau_r$ for $2 \leq r \leq 6$. Since $||v_n||_{E_\lambda} = 1$, going if necessary to a subsequence, we can assume that $v_n \rightharpoonup v$ in E_λ, $v_n \to v$ in $L^r(\mathbb{R}^3)$ for $2 \leq r < 6$ and $v_n \to v$ a.e. in $\mathbb{R}^3$. Set $\Omega = \{x \in \mathbb{R}^3 : v(x) \neq 0\}$. If $meas(\Omega) > 0$, then $\int_\Omega |v|^\theta dx > 0$. It follows from (f_1') and (f_2) that for any $M > \frac{C_0}{2\int_\Omega |v|^4 dx}$, there exists a constant $C_0(M) > 0$ such that

$$F(x,u) \geq M|u|^4 - C_0(M)|u|^2, \tag{2.23}$$

where

$$C_0 = \sup_{u \in E_\lambda \setminus \{0\}} \frac{\int_{\mathbb{R}^3} \omega |\phi_u| u^2 dx}{||u||_{E_\lambda}^4}.$$

Furthermore, we have

$$\frac{I(u_n)}{||u_n||_{E_\lambda}^4} = \frac{1}{2||u_n||_{E_\lambda}^2} - \frac{1}{2||u_n||_{E_\lambda}^4} \int_{\mathbb{R}^3} \omega \phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} \frac{F(x,u_n)}{||u_n||_{E_\lambda}^4} dx.$$

Then by (2.5), we deduce that

$$\lim \liminf_{n\to\infty} \int_{\mathbb{R}^3} \frac{F(x,u_n)}{||u_n||_{E_\lambda}^4} dx \leq \frac{C_0}{2}.$$

By (2.23), we have

$$\int_{\mathbb{R}^3} \frac{F(x,u_n)}{||u_n||_{E_\lambda}^4} dx \geq M||v_n||_4^4 - C_0(M) \frac{||v_n||_2^2}{||u_n||_{E_\lambda}^2}.$$

Therefore

$$\begin{aligned} \frac{C_0}{2} &\geq \liminf_{n\to\infty} \Big(\int_{\mathbb{R}^3} \frac{F(x,u_n)}{||u_n||_{E_\lambda}^4} dx + C_0(M) \frac{||v_n||_2^2}{||u_n||_{E_\lambda}^2} \Big) \\ &\geq \liminf_{n\to\infty} M||v_n||_4^4 \geq M \int_\Omega |v|^4 dx > \frac{C_0}{2}, \end{aligned}$$

which is a contradiction, then $meas(\Omega) = 0$, and as a result $v = 0$ a.e. in $\mathbb{R}^3$. Therefore, from (V_1), we have

$$||v_n||_2^2 = \int_{V(x)\geq 1} |v_n|^2 dx + \int_{V(x)<1} |v_n|^2 dx \leq \frac{1}{\lambda} ||v_n||_{E_\lambda}^2 + o(1) \leq \frac{2}{\lambda}$$

for n large enough. It follows from (f_1) and (f_7) that there exists a constant $c > 0$ such that

$$4F(x,u) - uf(x,u) \leq c|u|^2$$

for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$. Therefore, by Lemma 2.1, one has

$$\begin{aligned} 0 \leftarrow & \frac{1}{||u_n||^2_{E_\lambda}} \big[4I(u_n) - \langle I'(u_n), u_n\rangle\big] \\ = & \frac{1}{||u_n||^2_{E_\lambda}} \big[||u_n||^2_{E_\lambda} + \int_{\mathbb{R}^3} \big(f(x,u_n)u_n - 4F(x,u_n)dx\big) + \int_{\mathbb{R}^3} \phi^2_{u_n} u_n^2 dx\big] \\ \ge & 1 - c\int_{\mathbb{R}^3} |v_n|^2 dx \\ \ge & 1 - \frac{2c}{\lambda} \quad \text{as} \quad n \to \infty. \end{aligned}$$

Let $\lambda > 0$ be so large that the term $1 - \frac{2c}{\lambda} > 0$, then we get a contradiction. Hence $\{u_n\}$ is bounded in E_λ for large λ. Therefore, I possesses a critical value by Lemma 2.3, i.e., problem (1.1) has at least one nontrivial solution. The proof is complete. □

3. Existence of infinitely many nontrivial solutions

In this section, we consider the existence of infinitely many solutions of problem (1.1). We will give the proofs of Theorem 1.4 and Theorem 1.5. To complete the proof, we need the following results.

