J. Korean Math. Soc. **54** (2017), No. 3, pp. 967–986
https://doi.org/10.4134/JKMS.j160320
pISSN: 0304-9914 / eISSN: 2234-3008

SUMMING AND DOMINATED OPERATORS ON A CARTESIAN PRODUCT OF $c_0(\mathcal{X})$ SPACES

Gabriela Badea and Dumitru Popa

Abstract. We give the necessary condition for an operator defined on a cartesian product of $c_0(\mathcal{X})$ spaces to be summing or dominated and we show that for the multiplication operators this condition is also sufficient. By using these results, we show that $\Pi_s(c_0, \ldots, c_0; c_0)$ contains a copy of $l_s\left(l_2^m \mid m \in \mathbb{N}\right)$ for $s > 2$ or a copy of $l_s\left(l_1^m \mid m \in \mathbb{N}\right)$, for any $1 \leq s < \infty$. Also, $\Delta_{s_1,\ldots,s_n}(c_0, \ldots, c_0; c_0)$ contains a copy of $l_{v_n(s_1,\ldots,s_n)}$ if $v_n(s_1, \ldots, s_n) \leq 2$ or a copy of $l_{v_n(s_1,\ldots,s_n)}\left(l_2^m \mid m \in \mathbb{N}\right)$ if $2 < v_n(s_1, \ldots, s_n)$, where $\frac{1}{v_n(s_1,\ldots,s_n)} = \frac{1}{s_1} + \cdots + \frac{1}{s_n}$. We find also the necessary and sufficient conditions for bilinear operators induced by some method of summability to be 1-summing or 2-dominated.

1. Introduction and notation

In this paper we continue our study on the summing operators defined on a cartesian product of $c_0(\mathcal{X})$. While in [2] we deal with nuclear and multiple 1-summing operators on a cartesian product of $c_0(\mathcal{X})$, here we will address the dominated and summing operators defined on the same cartesian product. The summing operators as well as the dominated ones are two possible extensions to the multilinear settings of the linear summing operator, which were considered in order to find multilinear versions of the Pietsch domination theorem. In this paper, we will study simultaneously the dominated and the summing operators on a cartesian product of $c_0(\mathcal{X})$.

The notations and terminology used along the paper are standard in Banach space theory, as the reader can see in the famous monographs [5, 6, 15].

For $X_1, \ldots, X_n$, Y Banach spaces over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we consider the Banach space $\mathcal{L}(X_1, \ldots, X_n; Y)$ of all bounded n-linear operators, called simply multilinear operators, endowed with the operator norm

$$\|T\| = \sup_{\|x_1\|\leq 1,\ldots,\|x_n\|\leq 1} \|T(x_1, \ldots, x_n)\|.$$

Received May 4, 2016.

2010 *Mathematics Subject Classification.* Primary 47B10, 47L20; Secondary 46B45.

Key words and phrases. $(s; s_1, \ldots s_n)$-summing operator, $(s_1, \ldots s_n)$-dominated operator.

The first named author was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-RU-TE-2012-3-0042.

Let $\mathcal{X} = (X_k)_{k\in\mathbb{N}}$ be a sequence of Banach spaces. We denote by $c_0(\mathcal{X})$ the Banach space of all sequences $(x_k)_{k\in\mathbb{N}}$ with $x_k \in X_k$ for all $k \in \mathbb{N}$, $\|x_k\|_{X_k} \to 0$ as $k \to \infty$, endowed with the norm $\left\|(x_k)_{k\in\mathbb{N}}\right\|_{c_0(\mathcal{X})} = \sup_{k\in\mathbb{N}} \|x_k\|_{X_k}$, see [16, page 43]. Note that for $x \in c_0(\mathcal{X})$, $\|x\|_{c_0(\mathcal{X})} = \sup_{k\in\mathbb{N}} \|p_k(x)\|_{X_k}$, where $p_k : c_0(\mathcal{X}) \to X_k$ denotes the canonical mapping defined by $p_k(x_1, x_2, \ldots,) = x_k$, $k \in \mathbb{N}$. Also for $k \in \mathbb{N}$, we consider the canonical map $\sigma_k : X_k \to c_0(\mathcal{X})$ defined by $\sigma_k(x) = (0, \ldots, 0, \underbrace{x}_{k^{th}}, 0, \ldots)$. To avoid any possible confusion, if $\mathcal{X}_j = \left(X_k^j\right)_{k\in\mathbb{N}}$ $(1 \leq j \leq n)$ is a finite system of sequences of Banach spaces we write $\sigma_k^j : X_k^j \to c_0(\mathcal{X}_j)$ respectively $p_k^j : c_0(\mathcal{X}_j) \to X_k^j$ for the canonical mappings.

Let us recall the definition of a Banach ideal of operators, see [11] and also [9].

A subclass $\mathcal{A}$ of the class $\mathcal{L}$ of all bounded n-linear operators between Banach spaces is called an *ideal* if

(M_1) For all Banach spaces $X_1, \ldots, X_n, Y$ the component

$$\mathcal{A}(X_1, \ldots, X_n; Y) := \mathcal{L}(X_1, \ldots, X_n; Y) \cap \mathcal{A}$$

is a linear subspace of $\mathcal{L}(X_1, \ldots, X_n; Y)$.

(M_2) (the ideal property) If

$$X_1 \xrightarrow{A_1} Y_1, \ldots, X_n \xrightarrow{A_n} Y_n, Y_1 \times \cdots \times Y_n \xrightarrow{T} Z \xrightarrow{S} W,$$

where all A_j and S are bounded linear operators, $T \in \mathcal{A}(X_1, \ldots, X_n; Y)$, then the composition $S \circ T \circ (A_1, \ldots, A_n) \in \mathcal{A}(X_1, \ldots, X_n; W)$; $T \circ (A_1, \ldots, A_n) : X_1 \times \cdots \times X_n \to Z$ is defined by

$$T \circ (A_1, \ldots, A_n)(x_1, \ldots, x_n) = T(A_1(x_1), \ldots, A_n(x_n)).$$

(M_3) The mapping $P_{\mathbb{K}} : \mathbb{K}^n \to \mathbb{K}$, $P_{\mathbb{K}}(\lambda_1, \ldots, \lambda_n) = \lambda_1 \cdots \lambda_n$ belongs to $\mathcal{A}$.

A (ω-) *normed ideal* $(0 < \omega \leq 1)$ is a pair $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$, where $\mathcal{A}$ is an ideal and $\|\cdot\|_{\mathcal{A}} : \mathcal{A} \to [0, \infty)$ is an ideal (ω-) norm, i.e.,

(M_1') $\|\cdot\|_{\mathcal{A}}$ restricted to each component is a (ω-) norm.

(M_2') $\|S \circ T \circ (A_1, \ldots, A_n)\|_{\mathcal{A}} \leq \|S\| \|T\|_{\mathcal{A}} \|A_1\| \cdots \|A_n\|$ in the situation of (M_2).

(M_3') $\|P_{\mathbb{K}}\|_{\mathcal{A}} = 1$ in the situation of (M_3).

A *Banach ideal of operators* is a normed ideal of operators $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ with the property that all the components $(\mathcal{A}(X_1, \ldots, X_n; Y), \|\cdot\|_{\mathcal{A}})$ are Banach spaces.

Given $0 < p < \infty$ and a Banach space X, for a finite system $(x_i)_{1\leq i\leq n} \subset X$ we define $l_p(x_i \mid 1 \leq i \leq n) := \left(\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}}$ and $w_p\left((x_i)_{1\leq i\leq n}; X\right) := \sup_{\|x^*\|\leq 1} \left(\sum_{i=1}^n |x^*(x_i)|^p\right)^{\frac{1}{p}}$. If we consider the finite system of elements $(x_i)_{1\leq i\leq n}$

as a finite set A, then we denote by $w_p(A) = \sup_{\|x^*\|\le 1} \left(\sum_{i=1}^{n} |x^*(x_i)|^p \right)^{\frac{1}{p}}$. For $0 < p < \infty$, we use the common notation l_p for the space of all scalar sequences $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} |\lambda_n|^p < \infty$, $\|\lambda\|_p = (\sum_{n=1}^{\infty} |\lambda_n|^p)^{\frac{1}{p}}$. For $1 \le p < \infty$, we define p^* the conjugate of p, that is $\frac{1}{p} + \frac{1}{p^*} = 1$.

The following notation will be used to study simultaneously the $(s; s_1, \ldots, s_n)$-summing operators and the $(s_1, \ldots, s_n)$-dominated ones. Let n be a natural number. We define $v_n : [1, \infty)^n \to \left[\frac{1}{n}, \infty\right)$ by $\frac{1}{v_n(s_1,\ldots,s_n)} = \frac{1}{s_1} + \cdots + \frac{1}{s_n}$.

Let $s_1, \ldots, s_n \in [1, \infty)$ and $s \in (0, \infty)$ be such that $v_n(s_1, \ldots, s_n) \le s$. A bounded multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ is called $(s; s_1, \ldots, s_n)$-*summing* if there exists constant $C \ge 0$ such that for each $\left(x_i^j\right)_{1\le i\le m} \subset X_j$ $(1 \le j \le n)$ the following relation holds

$$l_s\left(T\left(x_i^1, \ldots, x_i^n\right) \mid 1 \le i \le m\right) \le C w_{s_1}\left(\left(x_i^1\right)_{1\le i\le m}\right) \cdots w_{s_n}\left(\left(x_i^n\right)_{1\le i\le m}\right).$$

In this case, $\pi_{s;s_1,\ldots,s_n}(T) := \inf\{C \,|\, C \text{ as above}\}$. The class of all $(s; s_1, \ldots, s_n)$-summing operators from $X_1 \times \cdots \times X_n$ into Y is denoted by

$$\Pi_{s;s_1,\ldots,s_n}(X_1, \ldots, X_n; Y).$$

It is well known that the class of all $(s; s_1, \ldots, s_n)$-summing operators is an ideal and for $s \ge 1, \pi_{s;s_1,\ldots,s_n}(\cdot)$ is a norm, while for $s < 1$, $\pi_{s;s_1,\ldots,s_n}(\cdot)$ is a s-norm.

For $s \in [1, \infty)$, a $(s; s, \ldots, s)$-summing operator will be called a s-*summing operator.*

If $s_1, \ldots, s_n \in [1, \infty)$ a $(v_n(s_1, \ldots, s_n); s_1, \ldots, s_n)$-summing operator $T : X_1 \times \cdots \times X_n \to Y$ is called $(s_1, \ldots, s_n)$-*dominated* and

$$\Delta_{s_1,\ldots,s_n}(T) = \pi_{v_n(s_1,\ldots,s_n);s_1,\ldots,s_n}(T).$$

We denote by $\Delta_{s_1,\ldots,s_n}(X_1, \ldots, X_n; Y)$ the class of all $(s_1, \ldots, s_n)$-dominated operators from $X_1 \times \cdots \times X_n$ into Y. If $v_n(s_1, \ldots, s_n) \ge 1$, then $\Delta_{s_1,\ldots,s_n}(\cdot)$ is a norm and if $v_n(s_1, \ldots, s_n) < 1$, $\Delta_{s_1,\ldots,s_n}(\cdot)$ is a $v_n(s_1, \ldots, s_n)$-norm.

We will need the following obvious result whose simple proof is left to the reader.

Remark 1. (i) If $T : X_1 \times \cdots \times X_n \to Y$ is $(s; s_1, \ldots, s_n)$-summing, then

$$\pi_{s;s_1,\ldots,s_n}(T) = \sup\left\{l_s\left(T\left(x_i^1, \ldots, x_i^n\right) \mid 1 \le i \le m\right)\right\},$$

where the supremum is taken over all systems $\left(x_i^j\right)_{1\le i\le m} \subset X_j$ such that $w_{s_j}\left(\left(x_i^j\right)_{1\le i\le m}\right) \le 1$ $(1 \le j \le n)$.

(ii) If $T : X_1 \times \cdots \times X_n \to Y$ is $(s; s_1, \ldots, s_n)$-summing, then for any $0 < \varepsilon < 1$ there exists $\left(x_i^j\right)_{1 \le i \le m} \subset X_j$ such that $w_{s_j}\left(\left(x_i^j\right)_{1 \le i \le m}\right) \le 1$ $(1 \le j \le n)$ and $(1 - \varepsilon)\,\pi_{s;s_1,\ldots,s_n}(T) \le l_s\left(T\left(x_i^1, \ldots, x_i^n\right) \mid 1 \le i \le m\right)$.

2. Summing and dominated multilinear operators on a cartesian product of $c_0(\mathcal{X})$ spaces

The next lemma is well-known, but for the sake of completeness we will include here its short proof.

Lemma 2. *Let $1 \le s < \infty$ and $(x_i)_{1 \le i \le k} \subset c_0(\mathcal{X})$. Then*

$$w_s\left((x_i)_{1 \le i \le k}; c_0(\mathcal{X})\right) = \sup_{n \in \mathbb{N}} w_s\left((p_n(x_i))_{1 \le i \le k}; X_n\right).$$

Proof. By a well-known relation, see [12, Lemma 1.14, page 40], we have

$$(*) \qquad w_s\left((x_i)_{1 \le i \le k}; c_0(\mathcal{X})\right) = \sup_{\|(\lambda_1,\ldots,\lambda_k)\|_{l_{s^*}^k} \le 1} \left\| \sum_{i=1}^k \lambda_i x_i \right\|_{c_0(\mathcal{X})}$$

and then

$$\begin{aligned} w_s\left((x_i)_{1 \le i \le k}; c_0(\mathcal{X})\right) &= \sup_{\|(\lambda_1,\ldots,\lambda_k)\|_{l_{s^*}^k} \le 1} \left(\sup_{n \in \mathbb{N}} \left\| p_n\left(\sum_{i=1}^k \lambda_i x_i\right) \right\|_{X_n} \right) \\ &= \sup_{n \in \mathbb{N}} \left(\sup_{\|(\lambda_1,\ldots,\lambda_k)\|_{l_{s^*}^k} \le 1} \left\| \sum_{i=1}^k \lambda_i p_n(x_i) \right\|_{X_n} \right) \\ &= \sup_{n \in \mathbb{N}} w_s\left((p_n(x_i))_{1 \le i \le k}; X_n\right) \text{ again by } (*). \end{aligned}$$

□

Our next result gives a necessary condition for a bounded multilinear operator defined on a cartesian product of $c_0(\mathcal{X})$ to be summing or dominated.

Theorem 3. *Let $n \in \mathbb{N}$, $1 \le s_1, \ldots, s_n < \infty$, $0 < s < \infty$ be such that $v_n(s_1, \ldots, s_n) \le s$ and $T : c_0(\mathcal{X}_1) \times \cdots \times c_0(\mathcal{X}_n) \to Y$ a bounded multilinear operator. If T is $(s; s_1, \ldots, s_n)$-summing, then all $T \circ \left(\sigma_k^1, \ldots, \sigma_k^n\right) : X_k^1 \times \cdots \times X_k^n \to Y$ are $(s; s_1, \ldots, s_n)$-summing and $\left(\pi_{s;s_1,\ldots,s_n}\left(T \circ \left(\sigma_k^1, \ldots, \sigma_k^n\right)\right)\right)_{k \in \mathbb{N}} \in l_s$.*

Moreover, $\left\|\left(\pi_{s;s_1,\ldots,s_n}\left(T \circ \left(\sigma_k^1, \ldots, \sigma_k^n\right)\right)\right)_{k \in \mathbb{N}}\right\|_s \le \pi_{s;s_1,\ldots,s_n}(T)$.

Proof. The fact that all $T \circ \left(\sigma_k^1, \ldots, \sigma_k^n\right)$ are $(s; s_1, \ldots, s_n)$-summing follows from the ideal property of the class of all $(s; s_1, \ldots, s_n)$-summing operators. Let $m \in \mathbb{N}$ and $0 < \varepsilon < 1$. For $1 \le k \le m$, by Remark 1, there exists

$F_k \subset \mathbb{N}$, F_k a finite set and $\left(x_{ki}^j\right)_{i\in F_k} \subset X_k^j$ such that $w_{s_j}\left(x_{ki}^j \mid i \in F_k\right) \leq 1$ for $1 \leq j \leq n$ and

$$\begin{aligned}&(1-\varepsilon)\,\pi_{s;s_1,\ldots,s_n}\left(T\circ\left(\sigma_k^1,\ldots,\sigma_k^n\right)\right)\\ &\leq l_s\left(T\circ\left(\sigma_k^1,\ldots,\sigma_k^n\right)\left(x_{ki}^1,\ldots,x_{ki}^n\right) \mid i\in F_k\right).\end{aligned}$$

Then

$$\begin{aligned}&(1-\varepsilon)\left(\sum_{k=1}^m\left[\pi_{s;s_1,\ldots,s_n}\left(T\circ\left(\sigma_k^1,\ldots,\sigma_k^n\right)\right)\right]^s\right)^{\frac{1}{s}}\\ \leq&\left(\sum_{k=1}^m\sum_{i\in F_k}\left\|T\circ\left(\sigma_k^1,\ldots,\sigma_k^n\right)\left(x_{ki}^1,\ldots,x_{ki}^n\right)\right\|^s\right)^{\frac{1}{s}}\\ =&\left(\sum_{k=1}^m\sum_{i\in F_k}\left\|T\left(\sigma_k^1\left(x_{ki}^1\right),\ldots,\sigma_k^n\left(x_{ki}^n\right)\right)\right\|^s\right)^{\frac{1}{s}}.\end{aligned}$$

Further, since T is $(s;s_1,\ldots,s_n)$-summing,

$$\begin{aligned}&\left(\sum_{k=1}^m\sum_{i\in F_k}\left\|T\left(\sigma_k^1\left(x_{ki}^1\right),\ldots,\sigma_k^n\left(x_{ki}^n\right)\right)\right\|^s\right)^{\frac{1}{s}}\\ =&\left(\sum_{i\in F_1}\left\|T\left(\sigma_1^1\left(x_{1i}^1\right),\ldots,\sigma_1^n\left(x_{1i}^n\right)\right)\right\|^s+\cdots+\sum_{i\in F_m}\left\|T\left(\sigma_m^1\left(x_{mi}^1\right),\ldots,\sigma_m^n\left(x_{mi}^n\right)\right)\right\|^s\right)^{\frac{1}{s}}\\ \leq&\ \pi_{s;s_1,\ldots,s_n}(T)\,w_{s_1}(A_1)\cdots w_{s_n}(A_n),\end{aligned}$$

where

$$\begin{aligned}A_1&=\left\{\sigma_1^1\left(x_{1i}^1\right)\mid i\in F_1\right\}\cup\left\{\sigma_2^1\left(x_{2i}^1\right)\mid i\in F_2\right\}\cup\cdots\cup\left\{\sigma_m^1\left(x_{mi}^1\right)\mid i\in F_m\right\}\\ &\subset c_0\left(\mathcal{X}_1\right)\\ &\vdots\\ A_n&=\left\{\sigma_1^n\left(x_{1i}^n\right)\mid i\in F_1\right\}\cup\left\{\sigma_2^n\left(x_{ni}^n\right)\mid i\in F_2\right\}\cup\cdots\cup\left\{\sigma_m^n\left(x_{mi}^n\right)\mid i\in F_m\right\}\\ &\subset c_0\left(\mathcal{X}_n\right).\end{aligned}$$

From Lemma 2 we have

$$w_{s_j}\left(A_j\right)=\sup_{a\in\mathbb{N}}w_{s_j}\left(p_a^j\left(A_j\right)\right)\ \text{ for } 1\leq j\leq n.$$

Let $a\in\mathbb{N}$. For instance, by using the relations $p_a^1\circ\sigma_b^1=0$ for $b\in\mathbb{N}$, $b\neq a$ and $p_a^1\circ\sigma_a^1=I_{X_a^1}$, we have

$$\begin{aligned}p_a^1\left(A_1\right)=&\left\{p_a^1\circ\sigma_1^1\left(x_{1i}^1\right)\mid i\in F_1\right\}\cup\left\{p_a^1\circ\sigma_2^1\left(x_{2i}^1\right)\mid i\in F_2\right\}\cup\cdots\\ &\cup\left\{p_a^1\circ\sigma_a^1\left(x_{ai}^1\right)\mid i\in F_a\right\}\end{aligned}$$

$$= \begin{cases} \left\{x_{ai}^1 \mid i \in F_a\right\} \cup \{0\} & \text{for } 1 \leq a \leq m \\ \{0\} & \text{for } a \geq m+1. \end{cases}$$

Then

$$w_{s_1}(A_1) = \max\left\{w_{s_1}\left(x_{1i}^1 \mid i \in F_1\right), \ldots, w_{s_1}\left(x_{mi}^1 \mid i \in F_m\right)\right\} \leq 1$$

since $w_{s_1}\left(x_{1i}^1 \mid i \in F_1\right) \leq 1, \ldots, w_{s_1}\left(x_{mi}^1 \mid i \in F_m\right) \leq 1$. In a similar way,

$$w_{s_n}(A_n) = \max\left\{w_{s_n}\left(x_{1i}^n \mid i \in F_1\right), \ldots, w_{s_n}\left(x_{mi}^n \mid i \in F_m\right)\right\} \leq 1.$$

Thus

$$(1-\varepsilon)\left(\sum_{k=1}^{m}\left[\pi_{s;s_1,\ldots,s_n}\left(T \circ \left(\sigma_k^1, \ldots, \sigma_k^n\right)\right)\right]^s\right)^{\frac{1}{s}} \leq \pi_{s;s_1,\ldots,s_n}(T).$$

Since $0 < \varepsilon < 1$ and $m \in \mathbb{N}$ are arbitrary, we obtain

$$\left(\sum_{k=1}^{\infty}\left[\pi_{s;s_1,\ldots,s_n}\left(T \circ \left(\sigma_k^1, \ldots, \sigma_k^n\right)\right)\right]^s\right)^{\frac{1}{s}} \leq \pi_{s;s_1,\ldots,s_n}(T),$$

which completes the proof. □

By Theorem 3 we obtain the following two consequences:

Corollary 4. *Let $T : c_0(\mathcal{X}_1) \times \cdots \times c_0(\mathcal{X}_n) \to Y$ be a bounded multilinear operator.*

(i) *Let $1 \leq s < \infty$. If T is s-summing, then all $T \circ \left(\sigma_k^1, \ldots, \sigma_k^n\right) : X_k^1 \times \cdots \times X_k^n \to Y$ are s-summing and $\left(\pi_s\left(T \circ \left(\sigma_k^1, \ldots, \sigma_k^n\right)\right)\right)_{k\in\mathbb{N}} \in l_s$. Moreover,*

$$\left\|\left(\pi_s\left(T \circ \left(\sigma_k^1, \ldots, \sigma_k^n\right)\right)\right)_{k\in\mathbb{N}}\right\|_s \leq \pi_s(T).$$

(ii) *Let $1 \leq s_1, \ldots, s_n < \infty$. If T is $(s_1, \ldots, s_n)$-dominated, then all $T \circ \left(\sigma_k^1, \ldots, \sigma_k^n\right) : X_k^1 \times \cdots \times X_k^n \to Y$ are $(s_1, \ldots, s_n)$-dominated and $\left(\Delta_{s_1,\ldots,s_n}\left(T \circ \left(\sigma_k^1, \ldots, \sigma_k^n\right)\right)\right)_{k\in\mathbb{N}} \in l_{v_n(s_1,\ldots,s_n)}$.*

Moreover, $\left\|\left(\Delta_{s_1,\ldots,s_n}\left(T \circ \left(\sigma_k^1, \ldots, \sigma_k^n\right)\right)\right)_{k\in\mathbb{N}}\right\|_{v_n(s_1,\ldots,s_n)} \leq \Delta_{s_1,\ldots,s_n}(T)$.