Lemma 3.1 ([17, Lemma 2.2])**.** *Let X be an infinitely dimensional Banach space and let $I \in C^1(X,\mathbb{R})$ be even, satisfy $(PS)_c$ condition, and $I(0) = 0$. If $X = Y \oplus Z$, where Y is finite dimensional and I satisfies*

(i) *There exists constants $\rho,\ \alpha > 0$ such that $I|_{\partial B_\rho \cap Z} \ge \alpha$;*

(ii) *For any finite dimensional subspace $\widetilde{X} \subset X$, there is $R = R(\widetilde{X}) > 0$ such that $I(u) \le 0$ on $\widetilde{X} \setminus B_R$.*

Then I possesses an unbounded sequence of critical values.

Let $\{e_j\}$ be a total orthonormal basis of $L^2(B_{R_0})$ (B_{R_0} appears in (V_2)) and define $X_j = \mathbb{R}e_j,\ j \in \mathbb{N}$,

$$Y_k = \oplus_{j=1}^k X_j,\ Z_k = \oplus_{j=k+1}^\infty X_j,\ k \in \mathbb{N}.$$

Set

$$E_\lambda(B_{R_0}) := \{u \in H^1(B_{R_0}) | \int_{B_{R_0}} \lambda V(x) u^2 dx < +\infty\}$$

with the norm

$$\|u\|_{E_\lambda(B_{R_0})} = \big(\int_{B_{R_0}} (|\nabla u|^2 + \lambda V(x)u^2) dx\big)^{\frac{1}{2}}.$$

Lemma 3.2. *Suppose that (V_1) is satisfied. Then for $2 \le r < 6$*

$$\beta_k := \sup_{u \in Z_k, \|u\|_{E_\lambda(B_{R_0})} = 1} \|u\|_{L^r(B_{R_0})} \to 0 \quad as \quad k \to +\infty.$$

Proof. The proof is similar to Lemma 3.2 of [12] or Lemma 3.2 of [16], so we omit it here. □

By Lemma 3.2, we can choose an integer $m \geq 1$ such that

$$\int_{B_{R_0}} u^2 dx \leq \frac{1}{2c_1} \int_{B_{R_0}} (|\nabla u|^2 + \lambda V(x) u^2) dx, \ \forall \ u \in Z_m \cap E_\lambda(B_{R_0}), \tag{3.1}$$

where c_1 appears in (f_1'). Let $\gamma(x) = 0$ if $|x| \leq R_0$ and $\gamma(x) = 1$ if $|x| \geq R_0$. Define

$$Y = \{(1-\gamma)u : u \in E_\lambda, (1-\gamma)u \in Y_m\} \tag{3.2}$$

and

$$Z = \{(1-\gamma)u : u \in E_\lambda, (1-\gamma)u \in Z_m\} + \{\gamma v : v \in E_\lambda\}. \tag{3.3}$$

Then Y and Z are subspaces of E_λ, and $E_\lambda = Y \oplus Z$.

Lemma 3.3. *Suppose that (V_1), (V_2) and (f_1') are satisfied. Then there exist constants $\rho, \ \alpha > 0$ such that $I|_{\partial B_\rho \cap Z} \geq \alpha$ for large λ.*

Proof. It follows from (3.1), (3.3) and (V_2) that

$$\begin{aligned} ||u||_2^2 &= \int\limits_{|x|<R_0} |u|^2 dx + \int\limits_{|x|\geq R_0} |u|^2 dx \\ &\leq \frac{1}{2c_1} ||u||_{E_\lambda}^2 + \frac{1}{\lambda d} \int\limits_{\{x\in\mathbb{R}^3 : V(x)>d\}} \lambda V(x) |u|^2 dx \\ &\leq \frac{1}{2c_1} ||u||_{E_\lambda}^2 + \frac{1}{\lambda d} ||u||_{E_\lambda}^2, \ \forall \ u \in Z. \end{aligned} \tag{3.4}$$