We need also the following kind of Nahoum's result, see [10, Theorem, page 5], [16, Lemma 23, page 274].

Proposition 5. *Let $s_1, \ldots, s_n \in [1, \infty)$, $s \in (0, \infty)$ be such that $v_n(s_1, \ldots, s_n) \leq s$ and $T : X_1 \times \cdots \times X_n \to Y$ a bounded multilinear operator. If there exist a sequence $(Z_k)_{k\in\mathbb{N}}$ of Banach spaces and a sequence of $(s; s_1, \ldots, s_n)$-summing operators $T_k : X_1 \times \cdots \times X_n \to Z_k$ such that all T_k are $(s; s_1, \ldots, s_n)$-summing, $\left(\pi_{s;s_1,\ldots,s_n}(T_k)\right)_{k\in\mathbb{N}} \in l_s$ and $\|T(x_1, \ldots, x_n)\|_Y^s \leq \sum_{k=1}^{\infty} \|T_k(x_1, \ldots, x_n)\|_{Z_k}^s$ for $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$, then T is $(s; s_1, \ldots, s_n)$-summing and*

$$\pi_{s;s_1,\ldots,s_n}(T) \leq \left\|\left(\pi_{s;s_1,\ldots,s_n}(T_k)\right)_{k\in\mathbb{N}}\right\|_s.$$

Proof. Let $\left(x_i^j\right)_{1\le i\le m} \subset X_j$ $(1 \le j \le n)$. Then

$$\begin{aligned}
&\sum_{i=1}^{m} \left\|T\left(x_i^1,\dots,x_i^n\right)\right\|_Y^s \\
\le\ &\sum_{i=1}^{m}\sum_{k=1}^{\infty} \left\|T_k\left(x_i^1,\dots,x_i^n\right)\right\|_{Z_k}^s \\
=\ &\sum_{k=1}^{\infty}\sum_{i=1}^{m} \left\|T_k\left(x_i^1,\dots,x_i^n\right)\right\|_{Z_k}^s \\
\le\ &\sum_{k=1}^{\infty} \left[\pi_{s;s_1,\dots,s_n}\left(T_k\right)\right]^s \left[w_{s_1}\left(x_i^1 \mid 1\le i\le m\right)\cdots w_{s_n}\left(x_i^n \mid 1\le i\le m\right)\right]^s.
\end{aligned}$$

By the definition of $(s;s_1,\dots,s_n)$-summing operators the proof is completed. □

Our next result proves that for the multiplication operator, the necessary condition from Theorem 3 is also a sufficient one.

First let us recall that for the sequences of Banach spaces $\mathcal{X}_j = \left(X_k^j\right)_{k\in\mathbb{N}}$ $(1 \le j \le n)$, $\mathcal{Y} = (Y_k)_{k\in\mathbb{N}}$ and for a sequence of bounded multilinear operators $\mathcal{V} = (V_k)_{k\in\mathbb{N}}, V_k : X_k^1 \times \cdots \times X_k^n \to Y_k$ such that $\sup_{k\in\mathbb{N}} \|V_k\| < \infty$, we define the multiplication operator $M_{\mathcal{V}} : c_0(\mathcal{X}_1) \times \cdots \times c_0(\mathcal{X}_n) \to c_0(\mathcal{Y})$ by $M_{\mathcal{V}}\left(\left(x_k^1\right)_{k\in\mathbb{N}}, \dots, \left(x_k^n\right)_{k\in\mathbb{N}}\right) := \left(V_k\left(x_k^1,\dots,x_k^n\right)\right)_{k\in\mathbb{N}}$.

Theorem 6. *Let $1 \le s_1,\dots,s_n < \infty$, $0 < s < \infty$ be such that $v_n(s_1,\dots,s_n) \le s$ and $M_{\mathcal{V}} : c_0(\mathcal{X}_1) \times \cdots \times c_0(\mathcal{X}_n) \to c_0(\mathcal{Y})$. The following assertions are equivalent*:

(i) *$M_{\mathcal{V}}$ is $(s;s_1,\dots,s_n)$-summing.*

(ii) *all V_k are $(s;s_1,\dots,s_n)$-summing and $\left(\pi_{s;s_1,\dots,s_n}(V_k)\right)_{k\in\mathbb{N}} \in l_s$.*

Moreover, $\pi_{s;s_1,\dots,s_n}(M_{\mathcal{V}}) = \left\|\left(\pi_{s;s_1,\dots,s_n}(V_k)\right)_{k\in\mathbb{N}}\right\|_s$.

Proof. (i)⇒(ii). Since, by (i), $M_{\mathcal{V}}$ is $(s;s_1,\dots,s_n)$-summing, from Theorem 3 it follows that all $M_{\mathcal{V}} \circ \left(\sigma_k^1,\dots,\sigma_k^n\right) : X_k^1 \times \cdots \times X_k^n \to c_0(\mathcal{Y})$ are $(s;s_1,\dots,s_n)$-summing, $\left(\pi_{s;s_1,\dots,s_n}\left(M_{\mathcal{V}} \circ \left(\sigma_k^1,\dots,\sigma_k^n\right)\right)\right)_{k\in\mathbb{N}} \in l_s$ and

$$\left\|\left(\pi_{s;s_1,\dots,s_n}\left(M_{\mathcal{V}} \circ \left(\sigma_k^1,\dots,\sigma_k^n\right)\right)\right)_{k\in\mathbb{N}}\right\|_s \le \pi_{s;s_1,\dots,s_n}(M_{\mathcal{V}}).$$

Since $M_{\mathcal{V}} \circ \left(\sigma_k^1,\dots,\sigma_k^n\right) = (0,\dots,0,\underbrace{\sigma_k^k \circ V_k}_{k^{th}},0,\dots)$ and $M_{\mathcal{V}} \circ \left(\sigma_k^1,\dots,\sigma_k^n\right)$ are $(s;s_1,\dots,s_n)$-summing we deduce that $\sigma_k^k \circ V_k$ are $(s;s_1,\dots,s_n)$-summing and $\pi_{s;s_1,\dots,s_n}\left(M_{\mathcal{V}} \circ \left(\sigma_k^1,\dots,\sigma_k^n\right)\right) = \pi_{s;s_1,\dots,s_n}\left(\sigma_k^k \circ V_k\right)$. Further, by the ideal property $p_k^k \circ \sigma_k^k \circ V_k = V_k$ are $(s;s_1,\dots,s_n)$-summing and $\pi_{s;s_1,\dots,s_n}(V_k) =$

$\pi_{s;s_1,\dots,s_n}\left(p_k^k\circ\sigma_k^k\circ V_k\right)\le\pi_{s;s_1,\dots,s_n}\left(\sigma_k^k\circ V_k\right)$. We deduce that

$$\left(\pi_{s;s_1,\dots,s_n}\left(V_k\right)\right)_{k\in\mathbb{N}}\in l_s \text{ and } \left\|\left(\pi_{s;s_1,\dots,s_n}\left(V_k\right)\right)_{k\in\mathbb{N}}\right\|_s\le\pi_{s;s_1,\dots,s_n}\left(M_{\mathcal{V}}\right).$$

(ii)⇒(i). We have $M_{\mathcal{V}}\left(x_1,\dots,x_n\right)=\left(V_k\left(p_k^1\left(x_1\right),\dots,p_k^n\left(x_n\right)\right)\right)_{k\in\mathbb{N}}$ and from $\left\|M_{\mathcal{V}}\left(x_1,\dots,x_n\right)\right\|_{c_0(\mathcal{Y})}\le\left\|M_{\mathcal{V}}\left(x_1,\dots,x_n\right)\right\|_s$ we deduce

$$\left\|M_{\mathcal{V}}\left(x_1,\dots,x_n\right)\right\|_{c_0(\mathcal{Y})}^s\le\sum_{k=1}^{\infty}\left\|V_k\circ\left(p_k^1,\dots,p_k^n\right)\right)\left(x_1,\dots,x_n\right)\right\|^s.$$

Since by (ii) and the ideal property $\pi_{s;s_1,\dots,s_n}\left(V_k\circ\left(p_k^1,\dots,p_k^n\right)\right)\le\pi_{s;s_1,\dots,s_n}(V_k)$ and $\left(\pi_{s;s_1,\dots,s_n}\left(V_k\right)\right)_{k\in\mathbb{N}}\in l_s$ from Proposition 5, $M_{\mathcal{V}}$ is $(s;s_1,\dots,s_n)$-summing and $\pi_{s;s_1,\dots,s_n}\left(M_{\mathcal{V}}\right)\le\left\|\left(\pi_{s;s_1,\dots,s_n}\left(V_k\right)\right)_{k\in\mathbb{N}}\right\|_s$. □

Corollary 7. (a) *Let $1\le s<\infty$ and $M_{\mathcal{V}}:c_0\left(\mathcal{X}_1\right)\times\cdots\times c_0\left(\mathcal{X}_n\right)\to c_0\left(\mathcal{Y}\right)$. The following assertions are equivalent*:

(i) *$M_{\mathcal{V}}$ is s-summing.*

(ii) *all V_k are s-summing and $\left(\pi_s\left(V_k\right)\right)_{k\in\mathbb{N}}\in l_s$.*

Moreover, $\pi_s\left(M_{\mathcal{V}}\right)=\left\|\left(\pi_s\left(V_k\right)\right)_{k\in\mathbb{N}}\right\|_{l_s}$.

(b) *Let $1\le s_1,\dots,s_n<\infty$ and $M_{\mathcal{V}}:c_0\left(\mathcal{X}_1\right)\times\cdots\times c_0\left(\mathcal{X}_n\right)\to c_0\left(\mathcal{Y}\right)$. The following assertions are equivalent*:

(i) *$M_{\mathcal{V}}$ is $(s_1,\dots,s_n)$-dominated.*

(ii) *all V_k are $(s_1,\dots,s_n)$-dominated and $\left(\Delta_{s_1,\dots,s_n}\left(V_k\right)\right)_{k\in\mathbb{N}}\in l_{v_n(s_1,\dots,s_n)}$.*

Moreover, $\Delta_{s_1,\dots,s_n}\left(M_{\mathcal{V}}\right)=\left\|\left(\Delta_{s_1,\dots,s_n}\left(V_k\right)\right)_{k\in\mathbb{N}}\right\|_{v_n(s_1,\dots,s_n)}$.

3. Copies of vector-valued sequences spaces in $\Pi_s\left(c_0,\dots,c_0;c_0\right)$

In this section our goal is to present some non-trivial examples of summing operators, as applications of the above results. Our first examples will be defined by using a technique named *Average* of a finite number of elements, introduced by the second named author in [13]. The idea of considering these averages was suggested by the well-known discrete form of the Rademacher means, namely the equality

$$\int_0^1\left\|\sum_{i=1}^m r_i\left(t\right)x_i\right\|dt=\frac{1}{2^m}\sum_{(\varepsilon_1,\dots,\varepsilon_m)\in\{-1,1\}^m}\left\|\varepsilon_1x_1+\cdots+\varepsilon_mx_m\right\|, \text{ see [5]}$$

and it became an useful tool to define and study various examples of summing operators.

Let us now fix some notations and recall this concept.

Let m be a natural number. For $\left(\lambda_1,\dots,\lambda_m\right)\in\mathbb{K}^m$ we define the finite system denoted by $Average_1\left(\lambda_i\mid 1\le i\le m\right)$ as being the system with 2^m elements obtained by arranging in the lexicographical order of $D_m=\{-1,1\}^m$ the elements $\varepsilon_1\lambda_1+\cdots+\varepsilon_m\lambda_m$ for $\left(\varepsilon_1,\dots,\varepsilon_m\right)\in D_m$. (On $\{-1,1\}$ we consider the natural order). Thus, as sets we have

$$Average_1\left(\lambda_i\mid 1\le i\le m\right)=\left\{\varepsilon_1\lambda_1+\cdots+\varepsilon_m\lambda_m\mid\left(\varepsilon_1,\dots,\varepsilon_m\right)\in D_m\right\}.$$

Next, if we denote the 2^m elements of the set $Average_1(\lambda_i \mid 1 \le i \le m)$ by $\{\beta_1, \beta_2, \dots, \beta_{2^m}\}$ and we apply the same procedure, we define

$$\begin{aligned} Average_2(\lambda_i \mid 1 \le i \le m) &= Average_1(\beta_i \mid 1 \le i \le 2^m) \\ &= \{\varepsilon_1\beta_1 + \cdots + \varepsilon_{2^m}\beta_{2^m} \mid (\varepsilon_1, \dots, \varepsilon_{2^m}) \in D_{2^m}\}. \end{aligned}$$

For more details about this technique and also several related results, see [14]. Let us note that

$$(1)\quad c_{\mathbb{K}} \|(\lambda_1, \dots, \lambda_m)\|_{l_1^m} \le \|Average_1(\lambda_i \mid 1 \le i \le m)\|_\infty \le \|(\lambda_1, \dots, \lambda_m)\|_{l_1^m}$$

(where $c_{\mathbb{K}} = 1$ in the real case and $c_{\mathbb{K}} = \frac{1}{2}$ in the complex case) and further by Khinchin's inequality

$$(2)\quad \begin{aligned} \frac{c_{\mathbb{K}}}{\sqrt{2}} \|(\lambda_1, \dots, \lambda_m)\|_{l_2^m} &\le \frac{1}{2^m} \|Average_2(\lambda_i \mid 1 \le i \le m)\|_\infty \\ &\le \|(\lambda_1, \dots, \lambda_m)\|_{l_2^m}. \end{aligned}$$

Let $(\alpha_{mi})_{1 \le i \le m,\, m \in \mathbb{N}}$ be a triangular matrix of scalars, $\alpha_m = (\alpha_{m1}, \dots, \alpha_{mm})$ and $\alpha = (\alpha_m)_{m \in \mathbb{N}}$. The sequence of averages

$$Average_1(\alpha_{11}), Average_1(\alpha_{21}, \alpha_{22}), \dots, Average_1(\alpha_{mi} \mid 1 \le i \le m), \dots$$

will be denoted by $(Average_1(\alpha_{mi} \mid 1 \le i \le m))_{m \in \mathbb{N}}$. From (1), we have that $(Average_1(\alpha_{mi} \mid 1 \le i \le m))_{m \in \mathbb{N}} \in c_0$ if and only if $|\alpha_{m1}| + \cdots + |\alpha_{mm}| \to 0$ i.e., $\alpha \in c_0(l_1^m \mid m \in \mathbb{N})$ and further we obtain

$$(3)\quad \begin{aligned} c_{\mathbb{K}} \|\alpha\|_{c_0(l_1^m \mid m \in \mathbb{N})} &\le \left\|(Average_1(\alpha_{mi} \mid 1 \le i \le m))_{m \in \mathbb{N}}\right\|_{c_0} \\ &\le \|\alpha\|_{c_0(l_1^m \mid m \in \mathbb{N})}. \end{aligned}$$

Next, the sequence of averages

$$Average_2\left(\tfrac{1}{2}\alpha_{11}\right), Average_2\left(\tfrac{1}{2^2}\alpha_{21}, \tfrac{1}{2^2}\alpha_{22}\right), \dots, Average_2\left(\tfrac{1}{2^m}\alpha_{mi} \mid 1 \le i \le m\right), \dots$$

will be denoted by $\left(Average_2\left(\frac{1}{2^m}\alpha_{mi} \mid 1 \le i \le m\right)\right)_{m \in \mathbb{N}}$. From (2), we have that $\left(Average_2\left(\frac{1}{2^m}\alpha_{mi} \mid 1 \le i \le m\right)\right)_{m \in \mathbb{N}} \in c_0$ if and only if $|\alpha_{m1}|^2 + \cdots + |\alpha_{mm}|^2 \to 0$, i.e., $\alpha \in c_0(l_2^m \mid m \in \mathbb{N})$ and further we obtain the inequality

$$(4)\quad \begin{aligned} \frac{c_{\mathbb{K}}}{\sqrt{2}} \|\alpha\|_{c_0(l_2^m \mid m \in \mathbb{N})} &\le \left\| \left(Average_2\left(\frac{1}{2^m}\alpha_{mi} \mid 1 \le i \le m\right)\right)_{m \in \mathbb{N}} \right\|_{c_0} \\ &\le \|\alpha\|_{c_0(l_2^m \mid m \in \mathbb{N})}. \end{aligned}$$

Let us recall that for $m \in \mathbb{N}$, $0 < p \le \infty$, $l_p^m := \left(\mathbb{K}^m, \|\cdot\|_p\right)$ and $(e_k)_{1 \le k \le m} \subset l_p^m$ is the canonical basis of the l_p^m. Let us also recall the concept of the nuclear multilinear operators, see [7, Definition 1.26]. A bounded multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ is called *nuclear* if there exist $\left(\psi_k^1\right)_{k \in \mathbb{N}} \subset$

$X_1^*, \ldots, (\psi_k^n)_{k\in\mathbb{N}} \subset X_n^*$, $(y_k)_{k\in\mathbb{N}} \subset Y$ such that $\sum_{k=1}^{\infty} \left\|\psi_k^1\right\| \cdots \|\psi_k^n\| \, \|y_k\| < \infty$ and

$$T(x_1, \ldots, x_n) = \sum_{k=1}^{\infty} \psi_k^1(x_1) \cdots \psi_k^n(x_n)\, y_k \quad \text{for } (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n.$$

Such a representation is called a nuclear representation of T. In this case $\|T\|_{nuc} = \inf\{\sum_{k=1}^{\infty} \left\|\psi_k^1\right\| \cdots \|\psi_k^n\| \, \|y_k\|\}$, where the infimum is taken over all nuclear representations of T. This class, denoted by $(\mathcal{N}, \|\cdot\|_{nuc})$, is the smallest Banach ideal among all other Banach ideals of multilinear operators (for the linear case see [11, Theorem 1.7.2, page 64] and for the multilinear case, see [1, Theorem 2]).

The first results of this section study the summing nature of the multiplication operator. These results will further be used in studying the summing nature of the operator defined by $Average_1$ and by $Average_2$. The next result extends Theorem 11 in [4].

Proposition 8. *Let $n \in \mathbb{N}$, $1 \le s_1, \ldots, s_n < \infty$, $0 < s < \infty$ be such that $v_n(s_1, \ldots, s_n) \le s$. Let $m \in \mathbb{N}$, $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{K}^m$. Then the multiplication operator $M_\alpha : l_\infty^m \times \cdots \times l_\infty^m \to l_1^m$, $M_\alpha(x_1, \ldots, x_n) = \alpha x_1 \cdots x_n = (\alpha_i \langle x_1, e_i\rangle \cdots \langle x_n, e_i\rangle)_{1\le i\le m}$ is $(s; s_1, \ldots, s_n)$-summing and*

(i) *if $0 < s < 1$, $\pi_{s;s_1,\ldots,s_n}(M_\alpha) = \|\alpha\|_s$;*

(ii) *if $1 \le s < \infty$, $\pi_{s;s_1,\ldots,s_n}(M_\alpha) = \|\alpha\|_1$.*

Proof. Let us first note that $\|M_\alpha\| = \|\alpha\|_1$.

(i) Let $0 < s < 1$. For all $(x_1, \ldots, x_n) \in l_\infty^m \times \cdots \times l_\infty^m$ we have

$$\begin{aligned}\|M_\alpha(x_1, \ldots, x_n)\|_1 &\le \|M_\alpha(x_1, \ldots, x_n)\|_s \\ &= \left(\sum_{i=1}^{m} |\alpha_i|^s \, |\langle x_1, e_i\rangle|^s \cdots |\langle x_n, e_i\rangle|^s\right)^{\frac{1}{s}}.\end{aligned}$$

By considering the rank one functionals $U_i : l_\infty^m \times \cdots \times l_\infty^m \to \mathbb{K}$, $U_i(x_1, \ldots, x_n) = \alpha_i \langle x_1, e_i\rangle \cdots \langle x_n, e_i\rangle$ $(1 \le i \le m)$, it follows that

$$\|M_\alpha(x_1, \ldots, x_n)\|_1^s \le \sum_{i=1}^{m} |U_i(x_1, \ldots, x_n)|^s.$$

By Proposition 5, we deduce that M_α is $(s; s_1, \ldots, s_n)$-summing and furthermore, $\pi_{s;s_1,\ldots,s_n}(M_\alpha) \le \left(\sum_{i=1}^{m} [\pi_{s;s_1,\ldots,s_n}(U_i)]^s\right)^{\frac{1}{s}} = \left(\sum_{i=1}^{m} |\alpha_i|^s\right)^{\frac{1}{s}} = \|\alpha\|_s$. For the reverse inequality, by the definition of the $(s; s_1, \ldots, s_n)$-summing operators and $w_{s_j}\left((e_i)_{1\le i\le m}; l_\infty^m\right) = 1 (1 \le j \le n)$, it follows that

$$\left(\sum_{i=1}^{m} \|M_\alpha(e_i, \ldots, e_i)\|^s\right)^{\frac{1}{s}} \le \pi_{s;s_1,\ldots,s_n}(M_\alpha),$$

hence $\|\alpha\|_s \leq \pi_{s;s_1,\ldots,s_n}(M_\alpha)$.