Therefore, by (2.1), (2.8) and (3.4), we have

$$\begin{aligned} I(u) &= \frac{1}{2} ||u||_{E_\lambda}^2 - \frac{1}{2} \int_{\mathbb{R}^3} \omega \phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u) dx \\ &\geq \frac{1}{2} ||u||_{E_\lambda}^2 - \frac{c_1}{2} ||u||_2^2 - \frac{c_2}{p} ||u||_p^p \\ &\geq \frac{1}{4} ||u||_{E_\lambda}^2 - \frac{c_1}{2\lambda} ||u||_{E_\lambda}^2 - \frac{c_2 \tau_p^p}{p} ||u||_{E_\lambda}^p \\ &\geq \frac{1}{8} ||u||_{E_\lambda}^2 - \frac{c_2 \tau_p^p}{p} ||u||_{E_\lambda}^p \end{aligned}$$

for n large enough. Since $2 < p < 6$, then there exist constants $\rho, \ \alpha > 0$ such that $I|_{\partial B_\rho \cap Z} \geq \alpha$. The proof is complete. □

Lemma 3.4. *Suppose that (f_1') and (f_2) are satisfied. Then for any finite dimensional subspace $\widetilde{E_\lambda} \subset E_\lambda$, there is $R = R(\widetilde{E_\lambda}) > 0$ such that $I(u) \leq 0$ on $\widetilde{E_\lambda} \setminus B_R$.*

Proof. For any finite dimensional subspace $\widetilde{E_\lambda} \subset E_\lambda$, by the equivalence of norms in the finite dimensional space, there is a constant $C(4) > 0$ such that

$$||u||_4^4 \geq C(4) ||u||_{E_\lambda}^4, \ \forall \ u \in \widetilde{E_\lambda}.$$

It follows from (f_1') and (2.7) that for any $M > \frac{C_0}{2C(4)}$ (where C_0 appears in (2.23)), there exists a constant $C(M) > 0$ such that

$$F(x,u) \geq M|u|^4 - C(M)|u|^2, \ \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

Then

$$\begin{aligned} I(u) &= \frac{1}{2}||u||^2_{E_\lambda} - \frac{1}{2}\int_{\mathbb{R}^3} \omega\phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx \\ &\leq \frac{1}{2}||u||^2_{E_\lambda} + \frac{C_0}{2}||u||^4_{E_\lambda} + C(M)||u||^2_2 - M||u||^4_4 \\ &\leq (\frac{1}{2} + C(M)\tau_2^2)||u||^2_{E_\lambda} - (MC(4) - \frac{C_0}{2})||u||^4_{E_\lambda} \end{aligned}$$

for all $u \in \widetilde{E_\lambda}$. Hence, there is a large $R = R(\widetilde{E_\lambda}) > 0$ such that $I(u) \leq 0$ on $\widetilde{E_\lambda} \setminus B_R$. The proof is complete. □

Proof of Theorem 1.4. Let $X = E_\lambda$, Y and Z be defined by (3.2) and (3.3), respectively. From (f_6), Lemma 2.5, Lemma 3.3, Lemma 3.4 and $I(0) = 0$, we know that I satisfies all the conditions of Lemma 3.1. Therefore, problem (1.1) has infinitely many nontrivial weak solutions. The proof is complete. □

Proof of Theorem 1.5. Let $X = E_\lambda$, Y and Z be defined by (3.2) and (3.3), respectively. From the proof of Theorem 1.3 and Theorem 1.4, we know that I satisfies all the conditions of Lemma 3.1. Therefore, problem (1.1) has infinitely many nontrivial weak solutions. The proof is complete. □

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Guofeng Che
School of Mathematics and Statistics
Central South University
Changsha, 410083 Hunan, P. R. China
E-mail address: cheguofeng222@163.com

Haibo Chen
School of Mathematics and Statistics
Central South University
Changsha, 410083 Hunan, P. R. China
E-mail address: math_chb@163.com