(ii) Let $1 \leq s < \infty$. From $M_\alpha(x_1,\ldots,x_n) = \sum_{i=1}^{m} \alpha_i \langle x_1, e_i\rangle \ldots \langle x_n, e_i\rangle e_i$, we deduce $\|M_\alpha\|_{nuc} \leq \sum_{i=1}^{m} |\alpha_i| = \|\alpha\|_1$. However, since for each Banach ideal of multilinear operators we have always $\|\cdot\| \leq \|\cdot\|_{\mathcal{A}} \leq \|\cdot\|_{nuc}$, see [11] and also [1], we get $\|\alpha\|_1 = \|M_\alpha\| \leq \|M_\alpha\|_{\mathcal{A}} \leq \|M_\alpha\|_{nuc} = \|\alpha\|_1$. Since $s \geq 1$, $\Pi_{s;s_1,\ldots,s_n}$ is a Banach ideal and the statement follows. □

Proposition 9. *Let n, m be natural numbers, $\alpha = (\alpha_1,\ldots,\alpha_m) \in \mathbb{K}^m$. Let $1 < p < \infty$ and let $M_\alpha : l_\infty^m \times \cdots \times l_\infty^m \to l_p^m$ be the multiplication operator $M_\alpha(x_1,\ldots,x_n) = \alpha x_1 \cdots x_n$ and $1 \leq s_1,\ldots,s_n < \infty$.*

(i) *If $v_n(s_1,\ldots,s_n) \leq p$, then M_α is $(s_1,\ldots,s_n)$-dominated and*

$$\Delta_{s_1,\ldots,s_n}(M_\alpha) = \|\alpha\|_{v_n(s_1,\ldots,s_n)}.$$

(ii) *If $p < v_n(s_1,\ldots,s_n)$, then M_α is $(s_1,\ldots,s_n)$-dominated and*

$$\Delta_{s_1,\ldots,s_n}(M_\alpha) = \|\alpha\|_p.$$

Proof. Since M_α is a finite rank operator it is $(s_1,\ldots,s_n)$-dominated. The crucial point is the evaluation of $\Delta_{s_1,\ldots,s_n}(M_\alpha)$.

(i) We have $M_\alpha(e_i,\ldots,e_i) = \alpha_i e_i$ and since $w_{s_j}\left((e_i)_{1\leq i\leq m}; l_\infty^m\right) = 1$ $(1 \leq j \leq n)$ by the definition of $(s_1,\ldots,s_n)$-dominated operators we deduce

$$\|\alpha\|_{v_n(s_1,\ldots,s_n)} \leq \Delta_{s_1,\ldots,s_n}(M_\alpha).$$

Also from $v_n(s_1,\ldots,s_n) \leq p$, for all $(x_1,\ldots,x_n) \in l_\infty^m \times \cdots \times l_\infty^m$ we have $\|M_\alpha(x_1,\ldots,x_n)\|_p \leq \|M_\alpha(x_1,\ldots,x_n)\|_{v_n(s_1,\ldots,s_n)}$ and then

$$\begin{aligned} & \|M_\alpha(x_1,\ldots,x_n)\|_p^{v_n(s_1,\ldots,s_n)} \\ \leq\ & \|M_\alpha(x_1,\ldots,x_n)\|_{v_n(s_1,\ldots,s_n)}^{v_n(s_1,\ldots,s_n)} \\ =\ & \sum_{i=1}^{m} |\alpha_i|^{v_n(s_1,\ldots,s_n)} |\langle x_1, e_i\rangle|^{v_n(s_1,\ldots,s_n)} \cdots |\langle x_n, e_i\rangle|^{v_n(s_1,\ldots,s_n)}. \end{aligned}$$

From here, by Proposition 5, we deduce that

$$\Delta_{s_1,\ldots,s_n}(M_\alpha) \leq \left(\sum_{i=1}^{m} |\alpha_i|^{v_n(s_1,\ldots,s_n)}\right)^{\frac{1}{v_n(s_1,\ldots,s_n)}} = \|\alpha\|_{v_n(s_1,\ldots,s_n)}.$$

(ii) From $p < v_n(s_1,\ldots,s_n)$ let us define $1 < t < \infty$ by $\frac{1}{p} = \frac{1}{v_n(s_1,\ldots,s_n)} + \frac{1}{t}$. Then, there exist $\beta = (\beta_1,\ldots,\beta_m) \in \mathbb{K}^m$ and $\gamma = (\gamma_1,\ldots,\gamma_m) \in \mathbb{K}^m$ such that $\alpha = \beta\gamma$, i.e., $\alpha_i = \beta_i\gamma_i$, $1 \leq i \leq m$ and $\|\alpha\|_p = \|\beta\|_{v_n(s_1,\ldots,s_n)} \|\gamma\|_t$. We deduce that $M_\alpha : l_\infty^m \times \cdots \times l_\infty^m \overset{M_\beta}{\to} l_{v_n(s_1,\ldots,s_n)}^m \overset{M_\gamma}{\to} l_p^m$ is a factorization of M_α, that is $M_\alpha = M_\gamma \circ M_\beta$.

By Proposition 5, $M_\beta : l_\infty^m \times \cdots \times l_\infty^m \to l_{v_n(s_1,\ldots,s_n)}^m$ is $(s_1,\ldots,s_n)$-dominated and $\Delta_{s_1,\ldots,s_n}(M_\beta) \leq \|\beta\|_{v_n(s_1,\ldots,s_n)}$. By the ideal property of $(s_1,\ldots,s_n)$-dominated operators, M_α is $(s_1,\ldots,s_n)$-dominated and

$$\Delta_{s_1,\ldots,s_n}(M_\alpha) \leq \Delta_{s_1,\ldots,s_n}(M_\beta)\|M_\gamma\| \leq \|\beta\|_{v_n(s_1,\ldots,s_n)}\|\gamma\|_t = \|\alpha\|_p .$$

Since $v_n(s_1,\ldots,s_n) > 1$ we always have $\|\alpha\|_p = \|M_\alpha\| \leq \Delta_{s_1,\ldots,s_n}(M_\alpha)$. □

The next result will be used in studying the summing nature of operators defined by *Average*$_2$.

Proposition 10. *Let $n \in \mathbb{N}$, $1 \leq s < \infty$, $m \in \mathbb{N}$, $\alpha = (\alpha_1,\ldots,\alpha_m) \in \mathbb{K}^m$ and $M_\alpha : l_\infty^m \times \cdots \times l_\infty^m \to l_2^m$, $M_\alpha(x_1,\ldots,x_n) = \alpha x_1 \cdots x_n$ be the multiplication operator. Then M_α is s-summing and*

(i) *if $s \leq 2$, $\pi_s(M_\alpha) = \|\alpha\|_s$;*
(ii) *if $2 < s$, $\pi_s(M_\alpha) = \|\alpha\|_2$.*

Proof. (i) Since $s \leq 2$, $\|\cdot\|_2 \leq \|\cdot\|_s$ and thus

$$\|M_\alpha(x_1,\ldots,x_n)\|_2^s \leq \|M_\alpha(x_1,\ldots,x_n)\|_s^s = \sum_{i=1}^m |\alpha_i|^s |\langle x_1, e_i\rangle|^s \cdots |\langle x_n, e_i\rangle|^s .$$

Then, by Proposition 5, M_α is s-summing and $\pi_s(M_\alpha) \leq \left(\sum\limits_{i=1}^m |\alpha_i|^s\right)^{\frac{1}{s}} = \|\alpha\|_s$. Now from $w_p\left((e_i)_{1\leq i\leq m}; l_\infty^m\right) = 1$, $M_\alpha(e_i,\ldots,e_i) = \alpha_i e_i$ $(1 \leq i \leq m)$ and the definition of s-summing operators, we deduce $\|\alpha\|_s \leq \pi_s(M_\alpha)$.

(ii) The case $n = 1$. We use that in the linear case, as is well-known and easy to prove, $\pi_2(M_\alpha : l_\infty^m \to l_2^m) = \|\alpha\|_2$; since $2 < s$, by the inclusion theorem from the linear case, we have $\|\alpha\|_2 = \|M_\alpha\| \leq \pi_s(M_\alpha : l_\infty^m \to l_2^m) \leq \pi_2(M_\alpha : l_\infty^m \to l_2^m) = \|\alpha\|_2$.

The case $n \geq 2$. We have $\|M_\alpha\| = \|\alpha\|_2$ and since always $\|M_\alpha\| \leq \pi_s(M_\alpha)$, we get $\|\alpha\|_2 \leq \pi_s(M_\alpha)$. The following reasoning was suggested to us by [3, Lemma 3.2]. Since $n \geq 2$, for $(x_1,\ldots,x_n) \in l_\infty^m \times \cdots \times l_\infty^m$ we have $\|M_\alpha(x_1,\ldots,x_n)\| \leq \|\alpha x_1\|\|x_2\|\cdots\|x_n\|$. Then for every $\left(x_i^1\right)_{1\leq i\leq k} \subset l_\infty^m, \ldots,$ $\left(x_i^k\right)_{1\leq i\leq k} \subset l_\infty^m$ we have

$$\begin{aligned}\left\|M_\alpha\left(x_i^1,\ldots,x_i^n\right)\right\| &\leq \left\|\alpha x_i^1\right\|\left\|x_i^2\right\|\cdots\left\|x_i^n\right\| \\ &\leq \left\|\alpha x_i^1\right\| w_s\left(\left(x_i^2\right)_{1\leq i\leq k}\right)\cdots w_s\left(\left(x_i^n\right)_{1\leq i\leq k}\right)\end{aligned}$$

and so

$$\begin{aligned}&\left(\sum_{i=1}^k \left\|M_\alpha\left(x_i^1,\ldots,x_i^n\right)\right\|^s\right)^{\frac{1}{s}} \\ \leq &\left(\sum_{i=1}^k \left\|\alpha x_i^1\right\|^s\right)^{\frac{1}{s}} w_s\left(\left(x_i^2\right)_{1\leq i\leq k}\right)\cdots w_s\left(\left(x_i^n\right)_{1\leq i\leq k}\right)\end{aligned}$$

$$\le \pi_s\left(M_\alpha : l_\infty^m \to l_2^m\right) w_s\left(\left(x_i^1\right)_{1\le i\le k}\right)\cdots w_s\left(\left(x_i^n\right)_{1\le i\le k}\right)$$
$$= \|\alpha\|_2\, w_s\left(\left(x_i^1\right)_{1\le i\le k}\right)\cdots w_s\left(\left(x_i^n\right)_{1\le i\le k}\right). \qquad \square$$

We are now able to present two results which give a complete characterization of the summing nature for some non-trivial operators defined by *Average*.

We denote by $(\alpha_{mi})_{1\le i\le m, m\in\mathbb{N}}$ an infinite triangular matrix of scalars, $\alpha_m = (\alpha_{m1}, \ldots, \alpha_{mm}) \in \mathbb{K}^n$, $\alpha = (\alpha_m)_{m\in\mathbb{N}}$ and $k_m = \frac{(m-1)m}{2}$ for all natural numbers m.

Proposition 11. *Let $n \in \mathbb{N}$ and $1 \le s_1, \ldots, s_n < \infty$, $0 < s < \infty$ be such that $v_n(s_1, \ldots, s_n) \le s$ and $(\alpha_{mi})_{1\le i\le m, m\in\mathbb{N}}$ be an infinite triangular matrix of scalars such that $\alpha \in l_\infty(l_1^m \mid m \in \mathbb{N})$. Let $Av_\alpha^1 : c_0 \times \cdots \times c_0 \to c_0$ be the operator defined by*

$$Av_\alpha^1(\xi_1, \ldots, \xi_n) = \left(Average_1\left(\alpha_{mi}\langle \xi_1, e_{i+k_m}\rangle \cdots \langle \xi_n, e_{i+k_m}\rangle \mid 1 \le i \le m\right)\right)_{m\in\mathbb{N}}.$$

Then: i) *for $0 < s < 1$, Av_α^1 is $(s; s_1, \ldots, s_n)$-summing if and only if $\alpha \in l_s(l_s^m \mid m \in \mathbb{N})$.*

ii) *for $1 \le s < \infty$, Av_α^1 is $(s; s_1, \ldots, s_n)$-summing if and only if $\alpha \in l_s(l_1^m \mid m \in \mathbb{N})$.*

Proof. Let $(\xi_1, \ldots, \xi_n) \in c_0 \times \cdots \times c_0$. Then, by (1), $Av_\alpha^1(\xi_1, \ldots, \xi_n) \in c_0$ if and only if $V_\alpha^1(\xi_1, \ldots, \xi_n) \in c_0(l_1^m \mid m \in \mathbb{N})$, where

$$V_\alpha^1(\xi_1, \ldots, \xi_n) = \big(\alpha_{m1}\langle \xi_1, e_{1+k_m}\rangle \cdots \langle \xi_n, e_{1+k_m}\rangle, \ldots, \alpha_{mm}\langle \xi_1, e_{k_{m+1}}\rangle \cdots \langle \xi_n, e_{k_{m+1}}\rangle\big)_{m\in\mathbb{N}}$$

and in this case, by (3), we have

$$c_{\mathbb{K}}\left\|V_\alpha^1(\xi_1, \ldots, \xi_n)\right\|_{c_0(l_1^m\mid m\in\mathbb{N})} \le \left\|Av_\alpha^1(\xi_1, \ldots, \xi_n)\right\|_{c_0} \le \left\|V_\alpha^1(\xi_1, \ldots, \xi_n)\right\|_{c_0(l_1^m\mid m\in\mathbb{N})}. \tag{5}$$

This means that Av_α^1 is well defined if and only if the operator $V_\alpha^1 : c_0 \times \cdots \times c_0 \to c_0(l_1^m \mid m \in \mathbb{N})$ defined by

$$V_\alpha^1(\xi_1, \ldots, \xi_n) = \big(\alpha_{m1}\langle \xi_1, e_{1+k_m}\rangle \cdots \langle \xi_n, e_{1+k_m}\rangle, \ldots, \alpha_{mm}\langle \xi_1, e_{k_{m+1}}\rangle \cdots \langle \xi_n, e_{k_{m+1}}\rangle\big)_{m\in\mathbb{N}}$$

is a bounded multilinear operator. Further, from (5), Av_α^1 is s-summing if and only if V_α^1 is s-summing.

Now, if we consider the identification $c_0 = c_0(l_\infty^m \mid m \in \mathbb{N})$, we observe that the operator $V_\alpha^1 : c_0(l_\infty^m \mid m \in \mathbb{N}) \times \cdots \times c_0(l_\infty^m \mid m \in \mathbb{N}) \to c_0(l_1^m \mid m \in \mathbb{N})$ is actually $M_{\mathcal{V}} : c_0(l_\infty^m \mid m \in \mathbb{N}) \times \cdots \times c_0(l_\infty^m \mid m \in \mathbb{N}) \to c_0(l_1^m \mid m \in \mathbb{N})$, $M_{\mathcal{V}} = (M_{\alpha_m})_{m\in\mathbb{N}}$, where $M_{\alpha_m} : l_\infty^m \times \cdots \times l_\infty^m \to l_1^m$ is the multiplication operator. Then Av_α^1 is $(s; s_1, \ldots, s_n)$-summing if and only if $M_{\mathcal{V}} : c_0(l_\infty^m \mid m \in \mathbb{N}) \times \cdots \times c_0(l_\infty^m \mid m \in \mathbb{N}) \to c_0(l_1^m \mid m \in \mathbb{N})$ is $(s; s_1, \ldots, s_n)$-summing, which by Theorem 6 is equivalent to $(\pi_{s;s_1,\ldots,s_n}(M_{\alpha_m})) \in l_s$. Now by Proposition 8, it follows that for $0 < s < 1$, $\alpha = (\alpha_m)_{m\in\mathbb{N}} \in l_s (l_s^m \mid m \in \mathbb{N}) = l_s$ and for $1 \le s < \infty$, $\alpha = (\alpha_m)_{m\in\mathbb{N}} \in l_s (l_1^m \mid m \in \mathbb{N})$. □

Before presenting a consequence of the above result, let us recall that a normed (ω-normed) space X contains a copy of the normed (ω-normed) space Y if there exist $T : Y \to X$ a linear operator and some constants c_1, $c_2 > 0$ such that $c_1 \|y\|_Y \le \|T(y)\|_X \le c_2 \|y\|_Y$ for $y \in Y$. From Proposition 11, we deduce:

Corollary 12. *Let $0 < s < \infty$ and $1 \le s_1, \ldots, s_n < \infty$. Then*

i) *for $0 < s < 1$, $\Pi_s (c_0, \ldots, c_0; c_0)$ contains a copy of $l_s (l_s^m \mid m \in \mathbb{N})$; for $1 \le s < \infty$, $\Pi_s (c_0, \ldots, c_0; c_0)$ contains a copy of $l_s (l_1^m \mid m \in \mathbb{N})$.*

ii) *for $v_n (s_1, \ldots, s_n) < 1$, $\Delta_{s_1,\ldots,s_n} (c_0, \ldots, c_0; c_0)$ contains a copy of*

$$l_{v_n(s_1,\ldots,s_n)} \left(l^m_{v_n(s_1,\ldots,s_n)} \mid m \in \mathbb{N} \right);$$

for $1 \le v_n (s_1, \ldots, s_n) < \infty$, $\Delta_{s_1,\ldots,s_n} (c_0, \ldots, c_0; c_0)$ contains a copy of

$$l_{v_n(s_1,\ldots,s_n)} (l_1^m \mid m \in \mathbb{N}).$$

Our next result studies the multilinear operator defined by $Average_2$. As consequences of this result, we will identify some other copies that $\Pi_s(c_0, \ldots, c_0; c_0)$ contains.

Proposition 13. *Let $1 \le s < \infty$, $1 \le s_1, \ldots, s_n < \infty$ and $(\alpha_{mi})_{1\le i\le m, m\in\mathbb{N}}$ be an infinite triangular matrix of scalars such that $\alpha \in l_\infty (l_2^m \mid m \in \mathbb{N})$. Let $Av_\alpha^2 : c_0 \times \cdots \times c_0 \to c_0$ be the operator defined by*

$$Av_\alpha^2 (\xi_1, \ldots, \xi_n) = \left(Average_2 \left(\tfrac{1}{2^m} \alpha_{mi} \langle \xi_1, e_{i+k_m} \rangle \cdots \langle \xi_n, e_{i+k_m} \rangle \mid 1 \le i \le m \right) \right)_{m\in\mathbb{N}}.$$

Then:

i) *Av_α^2 is s-summing if and only if $\alpha = (\alpha_m)_{m\in\mathbb{N}} \in l_s (l_s^m \mid m \in \mathbb{N}) = l_s$ if $1 \le s \le 2$ or $\alpha = (\alpha_m)_{m\in\mathbb{N}} \in l_s (l_2^m \mid m \in \mathbb{N})$ if $s > 2$.*

ii) *Av_α^2 is $(s_1, \ldots, s_n)$-dominated if and only if $\alpha = (\alpha_m)_{m\in\mathbb{N}} \in l_{v_n(s_1,\ldots,s_n)}$ if $v_n (s_1, \ldots, s_n) \le 2$, or $\alpha = (\alpha_m)_{m\in\mathbb{N}} \in l_{v_n(s_1,\ldots,s_n)} (l_2^m \mid m \in \mathbb{N})$ if $2 < v_n (s_1, \ldots, s_n)$.*

Proof. Let $(\xi_1, \ldots, \xi_n) \in c_0 \times \cdots \times c_0$. Then, by (2), $Av_\alpha^2 (\xi_1, \ldots, \xi_n) \in c_0$ if and only if $V_\alpha^2 (\xi_1, \ldots, \xi_n) \in c_0 (l_2^m \mid m \in \mathbb{N})$, where

$$\begin{aligned} V_\alpha^2 (\xi_1, \ldots, \xi_n) = {} & (\alpha_{m1} \langle \xi_1, e_{1+k_m} \rangle \cdots \langle \xi_n, e_{1+k_m} \rangle, \ldots, \\ & \alpha_{mm} \langle \xi_1, e_{k_{m+1}} \rangle \cdots \langle \xi_n, e_{k_{m+1}} \rangle)_{m\in\mathbb{N}} \end{aligned}$$

and in this case, by (4), it follows that

$$\begin{aligned} \frac{c_K}{\sqrt{2}} \left\| V_\alpha^2 (\xi_1, \ldots, \xi_n) \right\|_{c_0\left(l_2^m \mid m\in\mathbb{N}\right)} & \le \left\| Av_\alpha^2 (\xi_1, \ldots, \xi_n) \right\|_{c_0} \\ & \le \left\| V_\alpha^2 (\xi_1, \ldots, \xi_n) \right\|_{c_0\left(l_2^m \mid m\in\mathbb{N}\right)}. \end{aligned} \tag{6}$$

This means that Av_α^2 is well defined if and only if the operator $V_\alpha^2 : c_0 \times \cdots \times c_0 \to c_0\left(l_2^m \mid m \in \mathbb{N}\right)$ defined by

$$V_\alpha^2\left(\xi_1,\ldots,\xi_n\right) = \left(\alpha_{m1}\left\langle \xi_1, e_{1+k_m}\right\rangle \cdots \left\langle \xi_n, e_{1+k_m}\right\rangle, \ldots, \alpha_{mm}\left\langle \xi_1, e_{k_{m+1}}\right\rangle \cdots \left\langle \xi_n, e_{k_{m+1}}\right\rangle\right)_{m\in\mathbb{N}}$$

is a bounded multilinear operator. Further, from (6), Av_α^2 is $(s; s_1, \ldots, s_n)$-summing if and only if V_α^2 is $(s; s_1, \ldots, s_n)$-summing.

Now, if we consider the identification $c_0 = c_0\left(l_\infty^m \mid m \in \mathbb{N}\right)$, we observe that the operator $V_\alpha^2 : c_0\left(l_\infty^m \mid m \in \mathbb{N}\right) \times \cdots \times c_0\left(l_\infty^m \mid m \in \mathbb{N}\right) \to c_0\left(l_2^m \mid m \in \mathbb{N}\right)$ is actually $M_{\mathcal{V}} : c_0\left(l_\infty^m \mid m \in \mathbb{N}\right) \times \cdots \times c_0\left(l_\infty^m \mid m \in \mathbb{N}\right) \to c_0\left(l_2^m \mid m \in \mathbb{N}\right)$, $M_{\mathcal{V}} = \left(M_{\alpha_m}\right)_{m\in\mathbb{N}}$, where $M_{\alpha_m} : l_\infty^m \times \cdots \times l_\infty^m \to l_2^m$ is the multiplication operator.

Let us note that the conditions stated regarding the triangular matrix assure us that Av_α^2 is well defined.

i) We have, Av_α^2 is s-summing if and only if $M_{\mathcal{V}}$ is s-summing, which by Theorem 6 is equivalent to $\left(\pi_s\left(M_{\alpha_m}\right)\right) \in l_s$. Further, by Proposition 10 we have that if $1 \le s \le 2$, $\alpha = \left(\alpha_m\right)_{m\in\mathbb{N}} \in l_s\left(l_s^m \mid m \in \mathbb{N}\right)$ or if $s > 2$, $\alpha = \left(\alpha_m\right)_{m\in\mathbb{N}} \in l_s\left(l_2^m \mid m \in \mathbb{N}\right)$.

ii) We have, Av_α^2 is $(s_1, \ldots, s_n)$-dominated if and only if $M_{\mathcal{V}}$ is $(s_1, \ldots, s_n)$-dominated, which by Theorem 6 is equivalent to $\left(\Delta_{s_1,\ldots,s_n}\left(M_{\alpha_m}\right)\right) \in l_{v_n(s_1,\ldots,s_n)}$. Further, by Proposition 9, for $v_n\left(s_1, \ldots, s_n\right) \le 2$, we have $\alpha = \left(\alpha_m\right)_{m\in\mathbb{N}} \in l_{v_n(s_1,\ldots,s_n)}\left(l^m_{v_n(s_1,\ldots,s_n)} \mid m \in \mathbb{N}\right) = l_{v_n(s_1,\ldots,s_n)}$ and for $2 < v_n\left(s_1, \ldots, s_n\right)$, $\alpha = \left(\alpha_m\right)_{m\in\mathbb{N}} \in l_{v_n(s_1,\ldots,s_n)}\left(l_2^m \mid m \in \mathbb{N}\right)$. □

Hence, we deduce the following corollary:

Corollary 14. *Let $1 \le s < \infty$ and $1 \le s_1, \ldots, s_n < \infty$. Then*:

(i) $\Pi_s\left(c_0, \ldots, c_0; c_0\right)$ *contains a copy of* l_s *if* $1 \le s \le 2$ *or a copy of* $l_s\left(l_2^m \mid m \in \mathbb{N}\right)$ *if* $s > 2$.

(ii) $\Delta_{s_1,\ldots,s_n}\left(c_0, \ldots, c_0; c_0\right)$ *contains a copy of* $l_{v_n(s_1,\ldots,s_n)}$ *if* $v_n\left(s_1, \ldots, s_n\right) \le 2$ *or a copy of* $l_{v_n(s_1,\ldots,s_n)}\left(l_2^m \mid m \in \mathbb{N}\right)$ *if* $2 < v_n\left(s_1, \ldots, s_n\right)$.

4. Summing bilinear operators defined by some methods of summability

Our first result allows us to study the summing nature of some bilinear operators which are induced by some method of summability. In particular, we obtain the summing nature of the Cesàro operator on a cartesian product of $c_0(\mathcal{X})$.

Proposition 15. *Let $\mathcal{V} = \left(V_i\right)_{i\in\mathbb{N}}$, $V_i : X_i \times Y_i \to Z$ be a sequence of bounded bilinear operators such that $\sum_{i=1}^{\infty} V_i\left(x_i, y_i\right)$ is norm convergent for all*

$x = (x_i)_{i\in\mathbb{N}} \in c_0(\mathcal{X})$, $y = (y_i)_{i\in\mathbb{N}} \in c_0(\mathcal{Y})$ and let $S_{\mathcal{V}} : c_0(\mathcal{X}) \times c_0(\mathcal{Y}) \to Z$, $S_{\mathcal{V}}(x, y) = \sum\limits_{i=1}^{\infty} V_i(x_i, y_i)$. Then:

(i) $S_{\mathcal{V}}$ is 1-summing if and only if all V_i are 1-summing and $\sum\limits_{i=1}^{\infty} \pi_1(V_i) < \infty$. Moreover, $\pi_1(S_{\mathcal{V}}) = \sum\limits_{i=1}^{\infty} \pi_1(V_i)$.

(ii) $S_{\mathcal{V}}$ is 2-dominated if and only if all V_i are 2-dominated and $\sum\limits_{i=1}^{\infty} \Delta_2(V_i) < \infty$. Moreover, $\Delta_2(S_{\mathcal{V}}) = \sum\limits_{i=1}^{\infty} \Delta_2(V_i)$.

Proof. (i) Assuming that $S_{\mathcal{V}}$ is 1-summing, by Theorem 3, it follows that all $S_{\mathcal{V}} \circ (\sigma_i, \sigma_i)$ are 1-summing, $\sum\limits_{i=1}^{\infty} \pi_1(S_{\mathcal{V}} \circ (\sigma_i, \sigma_i)) < \infty$ and

$$\pi_1(S_{\mathcal{V}}) \leq \sum_{i=1}^{\infty} \pi_1(S_{\mathcal{V}} \circ (\sigma_i, \sigma_i)).$$

Then, from $S_{\mathcal{V}} \circ (\sigma_i, \sigma_i) = V_i$ we get that all V_i are 1-summing, $\sum\limits_{i=1}^{\infty} \pi_1(V_i) < \infty$ and $\pi_1(S_{\mathcal{V}}) \leq \sum\limits_{i=1}^{\infty} \pi_1(V_i)$. Conversely, let $\sum\limits_{i=1}^{\infty} \pi_1(V_i) < \infty$. Since $S_{\mathcal{V}}(x, y) = \sum\limits_{i=1}^{\infty} V_i(p_i(x), p_i(y))$ and $\pi_1(V_i \circ (p_i, p_i)) \leq \pi_1(V_i)$, from Proposition 5 we get that $S_{\mathcal{V}}$ is 1-summing and $\pi_1(S_{\mathcal{V}}) \leq \sum\limits_{i=1}^{\infty} \pi_1(V_i)$.

(ii) The proof is similar to the previous case, hence we omit it. □

In order to present our next example, we will use some methods of summability, whose definition we will further recall. An infinite matrix of scalar elements $(\alpha_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ is called *a method of summability* if given a sequence $(x_i)_{i\in\mathbb{N}} \in c_0$, all the series $\sum_{j=1}^{\infty} \alpha_{ij} x_j$ are convergent and the sequence $(y_i)_{i\in\mathbb{N}} \in c_0$, where $y_i = \sum_{j=1}^{\infty} \alpha_{ij} x_j$.

Moreover, it is well known (see [8, page 75]) that $(\alpha_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ is a method of summability if and only if

(i) there exists a positive constant M such that for each $i \in \mathbb{N}$, $\sum_{j=1}^{\infty} |\alpha_{ij}| \leq M$;

(ii) $\lim_{i\to\infty} \alpha_{ij} = 0$ for every $j \in \mathbb{N}$.

Let us note that a method of summability is regular in the sense of [8] if and only if $\lim_{i\to\infty} \sum_{j=1}^{\infty} \alpha_{ij} = 1$.

Proposition 16. *Let $(\lambda_i)_{i\in\mathbb{N}} \subset (0, \infty)$ be such that $\lambda_1 + \cdots + \lambda_n \nearrow \infty$, $(a_n)_{n\in\mathbb{N}} \subset (0, \infty)$ such that $a_n \nearrow \infty$ and furthermore the sequence $\left(\frac{\lambda_1 + \cdots + \lambda_n}{a_n}\right)_{n\in\mathbb{N}}$ is bounded. Let $\mathcal{V} = (V_i)_{i\in\mathbb{N}}$, $V_i : X_i \times Y_i \to Z$ be a sequence of bounded bilinear* *operators such that* $\sup_{i\in\mathbb{N}}\|V_i\| < \infty$ *and the bilinear operator* $T_{\mathcal{V}} : c_0(\mathcal{X}) \times c_0(\mathcal{Y}) \to c_0(Z)$,

$$T_{\mathcal{V}}(x,y) = \left(\frac{\lambda_1 V_1(x_1,y_1) + \cdots + \lambda_n V_n(x_n,y_n)}{a_n}\right)_{n\in\mathbb{N}}.$$

Then:

(i) $T_{\mathcal{V}}$ *is* 1*-summing if and only if all* V_i *are* 1*-summing and* $\sum_{i=1}^{\infty}\frac{\lambda_i\pi_1(V_i)}{a_i} < \infty$. *Moreover,* $\pi_1(T_{\mathcal{V}}) = \sum_{i=1}^{\infty}\frac{\lambda_i\pi_1(V_i)}{a_i}$.

(ii) $T_{\mathcal{V}}$ *is* 2*-dominated if and only if all* V_i *are* 2*-dominated and* $\sum_{i=1}^{\infty}\frac{\lambda_i\Delta_2(V_i)}{a_i} < \infty$. *Moreover,* $\Delta_2(T_{\mathcal{V}}) = \sum_{i=1}^{\infty}\frac{\lambda_i\Delta_2(V_i)}{a_i}$.

Proof. Since $(\alpha_{nk})_{(n,k)\in\mathbb{N}\times\mathbb{N}}$, $\alpha_{nk} = \begin{cases} \frac{\lambda_k}{a_n} \text{ for } k \le n \\ 0 \text{ for } k \ge n+1 \end{cases}$ is a method of summability, $T_{\mathcal{V}}$ is well defined. Also, let us note the following formal decomposition

$$\begin{aligned} T_{\mathcal{V}}(x,y) = & \left(\frac{\lambda_1 V_1(x_1,y_1)}{a_1}, \frac{\lambda_1 V_1(x_1,y_1)}{a_2}, \ldots, \frac{\lambda_1 V_1(x_1,y_1)}{a_n}, \ldots\right) \\ & + \left(0, \frac{\lambda_2 V_2(x_2,y_2)}{a_2}, \ldots, \frac{\lambda_2 V_2(x_2,y_2)}{a_n}, \ldots\right) + \cdots \end{aligned}$$

which suggests that

$$T_{\mathcal{V}}(x,y) = \sum_{i=1}^{\infty} S_i(x_i,y_i), \tag{7}$$

where

$$S_i : X_i \times Y_i \to c_0(Z),$$

$$S_i(x_i,y_i) = \left(0,\ldots,0, \frac{\lambda_i V_i(x_i,y_i)}{a_i}, \frac{\lambda_i V_i(x_i,y_i)}{a_{i+1}}, \frac{\lambda_i V_i(x_i,y_i)}{a_{i+2}}, \ldots\right).$$

Indeed, let $k \in \mathbb{N}$. For $x = (x_i)_{i\in\mathbb{N}} \in c_0(\mathcal{X})$, $y = (y_i)_{i\in\mathbb{N}} \in c_0(\mathcal{Y})$ we denote by

$$\begin{aligned} T_k(x,y) &= \sum_{i=1}^{k} S_i(x_i,y_i) \\ &= \left(\frac{\lambda_1 V_1(x_1,y_1)}{a_1}, \ldots, \frac{\sum_{i=1}^{k}\lambda_i V_i(x_i,y_i)}{a_k}, \frac{\sum_{i=1}^{k}\lambda_i V_i(x_i,y_i)}{a_{k+1}}, \ldots\right) \end{aligned}$$

the partial sum of the series. Then

$$\begin{aligned} & T_{\mathcal{V}}(x,y) - T_k(x,y) \\ = & \left(0,\ldots,0, \tfrac{\lambda_{k+1}V_{k+1}(x_{k+1},y_{k+1})}{a_{k+1}}, \tfrac{\lambda_{k+1}V_{k+1}(x_{k+1},y_{k+1}) + \lambda_{k+2}V_{k+2}(x_{k+2},y_{k+2})}{a_{k+2}}, \ldots\right), \end{aligned}$$

hence

$$\|T_{\mathcal{V}}(x,y)-T_k(x,y)\|_{c_0(Z)}=\sup_{i\in\mathbb{N}}\tfrac{\|\lambda_{k+1}V_{k+1}(x_{k+1},y_{k+1})+\cdots+\lambda_{k+i}V_{k+i}(x_{k+i},y_{k+i})\|}{a_{k+i}}.$$

Since

$$\begin{aligned}&\frac{\|\lambda_{k+1}V_{k+1}(x_{k+1},y_{k+1})+\cdots+\lambda_{k+i}V_{k+i}(x_{k+i},y_{k+i})\|}{a_{k+i}}\\
&\leq\frac{\lambda_{k+1}\|V_{k+1}\|\|x_{k+1}\|\|y_{k+1}\|+\cdots+\lambda_{k+i}\|V_{k+i}\|\|x_{k+i}\|\|y_{k+i}\|}{a_{k+i}}\\
&\leq\frac{\left(\sup_{i\in\mathbb{N}}\|V_i\|\right)(\lambda_{k+1}\|x_{k+1}\|\|y_{k+1}\|+\cdots+\lambda_{k+i}\|x_{k+i}\|\|y_{k+i}\|)}{a_{k+i}}\\
&\leq\frac{\lambda_{k+1}+\cdots+\lambda_{k+i}}{a_{k+i}}\left(\sup_{i\in\mathbb{N}}\|V_i\|\right)\left(\sup_{i\geq k+1}\|x_i\|\right)\left(\sup_{i\geq k+1}\|y_i\|\right)\\
&\leq\left(\sup_{n\in\mathbb{N}}\frac{\lambda_1+\cdots+\lambda_n}{a_n}\right)\left(\sup_{i\in\mathbb{N}}\|V_i\|\right)\left(\sup_{i\geq k+1}\|x_i\|\right)\left(\sup_{i\geq k+1}\|y_i\|\right).\end{aligned}$$

Thus

$$\begin{aligned}&\|T_{\mathcal{V}}(x,y)-T_k(x,y)\|_{c_0(Z)}\\
&\leq\left(\sup_{n\in\mathbb{N}}\frac{\lambda_1+\cdots+\lambda_n}{a_n}\right)\left(\sup_{i\in\mathbb{N}}\|V_i\|\right)\left(\sup_{i\geq k+1}\|x_i\|\right)\left(\sup_{i\geq k+1}\|y_i\|\right)\end{aligned}$$

and since $x=(x_i)_{i\in\mathbb{N}}\in c_0(\mathcal{X})$, $y=(y_i)_{i\in\mathbb{N}}\in c_0(\mathcal{Y})$, it follows that $\lim_{k\to\infty}\|T_{\mathcal{V}}(x,y)-T_k(x,y)\|_{c_0(Z)}=0$, hence the convergence of the series (7) is proved.

By Proposition 15, $T_{\mathcal{V}}$ is 1-summing (2-dominated) if and only if all S_i are 1-summing (2-dominated) and $\sum_{i=1}^{\infty}\pi_1(S_i)<\infty$ ($\sum_{i=1}^{\infty}\Delta_2(S_i)<\infty$). Moreover, $\pi_1(S_{\mathcal{V}})=\sum_{i=1}^{\infty}\pi_1(S_i)$ ($\Delta_2(C_{\mathcal{V}})=\sum_{i=1}^{\infty}\Delta_2(V_i)$). In order to evaluate $\pi_1(S_i)(\Delta_2(V_i))$, $S_i:X_i\times Y_i\to c_0(Z)$, for $(x,y)\in X_i\times Y_i$ note that

$$\begin{aligned}\|S_i(x,y)\|_{c_0(Z)}&=\left\|\left(0,\ldots,0,\frac{\lambda_iV_i(x,y)}{a_i},\frac{\lambda_iV_i(x,y)}{a_{i+1}},\frac{\lambda_iV_i(x,y)}{a_{i+2}},\ldots\right)\right\|_{c_0(Z)}\\
&=\sup\left\{\frac{\lambda_i\|V_i(x,y)\|}{a_i},\frac{\lambda_i\|V_i(x,y)\|}{a_{i+1}},\ldots\right\}\\
&=\|V_i(x,y)\|\sup\left\{\frac{\lambda_i}{a_i},\frac{\lambda_i}{a_{i+1}},\ldots\right\}=\frac{\lambda_i\|V_i(x,y)\|}{a_i}.\end{aligned}$$

Hence, $S_i:X_i\times Y_i\to c_0(Z)$ is 1-summing (2-dominated) if and only if $V_i:X_i\times Y_i\to Z$ is 1-summing (2-dominated). Moreover, $\pi_1(S_i)=\frac{\lambda_i\pi_1(V_i)}{a_i}$ ($\Delta_2(S_i)=\frac{\lambda_i\Delta_2(V_i)}{a_i}$). □

By considering, for instance, $\lambda_i = 1$, $a_i = i$, $i \in \mathbb{N}$, we obtain the conditions for the bilinear Cesàro operator on $c_0(\mathcal{X}) \times c_0(\mathcal{Y})$ to be 1-summing or 2-dominated. Hence we are adding a new result related to [2, Corollary 1] which studies the summing nature of the Cesàro operator on $c_0(\mathcal{X}) \times c_0(\mathcal{Y})$.

Corollary 17. *Let $\mathcal{V} = (V_i)_{i\in\mathbb{N}}$, $V_i : X_i \times Y_i \to Z$ be a sequence of bounded bilinear operators such that $\sup_{i\in\mathbb{N}} \|V_i\| < \infty$.*

(a) *Let $C_{\mathcal{V}} : c_0(\mathcal{X}) \times c_0(\mathcal{Y}) \to c_0(Z)$, $C_{\mathcal{V}}(x, y) = \left(\frac{V_1(x_1,y_1)+\cdots+V_n(x_n,y_n)}{n}\right)_{n\in\mathbb{N}}$ be the bilinear Cesàro operator. Then*:

(i) *$C_{\mathcal{V}}$ is 1-summing if and only if all V_i are 1-summing and $\sum_{i=1}^{\infty} \frac{\pi_1(V_i)}{i} < \infty$. Moreover, $\pi_1(C_{\mathcal{V}}) = \sum_{i=1}^{\infty} \frac{\pi_1(V_i)}{i}$.*

(ii) *$C_{\mathcal{V}}$ is 2-dominated if and only if all V_i are 2-dominated and $\sum_{i=1}^{\infty} \frac{\Delta_2(V_i)}{i} < \infty$. Moreover, $\Delta_2(C_{\mathcal{V}}) = \sum_{i=1}^{\infty} \frac{\Delta_2(V_i)}{i}$.*

(b) *Let $H_{\mathcal{V}} : c_0(\mathcal{X}) \times c_0(\mathcal{Y}) \to c_0(Z)$,*

$$H_{\mathcal{V}}(x, y) = \left(\frac{V_1(x_1, y_1) + \frac{1}{2}V_2(x_2, y_2) + \cdots + \frac{1}{n}V_n(x_n, y_n)}{\ln(n+1)}\right)_{n\in\mathbb{N}}.$$

Then:

(i) *$H_{\mathcal{V}}$ is 1-summing if and only if all V_i are 1-summing and $\sum_{i=1}^{\infty} \frac{\pi_1(V_i)}{i\ln(i+1)} < \infty$. Moreover, $\pi_1(H_{\mathcal{V}}) = \sum_{i=1}^{\infty} \frac{\pi_1(V_i)}{i\ln(i+1)}$.*

(ii) *$H_{\mathcal{V}}$ is 2-dominated if and only if all V_i are 2-dominated and $\sum_{i=1}^{\infty} \frac{\Delta_2(V_i)}{i\ln(i+1)} < \infty$. Moreover, $\Delta_2(H_{\mathcal{V}}) = \sum_{i=1}^{\infty} \frac{\Delta_2(V_i)}{i\ln(i+1)}$.*

Acknowledgement. We thank the referee for his/her constructive comments and remarks which improved the first version of the paper.

References

[1] G. Badea, *On nuclear and multiple summing bilinear operators on $c_0 \times c_0$*, Quaest Math. **33** (2010), no. 2, 253–261.

[2] G. Badea and D. Popa, *Swartz type results for nuclear and multiple 1-summing bilinear operators on $c_0(\mathcal{X}) \times c_0(\mathcal{Y})$*, Positivity **19** (2015), no. 3, 475–487.

[3] G. Botelho, *Cotype and absolutely summing multilinear mappings and homogeneous polynomials*, Proc. Royal Irish Acad. Sect. A **97** (1997), no. 2, 145–153.

[4] R. Cilia and J. M. Gutiérrez, *Nuclear and integral polynomials*, J. Aust. Math. Soc. **76** (2004), no. 2, 269–280.

[5] A. Defant and K. Floret, *Tensor norms and operator ideals*, Math. Studies 176, North-Holland, 1993.

[6] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely Summing Operators*, Cambridge Stud. Adv. Math. 43, Cambridge University Press, 1995.

[7] S. Dineen, *Complex analysis in locally convex spaces*, North Holland Math Studies, vol 57, North Holland, Amsterdam 1981.

[8] N. Dunford and J. T. Schwartz, *Linear Operators I*: *General Theory*, Pure and Applied Mathematics, vol. 7, Interscience, New York, 1958.

[9] K. Floret and D. García, *On ideals of polynomials and multilinear mappings between Banach spaces*, Arch. Math. (Basel) **81** (2003), no. 3, 300–308.

[10] A. Nahoum, *Applications radonifiantes dans l'espace des series convergentes. II*: *Les résultats*, Sém. Maurey-Schwartz 1972-1973, exp XXV (1973), https://www.numdam.org.

[11] A. Pietsch, *Ideals of multilinear functionals*, Proc. 2^{nd} Int. Conf. Operator Alg., Leibzig, Teubner Texte Math. **62** (1983), 185–199.

[12] ______, *Eigenvalues and s-numbers*, Cambridge Stud. Adv. Math. 13, Cambridge University Press, 1987.

[13] D. Popa, *Examples of operators on $C[0,1]$ distinguishing certain operator ideals*, Arch. Math. **88** (2007), no. 4, 349–357.

[14] ______, *Averages and compact, absolutely summing and nuclear operators on $C(\Omega)$*, J. Korean Math. Soc. **47** (2010), no. 5, 899–924.

[15] N. Tomczak-Jagermann, *Banach-Mazur distances and finite dimensional operator ideals*, Pitman Monographs, vol. 38, Harlow: Longman Scientific & Technical, 1989.

[16] P. Wojtaszczyk, *Banach spaces for analysts*, Cambridge Stud. Adv. Math. 25, Cambridge University Press, 1996.

Gabriela Badea
Department of Mathematics
Ovidius University of Constanta
Bd. Mamaia 124
900527 Constanta, Romania
E-mail address: gbadea@univ-ovidius.ro

Dumitru Popa
Department of Mathematics
Ovidius University of Constanta
Bd. Mamaia 124
900527 Constanta, Romania
E-mail address: dpopa@univ-ovidius.ro